Transient Resonances in the Inspirals of Point Particles into Black Holes

Éanna É. Flanagan and Tanja Hinderer

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Transient resonances in the inspirals of point particles into black holes

Éanna É. Flanagan
Center for Radiophysics and Space Research, Cornell University, Ithaca, NY 14853, USA

Tanja Hinderer
Theoretical Astrophysics, California Institute of Technology, Pasadena, CA 91125, USA

We show that transient resonances occur in the two body problem in general relativity, for spinning black holes in close proximity to one another when one black hole is much more massive than the other. These resonances occur when the ratio of polar and radial orbital frequencies, which is slowly evolving under the influence of gravitational radiation reaction, passes through a low order rational number. At such points, the adiabatic approximation to the orbital evolution breaks down, and there is a brief but order unity correction to the inspiral rate. The resonances cause a perturbation to orbital phase of order a few tens of cycles for mass ratios $\sim 10^{-6}$, make orbits more sensitive to changes in initial data (though not quite chaotic), and are genuine non-perturbative effects that are not seen at any order in a standard post-Newtonian expansion. Our results apply to an important potential source of gravitational waves, the gradual inspiral of white dwarfs, neutron stars, or black holes into much more massive black holes. Resonances effects will increase the computational challenge of accurately modeling these sources.

Introduction: The dynamics of a two-body system emitting gravitational radiation is an important problem in general relativity. Binary systems of compact bodies undergo a radiation-reaction-driven inspiral until they merge. There are three different regimes in the parameter space of these systems: (i) The *weak field, Newtonian regime* $r \gg 1$, where r is the orbital separation in units where $G = c = M = 1$ and M is the total mass [1]. (ii) The *relativistic, equal mass regime* $r \sim 1$, $\varepsilon \sim 1$ (where $\varepsilon = \mu/M$ is the mass ratio with μ being the reduced mass) [2]. (iii) The *relativistic, extreme mass ratio regime* $r \sim 1$, $\varepsilon \ll 1$, which is characterized by long, gradual inspirals on a timescale $\sim \varepsilon^{-1}$, and for which computational methods are currently under development.

In this Letter, we show that in the relativistic, extreme mass ratio regime, there are qualitatively new aspects to the two-body problem in general relativity, namely the effects of transient resonances. While resonances are a common phenomenon in celestial mechanics when three or more objects are involved [3], they can occur with just two objects in general relativity, due to its nonlinearity. They are non-perturbative effects for highly relativistic sources and are not seen at any order in standard post-Newtonian expansions of inspiral solutions. Their existence is closely related to the onset of chaotic dynamics, which has previously been shown to occur in general relativity in other, cosmological contexts [4].

The resonances have direct observational relevance: Compact objects ($1 \lesssim \mu/M_\odot \lesssim 10$, where $M_\odot$ is the Solar mass) inspiraling into much larger black holes are expected to be a key source for gravitational wave detectors. Advanced LIGO will potentially observe $3-30$ such events per year, with $50 \lesssim M/M_\odot \lesssim 1000$ [5], and future space-based detectors are expected to detect such inspirals with $10^4 \lesssim M/M_\odot \lesssim 10^7$ out to cosmological distances at a rate of ~ 50 per year [6]. The observed gravitational wave signal will be rich in information. For example, one will be able to extract a map of the spacetime geometry of the central object and test if it is consistent with general relativity's predictions for a black hole [5, 7]. Such mapping will require accurate theoretical models of the gravitational waveforms, which remain phase coherent with the true waveforms to an accuracy of ~ 1 cycle over the large number $\sim \varepsilon^{-1} \sim M/\mu \sim 10^2 - 10^6$ of cycles of inspiral. Over the past decade there has been a significant research effort aimed at providing such accurate models [8]. Resonances will complicate this enterprise, as we discuss below.

