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# Fractional resonances between waves and energetic particles in tokamak plasmas

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## Abstract

From numerical simulation and analytical modeling it is shown that fast ions can resonate with plasma waves at fractional values of the particle drift-orbit transit frequency when the plasma wave amplitude is sufficiently large. The fractional resonances, which are caused by a non-linear interaction between the particle orbit and the wave, give rise to an increased density of resonances in phase space which reduces the threshold for stochastic transport. The effects of the fractional resonances on spatial and energy transport are illustrated for energetic particle geodesic acoustic mode but they apply equally well to other types of MHD activity.

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Resonant interactions between waves and particles are ubiquitous in the physical world. Resonances between waves and drift-orbit transit frequencies of fast particles in magnetized plasmas play an important role in fast-ion transport and mode excitation. The well known linear resonance condition in Tokamaks between a wave with frequency  $\omega$  and the particle poloidal drift-orbit transit frequency,  $\omega_t$ , is given by:  $\omega = p\omega_t + n\omega_\phi$  with  $n$  the toroidal mode number of the wave,  $\omega_\phi$  the toroidal transit frequency, and  $p$  the bounce harmonic number which is an integer in this expression [3–5]. In this letter, however, we show that resonances between waves and particles can also occur when  $p$  is fractional. This increases the number of available wave-particle resonances significantly, and we explore the effects of fractional resonances on fast-ion transport. Although fractional resonances in non-linear plasma media have been noticed previously [1, 2] this letter is the first to show that fractional resonances can cause particle losses in a magnetic confinement device. The fractional wave-particle resonances were first found in simulations of particles interacting with an Energetic particle Geodesic Acoustic Mode (E-GAM) [6], and they can be explained with a theoretical model presented in this paper. The fractional resonances are not restricted to EGAMs but they also appear when other MHD activity is present such as Alfvén eigenmodes at realistic mode amplitudes.

Wave-particle interactions can be studied in detail for the E-GAM because it is a global toroidally symmetric ( $n = 0$ ) electrostatic mode with mode frequencies in the range of 10 to 30 kHz. It can be excited to large amplitudes: density fluctuations of  $\tilde{n}/n \approx 10\%$  have been found experimentally [7] and can expel large numbers of fast ions. In accordance with [6] we have modeled the E-GAM as a time-varying electrostatic potential (fig. 1a) which is a flux function. From this potential the radial electric field in the plasma was obtained (fig. 1d). The magnitude of the potential in the plasma center, 5 kV, was obtained by comparing the measured density fluctuations with the ones obtained from displacement induced by the mode [7].

The full orbit following code SPIRAL [8], which calculates single particle orbits in toroidal geometry by solving the Lorentz equations, was used to calculate particle trajectories in the

plasma in the presence of the E-GAM. Periodic excursions of the particle orbit and energy were found under the influence of a 15 kHz E-GAM with a central electrical potential of 5 kV as shown in Fig. 1 for a 50 keV deuteron launched on a passing orbit. Initially, the particle gains energy from the mode until its orbit becomes so large that the phase between the mode changes and the particle loses energy to the mode until it returns to its initial position and the whole process repeats itself.

In order to determine the resonances between the poloidal transit frequencies of the particles and a 15 kHz EGAM we have launched a number of deuterons with energies between 5 and 150 keV with an initial pitch ( $v_{\parallel}/v$ ) of 0.5 at a major radius of 2.02 m at the mid-plane. The selected energy range gives poloidal transit frequencies between 4 and 40 kHz. For each particle the maximum and minimum energy excursion was determined and plotted against the initial transit frequency for a number of values of the central electrical potential as shown in Fig. 2.

