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We demonstrate a simple pulse shaping technique designed to improve the fidelity of spin-dependent force operations commonly used to implement entangling gates in trapped-ion systems. This extension of the Mølmer-Sørensen gate can theoretically suppress the effects of certain frequency and timing errors to any desired order and is demonstrated through Walsh modulation of a two-qubit entangling gate on trapped atomic ions. The technique is applicable to any system of qubits coupled through collective harmonic oscillator modes.

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The use of spin-dependent forces to create entangled quantum systems has become widespread [1–4] and is currently the technique used for the highest fidelity multi-qubit operations [5]. This powerful technique, first proposed in [6–8], has been used to implement quantum algorithms [9], create large entangled states [10], test quantum fundamentals [11, 12], and perform simulations of quantum magnetism [13, 14] and quantum field theory [15]. As these types of experiments are scaled to larger numbers of qubits and more complex algorithms, the accumulation of gate errors will eventually require quantum error correction. Because of the large overhead required for quantum error correction, it is important that qubit operations be optimized passively in terms of speed and robustness to non-ideal control environments. In this paper, we show how ideas similar to the spin-echo pulse sequence [16] and those developed in the context of dynamical decoupling [17–19] can be used to optimize the Mølmer-Sørensen (MS) gate that is based on the spin-dependent force.

Spin-dependent force gates operate by coupling qubit states to a collective external degree of freedom referred to as a quantum bus. The coupling is switched on for an amount of time that introduces a particular phase between the spin states and leaves them disentangled from the external degree of freedom at the end of the gate. While the relative spin phase is robust due to geometric features [20], the disentanglement of the qubit space and the quantum bus at the end of the operation may be more susceptible to experimental errors and is equally crucial to achieving a high fidelity gate. Errors caused by noise on the energy splitting of the qubit can be suppressed by the insertion of an additional swapping pulse on the qubit states in the middle of a two qubit gate [21] or, as proposed in [22], by a π phase shift in the drive field. In this Letter, we present analytic results which extend these ideas and show how frequency and timing errors can theoretically be suppressed to any desired order with an optimized gate sequence that does not rely on the insertion of additional π pulses within the gate. Furthermore, the technique is demonstrated using atomic hyperfine qubits driven by a stimulated Raman process and shown to be more robust to certain errors than the operation described in the original proposal [6]. Similar to the single-qubit composite pulses [23] originally designed for NMR experiments now being in widespread use in other quantum information systems, this composite pulse should be applicable to any system of qubits coupled to a driven harmonic oscillator such as superconducting flux qubits [24] or cavity QED [22].

In trapped-ion systems, the spin-dependent force couples internal atomic states of neighboring ions through the collective modes of motion generated by the Coulomb interaction. In the MS scheme, a spin-dependent force is created by off-resonantly driving the red and blue sideband transitions simultaneously. The interaction Hamiltonian takes the form $\hat{H} = \Omega/2(\hat{\sigma}_+ e^{i\phi_s} + \hat{\sigma}_- e^{-i\phi_s})(\hat{a}e^{-i\delta t}e^{i\phi_m} + \hat{a}^\dagger e^{i\delta t}e^{-i\phi_m})$ where Ω is the sideband transition frequency, $\hat{\sigma}_\pm$ are the raising and lowering operators for the qubit, $\{\hat{a}^\dagger, \hat{a}\}$ are the creation and annihilation operators for the collective harmonic oscillator mode, and $\delta/2\pi$ is the symmetric detuning of the drive field from the sidebands [6]. The spin phase $\phi_s = (\phi_b + \phi_r)/2$ and the motional phase $\phi_m = (\phi_b - \phi_r)/2$ are determined by the phase of the red and blue drive fields. For the general case of N ions, the time-evolution operator is given by,

$$\hat{U}(t) = e^{-i\int_0^t dt'\hat{H}(t') - \frac{1}{2}\int_0^t dt'\int_0^{t'} dt''\left[\hat{H}(t'),\hat{H}(t'')\right]} \tag{1}$$

$$= e^{\hat{S}_N(\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a})}e^{-i\Phi(t)\hat{S}_N^2}, \tag{2}$$

where the total spin operator is given by $\hat{S}_N = \sum_{i=1}^N \sigma_+^{(i)} e^{i\phi_s} + \sigma_-^{(i)} e^{-i\phi_s}$, the time-dependent displacement coefficient is $\alpha(t) = \Omega/2\int_0^t dt' e^{-i\delta t'}e^{i\phi_m}$ and $\Phi(t)$ is a time-dependent phase that depends only on Ω and δ. When a collection of trapped ions that are each identically prepared in an eigenstate of $\hat{\sigma}_z$ evolves according to (2), the spin-dependent displacement operator splits the motional wavepacket into $N+1$ pieces that execute circular trajectories in phase space according to the definition of $\alpha(t)$. The term in (2) proportional to $\hat{S}_N^2$ imprints a relative phase on the various spin states, allowing the operation to be used as an entangling operation.