Method of Analysis: Over timescales short compared to the dephasing time $\sim \varepsilon^{-1/2}$, inspirals can be accurately modeled using black hole perturbation theory, with ε as the expansion parameter. The leading order motion is geodesic motion on the background Kerr metric. At the next order the motion is corrected by the particle's self-force or radiation reaction force, for which a formal expression is known [9], and which has been computed explicitly in special cases; see, e.g., the review [8]. Over the longer inspiral timescale $\sim \varepsilon^{-1}$, it is necessary to augment these methods with two-timescale expansions which are currently under development [10, 11]. In this framework the leading order motion is an adiabatic inspiral, and there are various post-adiabatic corrections.

Geodesic motion in the Kerr spacetime is an integrable dynamical system, and it is useful to use the corresponding generalized action-angle variables to parameterize the inspiral. The resulting equations are [17]:

$$\frac{dq_\alpha}{d\tau} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2), \tag{1a}$$

$$\frac{dJ_\nu}{d\tau} = \varepsilon G_\nu^{(1)}(q_\theta, q_r, \mathbf{J}) + \varepsilon^2 G_\nu^{(2)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^3). \tag{1b}$$

Here τ is Mino time [13] and J_ν are the conserved integrals of geodesic motion given by $J_\nu = (E/\mu, L_z/\mu, Q/\mu^2)$, where E is the energy, L_z is the angular momentum, and Q the Carter constant. The variables $q_\alpha = (q_t, q_r, q_\theta, q_\phi)$ are generalized angle variables conjugate to Mino time [10]. The right hand sides at $O(\varepsilon^0)$ describe geodesic motion, with fundamental frequencies ω_r, ω_θ and ω_ϕ. The forcing functions $g^{(1)}_\alpha$, $G^{(1)}_\nu$ and $G^{(2)}_\nu$ are due to the first order and second order self-forces, and are 2π-periodic in q_θ and q_r. The piece of $G^{(1,2)}_\nu$ that is even under $q_\theta \to 2\pi - q_\theta$, $q_r \to 2\pi - q_r$, and the piece of $g^{(1)}_\alpha$ that is odd, are the dissipative self-force, and the remaining piece is the conservative self-force [10].

In the limit $\varepsilon \to 0$, solutions to Eqs. (1) can be derived using the two-timescale method, which essentially consists of an ansatz for the dependence of the solutions on ε which is more complicated than a Taylor series expansion, that is justified *a posteriori* [10, 14]. The leading order solutions are given by the following *adiabatic prescription*: Drop the forcing terms $g^{(1)}_\alpha$ and $G^{(2)}_\nu$, and replace $G^{(1)}_\nu$ by its average over the 2-torus parameterized by q_θ, q_r. It is now known how to evaluate this averaged force explicitly [12, 13], although generic adiabatic inspirals have not yet been computed numerically.

Consider now post-adiabatic effects. The dynamical system (1) consists of a perturbed, integrable Hamiltonian system. Resonances in this general type of system have been studied in detail and are well understood [14], and we can apply the general theory to the present context. The existence of resonances in this context has previously been suggested by Refs. [15, 16]. We will present three different treatments of the resonances: (i) An intuitive, order of magnitude discussion, which is sufficient to deduce their key properties; (ii) A numerical treatment; and (iii) A sketch of a formal analytic derivation. A more detailed treatment will be presented in Ref. [17].

Order of Magnitude Estimates: Suppose that we have an adiabatic solution, which will be of the form $q_\alpha(\tau,\varepsilon) = \psi_\alpha(\varepsilon\tau)/\varepsilon$, $J_\nu(\tau,\varepsilon) = J_\nu(\varepsilon\tau)$. Consider now the post-adiabatic correction terms in Eqs. (1), near some arbitrarily chosen point $\tau = 0$. We expand q_θ as $q_\theta = q_{\theta 0} + \omega_{\theta 0}\tau + \dot\omega_{\theta 0}\tau^2 + O(\tau^3)$, where subscripts 0 denote evaluations at $\tau = 0$, and we expand q_r similarly. We also expand $G^{(1)}_\nu$ as a double Fourier series: $G^{(1)}_\nu(q_\theta, q_r, \mathbf{J}) = \sum_{k,n} G^{(1)}_{\nu\,kn}(\mathbf{J}) e^{i(kq_\theta + nq_r)}$, where the 00 term is the adiabatic approximation, and the remaining terms drive post-adiabatic effects. Inserting the expansions of q_θ and q_r, we find for the phase of the (k,n) Fourier component

$$(\text{constant}) + (k\omega_{\theta 0} + n\omega_{r0})\tau + (k\dot\omega_{\theta 0} + n\dot\omega_{r0})\tau^2 + \dots \quad (2)$$