At lowest mode amplitude ( $V_{central} = 0.005$  kV which is 0.1% of typical EGAM amplitudes) only the linear  $p = 1, 2$ , and 3 resonances are visible. When the mode amplitude is increased to  $V_{central} = 0.5$  kV several fractional resonances appear between the  $p = 1$  and 2 resonances while at 30 kHz the  $1/2$  resonance appears. The increased mode amplitude also broadens the resonances. This can be explained by the large excursions that the particle makes at those mode amplitudes. The time-averaged bounce period over one energy excursion cycle is now resonant with the mode frequency. At a central potential of 1.25 kV the threshold for resonance overlap between the linear and fractional resonances is passed as can be inferred from the spikes appearing in the energy traces near the the linear resonances. Finally, at  $V_{central} = 5$  kV, the experimental value of the central potential, the orbits in the frequency interval up to the  $p = 1$  has become fully stochastic due to resonance overlap while at higher frequencies additional new sub-harmonic resonances have appeared.

The fractional resonances found in the simulations can be understood by writing down the Hamiltonian,  $H = \varepsilon + e\delta\Phi$ , of a charged particle with charge  $e$ , and kinetic energy,  $\varepsilon$ , in a time-varying electrical potential:  $\delta\Phi = \Phi_0 \sin(\omega t - k_r r)$  with  $\omega$  the mode frequency

and  $k_r$  its wave number. Resonances correspond to secular behavior and are obtained when  $\langle dH/dt \rangle \neq 0$  with  $\langle \dots \rangle = \frac{1}{T} \int_0^T \dots dt$  and  $T$  the fast time scale. The unperturbed particle position is given by:  $r = r_0 + \rho_0$  with  $r_0$  the drift center and  $\rho_0$  the drift orbit radius and its radial drift velocity  $\dot{\rho}_0 = v_{dr} = v_d \sin(\theta)$ . The angle coordinate,  $\theta$ , and particle transit frequency,  $\dot{\theta} = \omega_t$ , are both functions of particle energy, pitch, and drift center. The mode induces a perturbation to the energy,  $\varepsilon = \varepsilon_0 + \delta\varepsilon$ , and drift center,  $r_c = r_0 + \delta r$ , whereby the effects in energy dominate over the change in drift orbit center. From the change of energy in time, given by  $\delta\dot{\varepsilon} = v_{dr} \cdot e\delta E$  ( $\delta E$  the electric field associated with the potential  $\delta\Phi$ ), which can be cast into:

$$\delta\dot{\varepsilon} = \frac{k_r v_{d0} e \Phi_0}{2} \sin(\theta_0 \pm (\omega t - k_r r))$$

whereby the  $\pm$  sign accounts for the fact that the mode forms beat waves with the particle transit frequency at the sum and difference frequency. The change of energy over time can be obtained:

$$\delta\varepsilon = \frac{k_r v_{d0} e \Phi_0}{2} \frac{\cos(\theta_0 \pm (\omega t - k_r r)) - \cos(\mp k_r r)}{\omega_{t0} \pm \omega}$$

from a simple integration over time and by using the leading term of the particle transit frequency:  $\dot{\theta}_0 = \omega_{t0}$ . The next step is to expand the perturbed particle transit frequency as:  $\dot{\theta} = \omega_{t0} + (d_\varepsilon \omega_{t0})\delta\varepsilon + (d_r \omega_{t0})\delta r$  (where  $d_x y \equiv dy/dx$ ). Integrating this expression with time gives:  $\theta_0 = \omega_{t0} t$  for the first term and  $\delta\theta = d_\varepsilon \omega_{t0} \int \delta\varepsilon dt$  for the second term while the third term is omitted because  $\delta r$  is small. In a similar manner the change in the drift-orbit radius is calculated as  $\dot{\rho} = v_d \sin(\theta) = (v_{d0} + (d_\varepsilon v_{d0})\delta\varepsilon + (d_r v_{d0})\delta r) \sin(\theta_0 + \delta\theta)$  with  $\dot{\rho} = \dot{\rho}_0 + \delta\dot{\rho}$  where  $\dot{\rho}_0 = v_{d0} \sin(\theta_0)$  and  $\delta\dot{\rho} = ((d_\varepsilon v_{d0})\delta\varepsilon + (d_r v_{d0})\delta r) \sin(\theta_0) + v_{d0} \cos(\theta_0) \delta\theta$ . Integrating over time gives:  $\rho_0 = -v_{d0} \cos(\theta_0)/\omega_{t0}$ , while  $\delta\rho$  is evaluated as:

$$\delta\rho = \sum_{i=2}^7 z_i \cos(\alpha_i) \quad (1)$$

where  $z_i$  and  $\alpha_i$  are given in Table I. Wave-particle resonances are now obtained from:  $\langle dH/dt \rangle = \omega e \Phi_0 \text{Re} \langle e^{i(\omega t - k_r(r_c + \rho_0 + \delta\rho))} \rangle \neq 0$ . The terms with  $\rho_0$  and  $\delta\rho$  contain cosine functions so we can use the Jacobi-Anger identity to express  $\langle dH/dt \rangle$  as product of Bessel

functions (with  $\psi = \omega t - k_r(r_c + \rho_0 + \delta\rho)$ ):

$$e^{i\psi} = \sum_{j,k,l,m,n,p,q} i^{j+2(k+l+q)} J_j(z_1) J_k(z_2) J_l(z_3) J_m(z_4) J_n(z_5) J_p(z_6) J_q(z_7) e^{i((k+l-m-n-p+q)k_r r + ((j+l+m+2(p+q))\omega_{t0} + (k+l-m+1)\omega)t - k_r r_c)} \quad (2)$$

where we have made use of the fact that  $k + l - m - n - p + q = 0$  which must hold for the spatial part when the time average of this expression is non-zero. In a similar way the temporal part of Eq. (2) should be zero at the resonance, so we get:

$$\frac{\omega}{\omega_{t0}} = -\frac{j + l + m + 2(p + q)}{(k + l - m + 1)} \quad (3)$$

From this expression we can immediately see that a multitude of fractional resonances between the mode frequency,  $\omega$ , and the initial particle transit frequency,  $\omega_{t0}$ , are obtained with appropriate choices of the indices  $j, k, l, m, p$ , and  $q$ . The linear resonance condition is recovered when the potential,  $\Phi$ , is vanishing small and therefore  $k, l, m, p$ , and  $q$  are zero.

It is interesting to note the analogy between the fractional resonances and resonant heating below the cyclotron frequency as described in [9, 10]. In the cyclotron case the heating,  $\delta W_c = v_\perp \cdot e\delta E_\perp$ , is caused by the product of the perpendicular velocity,  $v_\perp$ , and perpendicular electric field,  $\delta E_\perp$  while the cyclotron frequency,  $\omega_c$ , is constant. In the fractional resonances case the heating is given by:  $\delta \dot{\epsilon} = v_{dr} \cdot e\delta E$ , whereby  $v_{dr}$  plays the same role as  $v_\perp$  in the cyclotron case and the electric field,  $\delta E$ , is equivalent to  $\delta E_\perp$ . The particle transit frequency,  $\omega_t$ , plays a similar role as  $\omega_c$  but  $\omega_t$  depends on the drift velocity. Another difference between the cyclotron and fractional resonances case is, is that the sub-harmonic resonances appear at frequencies below the fundamental cyclotron resonance while in the fractional resonance case the sub-harmonic resonances appear above the fundamental resonance (Fig. 2a).

The fractional resonances increase the resonance density in phase space as shown in Fig. 3. This can lead to increased transport in two ways: i) stochastic transport can occur when the fractional and linear resonances overlap while ii) resonant transport can occur when the fractional resonances are well separated.