In order to prepare a pure spin state with this type of operation, the entanglement between the spin and motion must disappear at the end of the gate. When the gate time t_g is not equal to $2\pi j/\delta$, where j is any non-zero integer, the motional wavepackets do not trace out closed trajectories in phase space and therefore will not be fully disentangled from the spin state. The required level of precision grows with higher temperatures since the overlap between two states separated in phase space decreases exponentially with temperature. To see this, consider a qubit under the influence of the time evolution operator in (2). If the initial motional state is assumed to be a Gaussian state $\psi(x)$ with an uncertainty in position Δx and we describe a small timing or detuning error in the gate operation as an unintentional momentum displacement $\hbar q$, then the overlap between the two motional states is given by $\int_{-\infty}^{\infty} dx\, \psi^*(x)e^{-iqx}\psi(x) = \exp\left[-\frac{1}{2}(q\Delta x)^2\right]$. For a harmonic oscillator in a thermal state, Δx increases approximately as $\sqrt{T}$ for $k_B T > \hbar\omega$ meaning that the overlap between the two states decreases exponentially. As shown in Fig. 1(a), a small detuning error can largely be corrected with a second pulse whose phase has been shifted by π. We now discuss how to generalize this simple pulse sequence in order to suppress larger errors of this type.

Suppose there is a symmetric error Δ in the detuning such that $\delta = 2\pi/t_g + \Delta$ that could be the result of a change in the trapping frequency. The error in the operation results in some residual entanglement between the spin and motion that can be quantified by the magnitude of $\alpha_0(t_g) = \Omega/2\int_0^{t_g} dt e^{-i\delta t}$, (which goes to zero for $\Delta = 0$ at $t_g = 2\pi j/\delta$). We will show that by switching either ϕ_s or ϕ_m between 0 and π at times prescribed by certain Walsh functions, the effect of Δ on the magnitude of $\alpha(t_g)$ can be suppressed to any order. A Walsh function, denoted here as $\mathrm{W}(k,x)$, is a piecewise constant function that alternates between the values ±1 at certain values of x depending on the dyadic-ordered index k [25], (see Fig. 1(b)). If ϕ_r and ϕ_b shift together between 0 and π, then ϕ_s shifts between 0 and π, but ϕ_m remains constant and can be assumed to be 0 without the loss of generality. Note the effect of the phase shift $\phi_s = 0 \Rightarrow \phi_s = \pi$ is equivalent to shifting the motional phase $\phi_m = 0 \Rightarrow \phi_m = \pi$ while keeping ϕ_s constant. Both of these phase shifts are equivalent to the mapping $\hat{H} \Rightarrow -\hat{H}$, which can also be achieved with π pulses on the qubit states as done in [21]. Although the π phase shifts and π rotations are ideally equivalent, the phase shift switching time and precision is limited by electronics whereas the microwave rotations depend on qubit control that might be subject to the same noise source that generates Δ. If the times at which phase shifts occur are determined by the zero crossing times of $\mathrm{W}(k,t/t_g)$, then $\hat{S}_N = \mathrm{W}(k,t/t_g)\sum_i^N \sigma_+^{(i)} + \sigma_-^{(i)} \equiv \mathrm{W}(k,t/t_g)\hat{X}_N$. When modulating ϕ_s in this manner, the displacement

