Entanglement of a Quantum Field with a Dispersive Medium

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The entanglement of a quantum field with a dispersive medium

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In this letter we study the entanglement of a quantum radiation field interacting with a dielectric medium. In particular, we describe the quantum mixed state of a field interacting with a dielectric through plasma and Drude models, and show that these generate very different entanglement behavior, as manifested in the entanglement entropy of the field. We also present a formula for a "Casimir" entanglement entropy, i.e. the distance dependence of the field entropy. Finally, we study a toy model of the interaction between two plates. In this model, the field entanglement entropy is divergent, however, as in the Casimir effect, its distance dependent part is finite and the field matter entanglement is reduced when the objects are far.

The theory describing electromagnetic fluctuations in a linear material medium is an essential tool in understanding matter-radiation interaction. Classically, thermally excited fluctuations as well as purely quantum, zero temperature, fluctuations in macroscopic media have been widely studied in terms of susceptibilities (see e.g. [1] and Rytov et al. [22]). While the use of such response functions have been highly successful in describing various phenomena, the actual quantum *state* of the field is usually not described. Indeed, even at zero temperature, a field interacting with realistic materials (as opposed to idealized Dirichlet or Neumann boundaries), is in general not in a pure state and should be described in terms of a mixed state density matrix. In modern terms, this mixed state is viewed as a consequence of matter-field entanglement, and carries a non-vanishing von-Neuman entropy, sometimes called "entanglement entropy" (EE).

In this Letter we describe the quantum state associated with a field in a medium *given a dielectric function* ε. The effective action for such a field includes frequency dependence. Usually, such frequency dependent action is associated with response functions, however, here, we have to understand the action as specifying an *instantaneous* form for the photonic density matrix. We do this by describing the effective density matrix of the field.

As a measure of the field-matter entanglement, we use the entanglement entropy of the field. EE has been the subject of intense investigations in recent years, in a large variety of quantum systems, with applications to quantum information, condensed matter theory and high energy theory (For a review see e.g. [2]). In particular, the entropy of radiation coupled to matter has been considered in numerous works. Usually the focus is on a single degree of freedom coupled to a bath of oscillators: For example, the entropy of a spin in the spin-boson model within the frame work of the Caldeira-Legget model [3] was considered in [4] while the entanglement of a single radiation mode with an array of spins was studied in [5] for the Dicke model . On a different note, the EE of spatially separated intervals of vacuum (or ground state of a spin chains) has been considered in [6, 7]. Here we consider a situation distinct from these works, and more akin to the scenario considered in studies of Casimir and Van der Waals interactions between macroscopic bodies: that of a field in contact with macroscopic dispersive bodies.

Consider the electromagnetic field interacting with a dielectric medium at zero temperature. We assume that no external charges or currents are present and work in the gauge $A_0 = 0$. The long wave-length effective action,

$$S = \frac{1}{4\pi}\int \mathrm{d}^3x\mathrm{d}\omega \mathbf{A}^*_\omega(\mathbf{x})[\omega^2\varepsilon(\omega,\mathbf{x}) - \nabla\times\nabla]\mathbf{A}_\omega(\mathbf{x}), \quad (1)$$

encodes the field interaction with the material through the dielectric response function $\varepsilon(\omega, \mathbf{x})$. As we show bellow, the appearance of frequency dispersion in $\varepsilon(\omega, \mathbf{x})$ means that an action like (1) does not describe the state of the field as a ground state of a field Hamiltonian: indeed - the action is *non-local* in time and so cannot be quantized to yield a proper quantum Hamiltonian.

Clearly, one possible effect of coupling to the environment may be thermalization of our field. It is therefore tempting to try and describe the instantaneous state of such a field as effectively thermal i.e. $\rho \sim Z^{-1}e^{-\beta H_{eff}}$ for a reasonable effective field hamiltonian H_{eff}. Such effective "entanglement hamiltonians" have been the focus on recent studies due to their relation to the conformal edge spectra of fractional quantum hall systems [8, 9].