Normally, the second term is nonzero and thus the force oscillates on a timescale ~ 1, much shorter than the inspiral timescale $\sim 1/\varepsilon$, and so the force averages to zero. However, when the resonance condition $k\omega_{\theta 0} + n\omega_{r0} = 0$ is satisfied, the (k,n) force is slowly varying and cannot be neglected, and so gives an order-unity correction to the right hand side of Eq. (1b). The duration of the resonance is given by the third term in (2) to be $\tau_{\rm res} \sim 1/\sqrt{v\dot\omega} \sim 1/\sqrt{v\varepsilon}$, where $v = |k| + |n|$ is the order of the resonance; after times longer than this the quadratic term causes the force to oscillate and again average to zero. The net change in the action variables J_ν is therefore $\Delta J_\mu \sim \dot J\tau_{\rm res} \sim \varepsilon\tau_{\rm res} \sim \sqrt{\varepsilon/v}$. After the resonance, this change causes a phase error $\Delta\phi$ that accumulates over an inspiral, of order the total inspiral phase $\sim 1/\varepsilon$ times $\Delta J/J \sim \sqrt{\varepsilon/v}$, which gives $\Delta\phi \sim 1/\sqrt{v\varepsilon}$.

This discussion allows us to deduce several key properties of the resonances. First, corrections to the gravitational wave signal's phase due to resonance effects scale as the square root of the inverse of mass of the small body. These corrections thus become large in the extreme-mass-ratio limit, dominating over all other post-adiabatic effects, which scale as $\varepsilon^0 \sim 1$.

Second, they occur when ω_r/ω_θ is a low order rational number. There is a simple geometric picture corresponding to this condition [16, 18]: the geodesic orbits do not ergodically fill out the (q_θ, q_r) torus in space as generic geodesics orbits do, but instead form a 1 dimensional curve on the torus. This implies that the time-averaged forces for these orbits are not given by an average over the torus, unlike the case for generic orbits.

Third, they occur only for non-circular, non-equatorial orbits about spinning black holes. For other cases, the forcing terms $G^{(1)}_\nu$ depend only on q_θ, or only on q_r, but not both together, and thus the Fourier coefficient $G^{(1)}_{\nu\,kn}$ will vanish for any resonance.

Fourth, they are driven only by the spin-dependent part of the self-force, for the same reason: spherical symmetry forbids a dependence on q_θ in the zero-spin limit.

Fifth, they appear to be driven only by the dissipative part of the self-force, and not by the conservative part, again because the forcing terms do not depend on both q_θ and q_r. We have verified that this is the case up to the post-Newtonian order that spin-dependent terms have been computed [19], and we conjecture that it is true to all orders. The reason that this occurs is that the conservative sector of post-Newtonian theory admits three independent conserved angular momentum components; the ambiguities in the definition of angular momentum are associated with radiation, in the dissipative sector. As a consequence, the perturbed conservative motion is integrable to leading order in ε, and an integrable perturbation to a Hamiltonian cannot drive resonances.

Sixth, although the resonance is directly driven only by dissipative, spin dependent self-force, computing resonance effects requires the conservative piece of the first order self-force and the averaged, dissipative piece of the second order self-force. Those pieces will cause $O(1)$ corrections to the phases over a complete inspiral [10], and the kicks ΔJ_μ produced during the resonance depend on