Stochastic transport occurs at sufficiently high mode amplitudes as can be seen in Fig. 2d where a stochastic region is formed below 18 kHz in the region where the linear resonances were located. The appearance of the fractional resonances between the linear resonances decrease the threshold for stochastic orbits. Another important observation that can be made from Fig. 2d it that when the region has become stochastic the particles gain more energy from the mode than that they lose to the mode which leads to a damping of the mode. In carefully designed experiments with co and counter beam injection this damping may be observed experimentally.

Resonant transport was studied in simulations with three populations of 10000 particles each with their guiding centers at  $R, Z = (2.02, 0.0)$  m, an energy of 50 keV, and pitches of 0.40, 0.73, and 0.88, respectively, so that the particles are resonant with the fundamental resonance, the half resonance, and non-resonant respectively, as indicated in Fig. 3 by the black squares. After the particles were distributed uniformly over the equilibrium drift-orbit, the EGAM was switched on for 0.27 ms and the spread in guiding centers relative to the unperturbed drift orbit and the spread in energy was calculated as a function of mode amplitude (Fig. 4).

From Fig. 4 it can be seen that the particles that were launched at the fundamental resonance are affected from very low wave amplitudes by spreading them in space and energy. The particle population at the half resonance is only affected when the mode amplitude reaches a threshold of 1 kV which is 20% of the amplitude seen in experiments. The non-resonant population is hardly affected by the mode. These results show clearly that the inclusion of the fractional resonances enhance the fast-ion transport and a larger part of phase space is affected by the mode than only the regions around the linear resonances.

Preliminary evidence of fast-ion transport at fractional harmonics of the mode frequency is observed during strong EGAM activity in the DIII-D tokamak where the EGAM was strongly excited during the current ramp-up phase with equal amounts (2.2 MW) of co- and counter-current neutral beam injection (NBI) similar to the experiments reported in [7] whereby the EGAM is strongly driven by the counter NBI (particle energy: 81 keV) while

the co-beam (particle energy: 76 keV) is injected in a region of phase space where the  $p = 1/2$  resonance is residing (fig. 3). In Fig. 5 the spectrum of EGAM oscillations as measured with a Mirnov coil is compared with the spectrum of losses to a fast-ion detector (FILD) [11]. A peak in the loss spectrum occurs at the half harmonic that does not appear in the instabilities spectrum which is consistent with the theoretical prediction that  $\dot{\rho}$  should contain fractional harmonic oscillations at large mode amplitude.

From numerical simulation and analytical modeling it was found that fast ions can resonate with plasma waves at fractional values of the particle drift-orbit transit frequency when the plasma wave amplitude is sufficiently large. The fractional resonances, which are caused by a non-linear interaction between the particle orbit and the wave, give rise to an increased density of resonances in phase space which reduces the threshold for stochastic fast-ion transport.

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TABLE I: The coefficients used in Eq. (1) and Eq. (2).

$i$	$z_i$	$\alpha_i$	index
1	$-\frac{v_{d0}}{\omega_{t0}}$	$\omega_{t0}t$	$j$
2	$\left(\frac{k_r v_{d0} e\Phi}{4}\right) \left(v'_{d\varepsilon} - \frac{v_{d0} \omega'_{t\varepsilon}}{\omega_{t0}}\right) \left(\frac{4\omega^2 - 28\omega_{t0}^2}{(4\omega_{t0}^2 - \omega^2)(\omega_{t0}^2 - \omega^2)}\right)$	$k_r r$	$k$
3, 4	$\left(\frac{k_r v_{d0} e\Phi}{4}\right) \left(v'_{d\varepsilon} - \frac{v_{d0} \omega'_{t\varepsilon}}{\omega_{t0}}\right) \left(\frac{2}{(\omega_{t0}^2 - \omega^2)}\right)$	$\omega_{t0}t \pm k_r r$	$l, m$
5	$\left(\frac{k_r v_{d0} e\Phi}{4}\right) \left(\frac{2v'_{d\varepsilon}}{\omega_{t0}^2 - \omega^2} - \frac{4\omega_{t0} v_{d0} \omega'_{t\varepsilon}}{(\omega_{t0}^2 - \omega^2)^2}\right)$	$\omega t - k_r r$	$n$
6, 7	$\left(\frac{k_r v_{d0} e\Phi}{4}\right) \left(\frac{v'_{d\varepsilon}}{(2\omega_{t0} \pm \omega)(\omega_{t0} \pm \omega)} + \frac{v_{d0} \omega'_{t\varepsilon}}{(2\omega_{t0} \pm \omega)(\omega_{t0} \pm \omega)^2}\right)$	$(2\omega_{t0} \pm \omega)t \mp k_r r$	$p, q$