We show, however, that in the field-matter system, the field cannot be viewed as simply thermal. Rather, different ranges of momenta feel different effective temperatures. While the occupation probability of UV photons vanishes with high momenta, we still find that the field in (1) carries a logarithmic UV divergent quantum "zero point" entropy and show it's cutoff dependence in (9).

Finally, we consider, *a la* Casimir, the distance dependence of the entropy associated with field interaction with a pair of objects separated by vacuum, and show a toy model in which the "Casimir" entanglement entropy is UV finite and decays with distance. Physically, this means that the field becomes more entangled with the plates as they come closer.

Here we study a simplified scalar field version of the electromagnetic field action (1).

$$S = \frac{1}{4\pi}\int \mathrm{d}^3x\mathrm{d}\omega \phi^*_\omega(\mathbf{x})[\omega^2\varepsilon(\omega,\mathbf{x}) - \nabla^2]\phi_\omega(\mathbf{x}). \quad (2)$$

Note that the UV dependence study bellow is valid for the full electromagnetic problem, since in homogenous systems (1) separates naturally into two scalar fields describing the two polarization modes. When the permittivity is independent of ω, the action is local in time, and one can easily quantize the associated scalar action *assuming* the conjugate momentum π_ϕ can be expressed in terms of $\dot{\phi}$ and doesn't depend on external fields. Such an action follows from the Hamiltonian $H = \frac{1}{4\pi}\int d^3x[\frac{\pi^2}{\varepsilon(\mathbf{x})} + (\nabla\phi)^2]$. It describes, at zero temperature, a pure state, and as such will have no entropy [31].

The situation is fundamentally different if ε is ω dependent. The non-locality of the action (2) in time signals that our system is coupled to external degrees of freedom which have been integrated out, yielding a non trivial temporal response kernel. In such a case, the system cannot be in a pure state implying that our radiative system is entangled with the matter fields.

To proceed, let us briefly review the method of calculating entropies of gaussian states (For details see, e.g. [10, 11]). The calculation is facilitated by the fact that all information lies in the two point functions of the field. For a scalar field with n degrees of freedom ϕ_n and conjugate momenta π_n, one defines the vector: $(O_1, ..O_{2n}) = (\phi_1, \pi_1, ..\phi_n, \pi_n)$. The state of the field is then determined by the covariance matrix:

$$\gamma_{jk} = 2Re\langle(O_j - \langle O_j\rangle)(O_k - \langle O_k\rangle)\rangle. \qquad (3)$$

γ can be brought into a Williamson normal form $W\gamma W^T = diag(\mu_1, ...\mu_n, \mu_1, ...\mu_n)$ by means of a symplectic transformation $W \in Sp(2n)$ preserving the canonical commutation relations $[O_j, O_k] = i\sigma_{j,k}$, where σ is the $2n \times 2n$ matrix

$$\sigma = \oplus_{j=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad W^t\sigma W = \sigma.$$

The occupation probability of the normal modes are directly related to the symplectic eigenvalues μ_i of γ. Finally, the density matrix can be written as:

$$\rho = Z^{-1}e^{-\sum_i u_i\Phi_i^+\Phi_i} \; ; \; u_i = \log\frac{\mu_i + 1}{\mu_i - 1} \qquad (4)$$

where Φ_i^+ is a bosonic creation operators for the normal mode i. The entropy is given by:

$$\mathcal{S} = \sum_i h(\mu_i) \; ; \; h(\mu) = \tfrac{\mu+1}{2}\log\tfrac{\mu+1}{2} - \tfrac{\mu-1}{2}\log\tfrac{\mu-1}{2} \qquad (5)$$

An additional, technical simplification occurs, if we assume that no $\langle\phi\pi\rangle$ correlations are present, it is then known that the symplectic eigenvalues are square roots of eigenvalues of $\Gamma = \mathcal{GH}$, where $\mathcal{G}$ and $\mathcal{H}$ are field and field momentum two point functions, respectively.