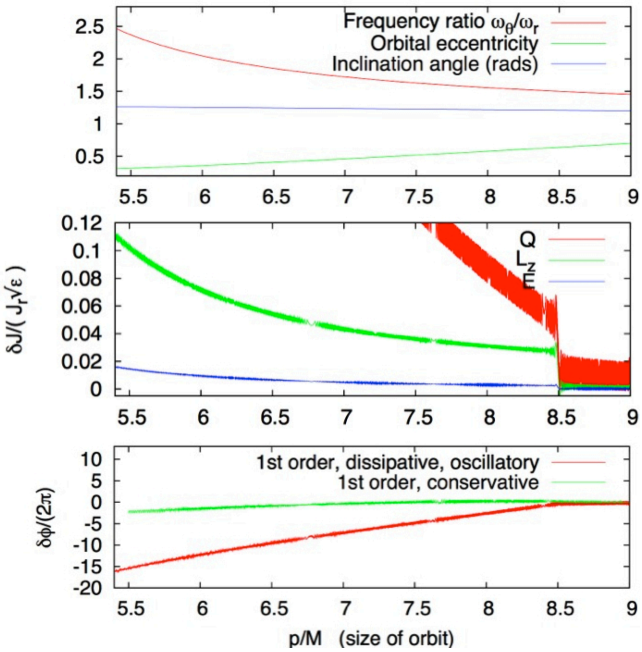

FIG. 1: [Top] The adiabatic inspiral computed from our approximate post-Newtonian self-force, for a mass ratio $\varepsilon = \mu/M = 3 \times 10^{-6}$, with black hole spin parameter $a = 0.95$, with initial conditions semilatus rectum $p = 9.0M$, eccentricity $e = 0.7$, and orbital inclination $\theta_{\rm inc} = 1.20$. The bottom curve is e, the middle curve is $\theta_{\rm inc}$, and the top curve is ratio of frequencies ω_θ/ω_r, shown as functions of p. [Middle] The fluctuating, dissipative part of the first order self-force causes a strong resonance when $\omega_\theta/\omega_r = 3/2$ at $p = 8.495$. Shown are the corrections to the energy E, angular momentum L_z and Carter constant Q, as functions of p, scaled to their values at resonance, and divided by the square root $\sqrt{\mu/M}$ of the mass ratio. The sudden jumps at the resonance are apparent, with the largest occurring for the Carter constant. [Bottom] The lower curve is the correction to the number of cycles $\phi/(2\pi)$ of azimuthal phase of the inspiral caused by the fluctuating, dissipative part of the first order self-force. The sharp downward kick due to the resonance at $p = 8.495$ can be clearly seen. The resonant corrections to the number of cycles of r and θ motion are similar. These phase shifts scale as $\sqrt{M/\mu}$. The upper curve is the post-adiabatic phase correction due to the conservative piece of the first order self-force, which is considerably smaller and is independent of the mass ratio.

the $O(1)$ phases at the start of the resonance.

Seventh, resonances give rise to increased sensitivity to initial conditions, analogous to chaos but not as extreme as chaos, because at a resonance information flows from a higher to a lower order in the perturbation expansion. For example, we have argued that changes to the phases at $O(1)$ prior to the resonance will affect the post-resonance phasing at $O(1/\sqrt{\varepsilon})$. Similarly changes to the phases at $O(\sqrt{\varepsilon})$ before resonance will produce $O(1)$ changes afterwards. With several successive resonances, a sensitive dependence on initial conditions could arise.

Numerical Integrations: The scaling relation $\Delta\phi \propto 1/\sqrt{\varepsilon}$ suggests the possibility of phase errors large compared to unity that impede the detection of the gravitational wave signal. To investigate this possibility, we numerically integrated the exact Kerr geodesic equations supplemented with approximate post-Newtonian forcing terms. While several such approximate inspirals have been computed previously [20], none have encountered resonances, because resonances require non-circular, non-equatorial orbits about a spinning black hole with non orbit-averaged forces, which have not been simulated before.