## FIGURE CAPTIONS

FIG. 1: (Color On-line ONLY) (a) EGAM potential. (b) Energy excursion of a 50 keV deuteron launched at  $R=2.02$  m,  $Z=0.0$  m, and pitch of 0.5 as function of time in the presence of a 15 kHz EGAM. The particle orbit during one energy cycle (0.33 ms) without (b) and with (c) the EGAM present. Orange line: full orbit; blue line: guiding center.

FIG. 2: Minimum and maximum particle energy excursion as function of the initial transit frequency for various values of the electrical potential of the EGAM. (a) At 0.05 kV only the linear resonances are present. (b) At 0.5 kV fractional resonances appear. (c) At 1.25 keV the threshold for mode overlap between linear and fractional resonances is passed while (d) at 5 keV the region up to the fundamental resonance has become stochastic and sub-harmonic resonances are developing strongly. Integer and a number of fractional resonances are indicated.

FIG. 3: (Color On-line ONLY) Energy-pitch phase space with linear [solid (green) line] and a small number of fractional resonances [dashed (red) line] shown. The locations of the three ensembles that were used to investigate the effects of the fundamental, half, and non resonances on particle transport are indicated with the black squares.

FIG. 4: Standard deviation of the radial (a) and energy (b) excursion of an ensemble of ions as a function of EGAM mode amplitude for the fundamental resonance, the half resonance and non-resonant particles.

FIG. 5: (Color On-line ONLY) Magnetic [dashed (black) line] and FILD [solid (red) line] spectrum. Losses at half the EGAM frequency are only observed with FILD.

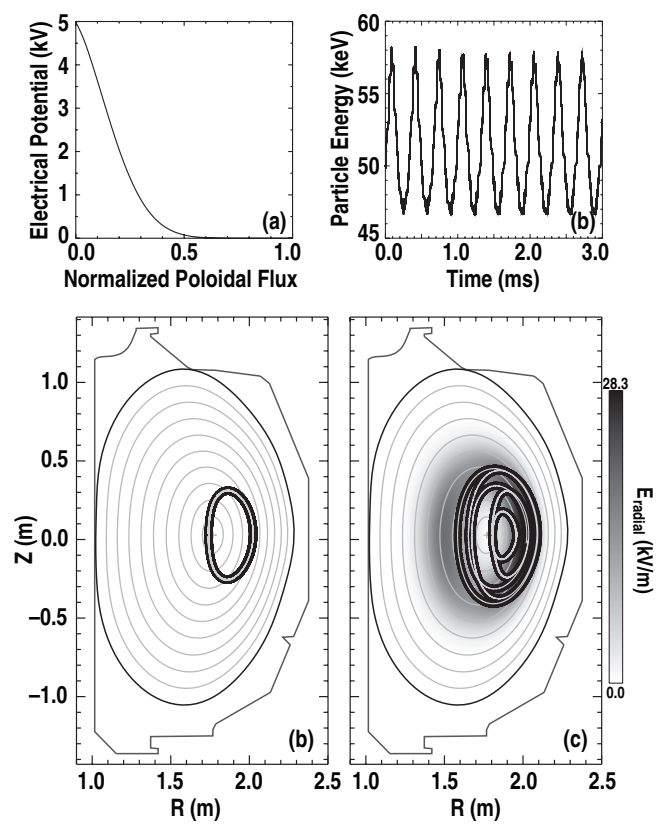


Fig. 1, G.J. Kramer, Phys. Rev. Lett.