For the translationally invariant medium the symplectic eigenvalues can be labeled by momentum and given by $\mu_k = 2\pi^{-1}(g_\mathbf{k}h_\mathbf{k})^{1/2}$, where $g_\mathbf{k}h_\mathbf{k}$ is the product $\mathcal{GH}$ in $\mathbf{k}$ space given by $g_k = \langle\phi^2\rangle_\mathbf{k} = \int d\mathbf{x}e^{i\mathbf{x}\cdot\mathbf{k}}\langle\phi(\mathbf{x}, t')\phi(\mathbf{0}, t)\rangle|_{t'\to t_+}$ and similarly $h_k = \langle\pi^2\rangle_\mathbf{k}$. The field entropy *per unit volume* can be written as:

$$\mathcal{S}_{field} = -V^{-1}\mathrm{Tr}\rho_{field}\log\rho_{field} = \int d\mathbf{k}h(\mu_\mathbf{k}) \qquad (6)$$

For concreteness, let us choose a typical dielectric function ε to use in (7) and (8). We take: $\varepsilon(\omega, \mathbf{x}) = 1 + 4\pi\chi(\omega, \mathbf{x})$. With a typical susceptibiliy of the form $\chi_b = \frac{\omega_p^2}{(\omega_0^2 - \omega^2 - i\gamma_p\omega)}$ (for a conductor we will add a Drude function $\chi_c = \chi_b + i\frac{\omega_c^2}{\omega(\gamma_c - i\omega)}$). The action (2) allows us to compute, per k mode:

$$g_k = \int_0^\infty d\omega\frac{1}{\omega^2\varepsilon(\mathbf{k}, i\omega) + k^2} \qquad (7)$$

and, using time point splitting,

$$h_k = \int_0^\infty d\omega\frac{1}{\omega^2\varepsilon(\mathbf{k}, i\omega) + k^2}(k^2 + 4\pi\omega^2\chi(i|\omega|)). \qquad (8)$$

Using, the integrals (7),(8) and (4) we represent the density matrix as $\rho \sim exp(-\sum_\mathbf{k} u_\mathbf{k}\Phi_\mathbf{k}^+\Phi_\mathbf{k})$.

One may try to interpret this state as thermal, i.e.: $\rho \sim exp(-\beta H_{eff})$. The effective field hamiltonian is then given by: $H_{eff} = \int d\mathbf{x}d\mathbf{x}'\hat{u}(\mathbf{x} - \mathbf{x}')\Phi_\mathbf{x}^+\Phi_{\mathbf{x}'}$ where $\hat{u} = \beta^{-1}\int d\mathbf{k}u_\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$. However, we find that generically the fourier transformed $\hat{u}$ gives us a non-local H_{eff}.

Alternatively, we may interpret $u_\mathbf{k}$ as $u_\mathbf{k} = \beta_k E_{k,free}$ where $E_{k,free}$ are the photon energies without the interaction. Thus, different momenta $\mathbf{k}$ feel different effective temperatures. Estimating $u_\mathbf{k}$ from the integrals (7),(8), we find for the soft modes, $u_\mathbf{k} = \beta_k E_{k,free} \propto \sqrt{k}$ and so the effective temperature:

$$T_k \equiv \frac{k}{\beta_k} \propto \sqrt{k} \quad as \quad k \to 0.$$

The energy and number of occupied soft modes per unit volume up to a given k_m are proportional to $k_m^{d+1/2}$ and $k_m^{d-1/2}$, respectively, are finite and small. The number variance of modes up to k_m is $\langle\delta N^2\rangle \sim \int d^dkn_k(1+n_k) \sim k_m^{d-1}$ in $d > 1$. However, for $d = 1$ we find a curious infrared divergence: $\langle\delta N^2\rangle = -\log\varepsilon_I$, where ε_I is an infra-red cutoff, inversely proportional to the system size.