For the numerical integrations we use instead of q_α the variables $\bar{q}_\alpha = (\bar{q}_t, \bar{q}_r, \bar{q}_\theta, \bar{q}_\phi) = (t, \psi, \chi, \phi)$ where ψ and χ are the angular variables for r and θ motion defined in Ref. [13]. The equations of motion (1) in these variables are

$$t_{,\tau} = \bar{\omega}_t(\bar{q}_\theta, \bar{q}_r, \mathbf{J}), \quad \phi_{,\tau} = \bar{\omega}_\phi(\bar{q}_\theta, \bar{q}_r, \mathbf{J}), \tag{3a}$$

$$\bar{q}_{\theta,\tau} = \bar{\omega}_\theta(\bar{q}_\theta, \mathbf{J}) + \varepsilon h_\theta^{(1)}(\bar{q}_\theta, \bar{q}_r, \mathbf{J}) + O(\varepsilon^2), \tag{3b}$$

$$\bar{q}_{r,\tau} = \bar{\omega}_r(\bar{q}_r, \mathbf{J}) + \varepsilon h_r^{(1)}(\bar{q}_\theta, \bar{q}_r, \mathbf{J}) + O(\varepsilon^2), \tag{3c}$$

$$J_{\nu,\tau} = \varepsilon H_\nu^{(1)}(\bar{q}_\theta, \bar{q}_r, \mathbf{J}) + O(\varepsilon^2). \tag{3d}$$

Here τ is Mino time [13], the frequencies $\bar{\omega}$ are given in [13], and $h_\alpha^{(1)}$ and $H_\nu^{(1)}$ are given in terms of the components of the 4-acceleration in [21].

We parameterize the three independent components of the acceleration in the following way: $a^\alpha = a^{\hat{r}} e_{\hat{r}}^\alpha + a^{\hat{\theta}} e_{\hat{\theta}}^\alpha + a_\perp \epsilon^\alpha{}_{\beta\gamma\delta} u^\beta e_{\hat{r}}^\gamma e_{\hat{\theta}}^\delta + (a^{\hat{r}} u_{\hat{r}} + a^{\hat{\theta}} u_{\hat{\theta}}) u^\alpha$, where $\vec{u}$ is the 4-velocity and $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ are unit vectors in the directions of ∂_r and ∂_θ. We compute the dissipative pieces of $a^{\hat{r}}$, $a^{\hat{\theta}}$ and $a_\perp$ from the results of [22], as functions of $\tilde{r} = r + a^2/(4r)$, $E_n = E - 1$, and $\bar{K} = Q + a^2 L_z^2 + a^2 E_n$, and then expand to $O(a^2)$ and to the leading post-Newtonian order at each order in a [17]. We also add the conservative component, expressed similarly and computed to $O(a)$ and to the leading post-Newtonian order [23]; see Ref. [17].

We numerically integrate Eqs. (3) twice, once using the adiabatic prescription, and once exactly, and then subtract at fixed t to obtain the post-adiabatic effects. The adiabatic prescription involves numerically integrating the right hand sides over the torus parameterized by q_θ, q_r at each time step, where $q_r = F_r(\bar{q}_r)/F_r(2\pi)$, $F_r(\bar{q}_r) = \int_0^{\bar{q}_r} d\bar{q}_r/[\bar{\omega}_r(\bar{q}_r, \mathbf{J})]$, with a similar formula for q_θ. This is numerically time consuming, but the adiabatic integration can take timesteps on the inspiral timescale $\sim 1/\varepsilon$ rather than the dynamical timescale ~ 1.

Typical results are shown in Fig. 1, which shows the adiabatic inspiral for a mass ratio of $\varepsilon = 3 \times 10^{-6}$ with $a = 0.95$, in terms of the relativistic eccentricity e, semilatus rectum p and orbital inclination $\theta_{\rm inc}$, which are functions of E, L_z and Q [24]. This example has a strong resonance at $\omega_\theta/\omega_r = 3/2$, that generates jumps in the conserved quantities of order a few percent times $\sqrt{\varepsilon}$, and causes phase errors over the inspiral of order 20 cycles. Phase errors of this magnitude will be a significant impediment to signal detection with matched filtering. We find that the resonance effects are dominated by the $O(a^2)$ terms, and the effect of the $O(a)$ terms are small. Additional resonances can occur at higher values of p, but the dominant resonances are likely to be the low-order ones in the relativistic regime of small p.