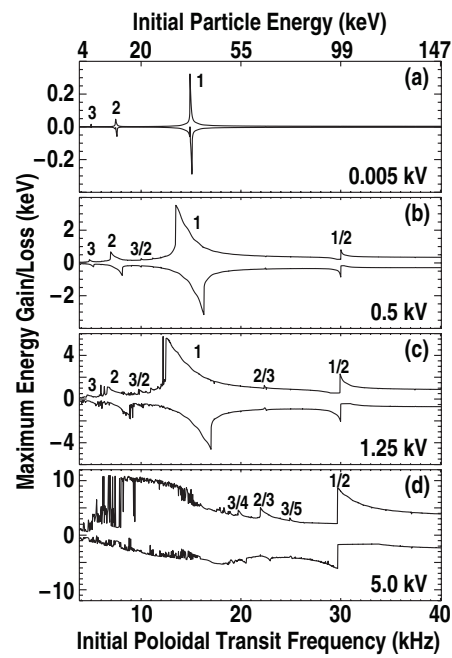


Fig. 2, G.J. Kramer, Phys. Rev. Lett.

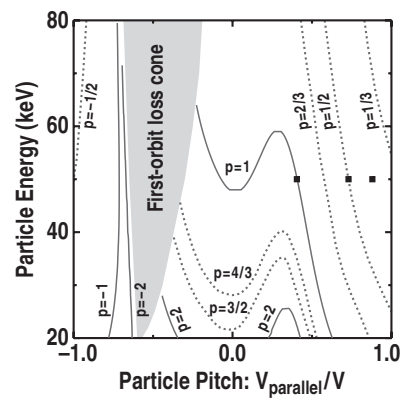


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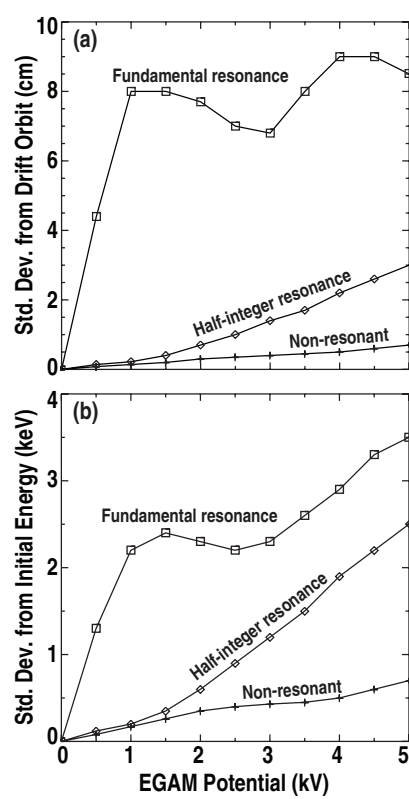


Fig. 4, G.J. Kramer, Phys. Rev. Lett.

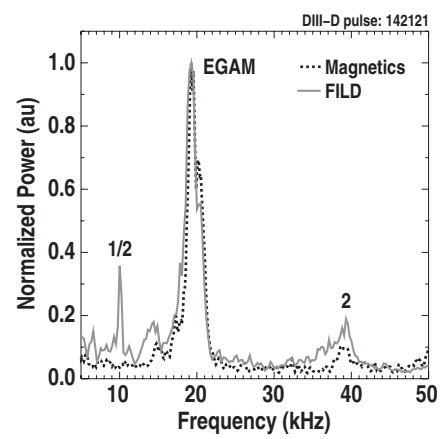


Fig. 5, G.J. Kramer, Phys. Rev. Lett.