Next, we find that the field entropy (6) suffers from a UV divergence. To do so, we estimate the symplectic eigenvalues $\mu_\mathbf{k}$ to lowest order in ω_p, ω_c and use (6). We find that $\mu_\mathbf{k} \sim 1 + \pi^{-1}2\omega_p^2\gamma_p\log k + ..$ for $k \gg 1$, and integration over momenta yields (in 3d, $\hbar = c = 1$ units):

$$\mathcal{S}_{field} = \begin{cases} (\omega_p^2\gamma_p + \omega_c^2\gamma_c)\log(\Lambda)^3 & \\ \omega_p^2\omega_0\log(\Lambda)^2 & \gamma_c = \gamma_p = 0 \\ 0 & plasma\ model \end{cases}, \qquad (9)$$

where Λ is a high momentum (UV) cutoff. Numerically, the approximations used in (9) actually recover the correct Λ dependence even for large values of ω_p, ω_c, since the dielectric response decays at the large k, ω limit.

It is interesting to observe the special place of the *"pure plasma"* limit response function. We can easily understand the result (9) as follows. An idealized plasma simply adds a finite mass to the field inside the region it occupies, completely expelling frequencies smaller than ω_p. Explicitly, substituting the plasma permittivity limit form $\chi = -\frac{\omega_c^2}{\omega^2}$ in the action (2), it produces a mass term for ϕ. Thus, the resulting action is consistent with a hermitian field Hamiltonian, and as such, at zero temperature, to a pure state. Interestingly, the use of a pure plasma in the computations of Casimir energy and Entropy has been at the heart of a recent debate [12–15]. note that the distinction between the Casimir entropy in the two models is manifested in the full quantum entropy computed herein [32].

Having shown that the entropy $\mathcal{S}_{field}$ is UV divergent, it is natural to ask, in analogy with the Casimir effect, what is the distance dependence of the entropy of interaction with two distinct bodies A and B? Is it UV finite? To answer such questions, we define:

$$\mathcal{S}_R(A,B) = \mathcal{S}(A\cup B) - \mathcal{S}(A) - \mathcal{S}(B), \qquad (10)$$

$\mathcal{S}_R$ should not be confused with the Casimir entropy, defined as $\mathcal{S}_C(A,B) = -\lim_{T\to 0}\partial_T F_C$, where F_C is the Casimir free energy, obtained by subtracting all distance independent terms from the free energy of the EM field in the presence of the bodies A, B. Also, $\mathcal{S}_R$ is distinct from the "relative entropy" of probability theory, as we are comparing different systems, and not merely different statistical information about the same system.

The relevance of Casimir entropy $\mathcal{S}_C$ to understanding thermal corrections of the Lifshitz formula has been pointed out in many papers (see e.g. [12, 16, 17]), where it was noticed that as $T \to 0$, $\mathcal{S}_C$ may not go to zero when using the Drude model (as might be expected by the Nernts theorem). It is interesting to note that while the Casimir EE $\mathcal{S}_R$ is distinct from $\mathcal{S}_C$, a similar behavior is observed in (9). In addition, at high temperatures we expect $\mathcal{S}_R = \mathcal{S}_C$, as most of the field entropy will be thermal (Technically, the relevant Green's function gets its major contribution from the $\omega = 0$ Matsubara pole), in addition, the Casimir force is entirely entropic [18].

In recent years, numerous results for Casimir interaction between objects have been obtained using a TGTG representation of the energy [19]. In these methods one separates between T operators associated with local properties of each body, and a free green's functions interpolating between them. See, e.g. ([20–22]). A natural question is: Can we describe $\mathcal{S}_R$ in similar terms? Using the methods of [22], assuming that $\phi\pi$ correlators vanish, we find an analogous expression for $\mathcal{S}_R(A,B)$:

$$\mathcal{S}_R(A,B) = -\frac{1}{\pi}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} dx \frac{\log\left[\frac{\sqrt{x}+\frac{1}{2}}{\sqrt{x}-\frac{1}{2}}\right]}{2\sqrt{x}} \mathrm{Tr}\log \left(1 - \frac{1}{1-K_A}\left(K_A K_B + K_{\mathrm{AUB}} - K_A - K_B\right)\frac{1}{1-K_B}\right) \qquad (11)$$

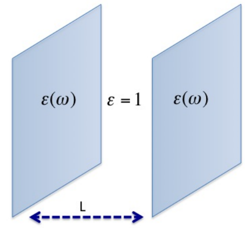

FIG. 1: "Casimir" entanglement entropy $\mathcal{S}_{\mathcal{R}}$ of plates.