Analytic Derivation: In terms of the slow time variable $\tilde{\tau} = \varepsilon\tau$, the solutions of the dynamical system (1) away from resonances can be expressed as an asymptotic expansion in ε at fixed $\tilde{\tau}$ [10, 14]:

$$q_\alpha(\tau,\varepsilon) \;=\; \frac{1}{\varepsilon}\left[\psi_\alpha^{(0)}(\tilde{\tau}) + \sqrt{\varepsilon}\psi_\alpha^{(1/2)}(\tilde{\tau}) + O(\varepsilon)\right], \quad \text{(4a)}$$

$$J_\nu(\tau,\varepsilon) \;=\; \mathcal{J}_\nu^{(0)}(\tilde{\tau}) + \sqrt{\varepsilon}\mathcal{J}_\nu^{(1/2)}(\tilde{\tau}) + O(\varepsilon). \quad \text{(4b)}$$

The leading order terms give the adiabatic approximation described above, and satisfy [10] $\psi^{(0)}_{\alpha,\tilde{\tau}} = \omega_\alpha[\boldsymbol{\mathcal{J}}^{(0)}]$, $\mathcal{J}^{(0)}_{\nu,\tilde{\tau}} = \langle G_\nu^{(1)}\rangle[\boldsymbol{\mathcal{J}}^{(0)}]$, where the angular brackets denote an average over the (q_r, q_θ) torus. The subleading, post-1/2-adiabatic order terms satisfy $\mathcal{J}^{(1/2)}_{\nu,\tilde{\tau}} - \langle G_\nu^{(1)}\rangle_{,J_\mu}\mathcal{J}_\mu^{(1/2)} = \Delta J_\nu^{(1/2)}\delta(\tau)$, $\psi^{(1/2)}_{\alpha,\tilde{\tau}} = \omega_{\alpha,J_\mu}\mathcal{J}_\mu^{(1/2)}$, where the δ-function source term arises at a resonance, taken to occur at $\tau = 0$.

Near the resonance we use an ansatz for the solutions which is an asymptotic expansion in $\sqrt{\varepsilon}$ at fixed $\hat{\tau} = \sqrt{\varepsilon}\tau$, and then match these solutions onto pre-resonance and post-resonance solutions of the form (4) [14]. To linear order in the force Fourier coefficients (an approximation which is valid here to within a few percent [17]), the jumps in the action variables for a resonance (k, n) can be computed by substituting the adiabatic solutions into the right hand side of Eqs. (1) and solving for the perturbation to the action variables. The result is

$$\Delta J_\nu^{(1/2)} = \sum_{s\neq 0}\sqrt{\frac{2\pi}{|\alpha s|}}\exp\left[\operatorname{sgn}(\alpha s)\frac{i\pi}{4} + is\chi_{\rm res}\right]G^{(1)}_{\nu\, sk,sn},$$

where $\chi_{\rm res} = kq_\theta + nq_r$, $\alpha = k\omega_{\theta,\tilde{\tau}} + n\omega_{r,\tilde{\tau}}$ and all quantities are evaluated at the resonance $\tau = 0$ using the adiabatic solution. In Ref. [17] we give the exact expression for this quantity that does not linearize in the force Fourier coefficients. We note that evaluating the phase $\chi_{\rm res}$ requires knowledge of the second subleading, $O(1)$ phase in Eq. (4a), which in turn requires knowledge of the force components $g_\alpha^{(1)}$, $G_\nu^{(1)}$ and $\langle G_\nu^{(2)}\rangle$ in Eq. (1) [10]. In addition, to obtain the phase to $O(1)$ accuracy after the resonance, it is necessary to also compute the subleading, $O(\varepsilon)$ jumps in J_ν and $O(1)$ jumps in q_α, which are given in [17].

Discussion: The dynamics of binary systems in general relativity is richer than had been appreciated. Transient resonances occurring during the inspiral invalidate the adiabatic approximation and give rise to corrections to the orbital phase that can be large compared to unity. It will be necessary to incorporate resonances into theoretical models of the gravitational waveforms for inspirals of compact objects into massive black holes, an important gravitational wave source. This will require knowledge of the second order gravitational self force and will be challenging.

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