Here $K_A = \frac{1}{x-1/4+is}\left(\Gamma_A - 1/4\right)$ is defined via $\Gamma(x,x') = i\int\langle\phi(x)\phi(x'')\rangle\langle\pi(x'')\pi(x')\rangle \mathrm{d}x''$, and similar expressions hold for K_B, K_{AUB}. Expression (11) is similar to the TGTG formulas in it's form: an integral over the TrLog of a combination of Green's functions. It differs from such formulas in several aspects: The integration variable x is not a frequency variable, but rather an auxiliary spectral variable, the presence of the term K_{AUB} doesn't allow for full separation into local object properties and free propagators and the non-analyticity at $x = 1/4$. All of these make the formula harder to use than the TGTG formulas. Nevertheless, it is a useful starting point for multiple scattering expansions.

Here, we defer a detailed study of the formula (11) and proceed instead to study the dependence of entanglement on the distance between bodies in a simple toy model:

Consider a field interacting with a medium, defined through the following (Wick rotated) action:

$$S_{\mathrm{pure}} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Big\{\int d^3x[\phi^*\left(-\omega^2+\nabla^2\right)\phi_\omega] - \int_B d^3x[\frac{(\omega^2+{\omega_0}^2)}{8\pi^2}{\psi_\omega}^*\psi_\omega + \omega_p\omega({\psi_\omega}^*\phi_\omega + \psi_\omega{\phi_\omega}^*)]\Big\} \qquad (12)$$

where ψ is a matter field which is confined to the body(s) B. This action may be viewed as a *purification* of the state ρ_ϕ of ϕ, i.e. we write our state as $\rho_\phi = \mathrm{Tr}_\psi|\Omega\rangle\langle\Omega|$ for some pure state $|\Omega\rangle$ in the larger ϕ, ψ space. The action (12) corresponds to the form $\chi(\omega) = \frac{{\omega_p}^2}{{\omega_0}^2-\omega^2}$ of the response of ϕ to a transparent, but dispersive medium.

Finally, to render the problem exactly solvable, we will assume that the $\phi\psi$ coupling in (12) comes from the form $\omega_p\int\psi\dot{\phi}$ in the classical Lagrangian. Note, that while on the classical level this Lagrangian coupling is equivalent to $-\omega_p\int\phi\dot{\psi}$, the actual quantum mechanical transformation required to affect this for gauge fields is of the Power-Zineau-Wooley (PZW) type used to interchange the minimal coupling $\mathbf{A}\cdot\mathbf{J}$ to $\mathbf{E}\cdot\mathbf{P}$ in quantum electrodynamics [23]. However, PZW mixes radiation and dipole degrees of freedom and does not preserve entanglement properties.

To compute the entropy of the field ϕ more efficiently, we use that $\mathcal{S}_\phi = \mathcal{S}_\psi$ (noting that (12) describes a pure state). $\mathcal{S}_\psi$ is obtained from the effective action for ψ:

$$S_{\rm eff}[\psi] = -\frac{1}{4\pi}\int_B d^3x d^3x' [\omega_p^2\omega^2\langle \mathbf{x}|G_0^B|\mathbf{x}'\rangle + \delta(\mathbf{x}-\mathbf{x}')\frac{\omega^2+\omega_0{}^2}{8\pi^2}]\psi_\omega{}^*(\mathbf{x})\psi_\omega{}^*(\mathbf{x}'), \tag{13}$$

where G_0^B is the restriction of the free Green's function $G_0 = \frac{1}{-\omega^2+\nabla^2}$ to the bodies B.

The two point function of ψ is:

$$\langle\psi(\mathbf{x})\psi(\mathbf{x}')\rangle = \int_0^\infty \langle \mathbf{x}|\frac{1}{\frac{\omega^2+\omega_0{}^2}{8\pi^2}+\omega_p{}^2\omega^2 G_0^B}|\mathbf{x}'\rangle\frac{d\omega}{2\pi} \tag{14}$$

with a similar expression holding for the conjugate momentum of ψ (obtained from the lagrangian in $S_{\rm pure}$):

$$\mathcal{H}_{\mathbf{x}\mathbf{x}'} = \langle \pi_{\psi(\mathbf{x})}\pi_{\psi(\mathbf{x}')}\rangle = -\frac{1}{16\pi^4} \int_0^\infty\langle\mathbf{x}|\frac{\omega^2}{\frac{\omega^2+\omega_0{}^2}{8\pi^2}+\omega_p{}^2\omega^2G_0^B} - 8\pi^2\delta(\mathbf{x}-\mathbf{x}')|\mathbf{x}'\rangle\frac{d\omega}{2\pi} \tag{15}$$

Next, we study the entropy generated by parallel planes separated by a distance L, as illustrated in Fig.1. To control UV behavior we place the system on a lattice, and use the discrete analogue of the free Green's function (in 1D) as: $G_0(n,m,k_\perp) = \frac{e^{-q|n-m|}}{2\sinh q}$, where: $\cosh q = 1+\frac{\omega^2+k_\perp^2}{2}$ ($k_\perp$ is parallel to the plates).

We first consider the 1D case. In 1D, the planes reduce to a pair of sites, and the kernel $(\frac{\omega^2+\omega_0{}^2}{8\pi^2}+\omega_p{}^2\omega^2G_0^B)^{-1}$ in eqs. (14,15) is a 2X2 matrix, amenable to an analytic treatment. Estimating the two point functions (14) and (15) using Watson's lemma, we find, at large distances $L \gg 1$ (assuming $\omega_0 > \omega_p$ and up to order ω_p^2):

$$\mathcal{G}_{nm} = \langle\psi(n)\psi(m)\rangle \sim \frac{1}{2\pi}\begin{pmatrix} A_G & -\frac{32\ \pi^4\omega_p{}^2}{\omega_0{}^4L^2} \\ -\frac{32\ \pi^4\omega_p{}^2}{\omega_0{}^4L^2} & A_G\end{pmatrix};$$
$$A_G = \frac{4\pi^3}{\omega_0} - \frac{32\pi^4\omega_p^2(\omega_0^2\arccos[\frac{2}{\omega_0}]-2\sqrt{\omega_0^2-4})}{\omega_0^2(\omega_0^2-4)^{3/2}}. \tag{16}$$

Similarly, the momentum correlation function $\mathcal{H}_{nm}$ is

$$\mathcal{H}_{nm} \sim \frac{-1}{32\pi^5}\begin{pmatrix} A_H & -\frac{192\pi^4\omega_p^2}{\omega_0^4L^4} \\ -\frac{192\pi^4\omega_p^2}{\omega_0^4L^4} & A_H\end{pmatrix}; \tag{17}$$
$$A_H = \frac{\omega_0}{4\pi} + \frac{32\pi^4\omega_p^2\left(2\sqrt{\omega_0^2-4}+(\omega_0^2-8)\arccos[\frac{2}{\omega_0}]\right)}{(\omega_0^2-4)^{3/2}}.$$

The dominant terms in the product $\mathcal{GH}$ are proportional to $\theta \equiv \frac{1}{64\pi^6}A_GA_H$, and are independent of distance.

As expected on physical grounds, the entropy strongly depends on the pinning frequency ω_0. We note that the entropy is maximized when ω_0 is small as possible, i.e. $\omega_0 \sim \omega_p$. In this limit the ψ field is only weakly restrained to $\psi = 0$ values, and thus generates large entropy for the ϕ field. In the weak coupling limit $\omega_p \ll 1$ we find the eigenvalues of $\mathcal{GH}$ to behave as $\theta\left(\omega_0=\omega_p\right) \sim \frac{1}{4}+2\pi\omega_p\log\frac{4}{e\omega_p}$. In the opposite limit, when $\omega_0 \gg \omega_p$ the ψ field is essentially "pinned" down, and we have $\theta \sim \frac{1}{4}+\frac{8\pi\omega_p^2}{\omega_0^3}-\frac{8\left(\pi^2\omega_p^2\right)}{\omega_0^4}$. As $\omega_0\to\infty$ we recover the minimal possible value of the product $\mathcal{GH}$ allowed by the uncertainty principle.

Finally, the symplectic eigenvalues of the covariance matrix are found, to leading orders in $\frac{1}{L},\omega_p$, to be

$$\lambda_\pm = \sqrt{\theta}\pm A_H\omega_p^2(4\pi^2\omega_0^4\sqrt{\theta}L^2)^{-1} \tag{18}$$

and the entropy using (5) for large L to behave as:

$$\mathcal{S}_\phi = \mathcal{S}(L\to\infty)+32\pi^7\omega_p^2\omega_0^{-3}L^{-4}. \tag{19}$$

At dimensions $D > 1$, the momentum $k_\perp$ is a good quantum number. We proceed to compute the entropy per $k_\perp$ by estimating $\langle\psi(x,k_\perp)\psi(x',k_\perp)\rangle$ and $\langle\pi_{\psi(x,k_\perp)}\pi_{\psi(x',k_\perp)}\rangle$. We find that the L dependent, off-diagonal elements in (16,17) are exponentially decaying as $e^{-L|k_\perp|}$. Integrating over $k_\perp$ yields

$$\mathcal{S}_\phi = \mathcal{S}(L\to\infty)+C_d(\omega_0,\omega_p)L^{-3-d} \tag{20}$$

where $C_d(\omega_0,\omega_p)$ is a constant.

Discussion: We studied the quantum state of a field in the presence of a dielectric material. We found that such a field is described by a density matrix whose Von-Neumann entropy diverges as described by eq. (9). The state cannot be considered as thermal, but rather the photons have a k dependent effective temperature. The occupation number of modes behaves as $n_k \sim 1/\sqrt{k}$ per unit volume for "soft" photons. A similar situation may arise when considering the phonons in solid. This effect is reminiscent of the infra red problem in quantum electrodynamics where, infinite numbers of soft photons are generated in transition amplitudes (see, e.g. [24]). The situation here is different in that the system is in equilibrium, however we are integrating over dipole transitions in the material.

To find long distance features, unaffected by UV divergence, we considered $\mathcal{S}_\mathcal{R}$, the distance dependent part of the entropy of two objects. In eq. (11) we present a novel general formula for the "Casimir Entanglement Entropy". Finally, we considered a toy model for the entropy generated in the field due to interaction between two thin planes. In this model $\mathcal{S}_\mathcal{R}$ shows a $L^{-(3+d)}$ decay: Thus, the field matter EE is reduced when the objects are far. Further investigation is needed to determine the power law in more realistic models.

The measurement of the full quantum entropy is in general quite hard, but possible to achieve in certain situations [25–28]. In the case of gaussian fields with translational invariance, it can be computed from mode occupations. We mention that many of the properties discussed are also valid for other fields, such as phonons in a solid. For example, occupation numbers of phonons in ultracold atoms or trapped ion systems may be measurable in experiments [29, 30], and may reveal some of the features described here.

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[31] The x dependence of ε does not spoil this property

[32] It is possible to incorporate the high momentum cutoff more naturally by including spatial dispersion in $\varepsilon(i\omega, k) = \varepsilon_0 \left(1 + \frac{f}{\mathrm{A\ k}^2 + \gamma\omega + \omega^2 + \omega_0^2}\right)$. This expression does not appear to significantly affect cutoff dependence.