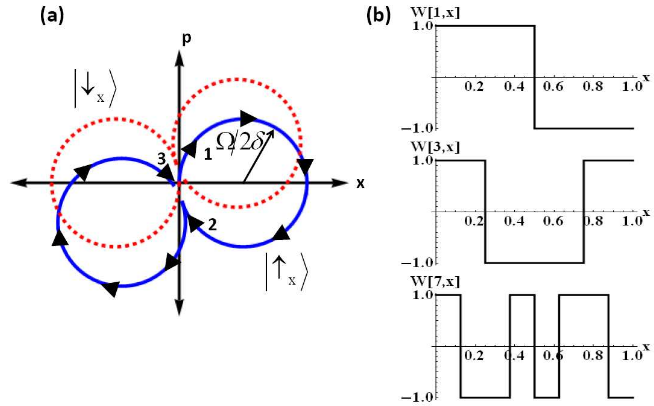

FIG. 1. (a)The phase space trajectory of the motional wavepackets of a single ion during a $\mathrm{W}(1,t/t_g)$ spin-dependent force operation with a small detuning error, $\Delta/\delta \ll 1$. The solid and dashed curves show the two different trajectories taken by the two different wavepackets associated with spin up and spin down in the $\hat{S}_1$ basis with only the spin up trajectory being labeled for clarity. After being initialized to a state centered at the origin, the spin-up wave-packet begins its clockwise motion near the point labeled 1. Halfway through the operation, near the point 2, the phase of the drive field is advanced by π, changing the direction of the applied force. At the end of the gate, near point 3, the wave-packet ends up much closer to the origin than the turning point near point 2. Therefore, the two wave-packets have more overlap at the end of the gate which is the key to achieving a higher fidelity operation. (b)The Walsh functions $\mathrm{W}(1,x)$, $\mathrm{W}(3,x)$ and $\mathrm{W}(7,x)$ are shown. Notice that $\mathrm{W}(3,x)$ can be constructed as two sequential $\mathrm{W}(1,x)$ functions with a phase flip on the second pulse. Likewise, $\mathrm{W}(7,x)$ can be constructed as sequential $\mathrm{W}(3,x)$ pulses with a phase flip on the second pulse.

operator in (2) becomes,

$$\hat{D}_k(t_g) = e^{-i\hat{X}_N\frac{\Omega}{2}\int_0^{t_g} dt\mathrm{W}(\mathrm{k,t/t_g})(e^{-i(\delta+\Delta)t}\hat{a}^\dagger + e^{i(\delta+\Delta)t}\hat{a})}. \tag{3}$$

By choosing a Walsh function with index $k = 2^n - 1$ where n is an integer and a detuning $\delta = 2^{n+1}\pi/t_g$, phase flips only occur at integer multiples of $2\pi/\delta$ and the effect of Δ can be suppressed to any order. This statement rests on the following equality,

$$\int_0^1 dx\mathrm{W}(2^n-1,x)e^{\pm i2^{n+1}\pi x}\sum_{l=0}^n a_l x^l = 0, \tag{4}$$

where a_l is a constant, (proof in the supplemental material). If the function $e^{\pm i\Delta t}$ is expanded in a Taylor series, the identity in (4) ensures that the displacement operator $\hat{D}_{2^n-1}(t_g) = \hat{\mathrm{I}} + \mathcal{O}(\Delta^{n+1})$, where $\hat{\mathrm{I}}$ is the identity operator.

To demonstrate the power of the composite pulse sequence, we use a qubit defined as the clock states in the $S_{1/2}$ hyperfine manifold of a Yb^+ ion in an RF Paul trap which can be initialized and read out us-

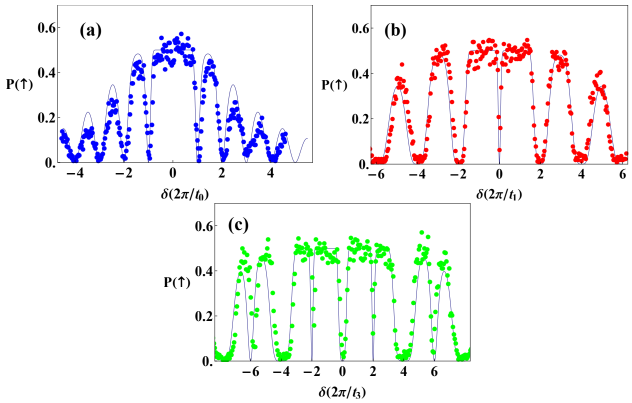

FIG. 2. A single ion prepared in $|\downarrow\rangle$ and an average excitation number $\bar{n} \approx 7$ is subjected to the standard and composite spin-dependent force operations and then measured in the $\hat{\sigma}_z$ basis. The data shown are plotted together with theoretical curves assuming an initial thermal state of motion. (a) The data show the probability of finding the ion in $|\uparrow\rangle$ as a function of the symmetric detuning δ for $t_0 = 100$ μsec. On resonance, $\delta = 0$, the motional wave-packets quickly become entangled with the spin state, resulting in a maximally mixed spin state. For finite δ, the wavepackets trace out circles in phase space resulting in revivals of the initial spin state when $\delta t_0/2\pi$ is a non-zero integer. (b) The spin-dependent force operation is implemented using $\mathrm{W}(1, t/t_1)$ for the phase $\phi_s(t)$ with $t_1 = \sqrt{2}t_0$. (c) $\mathrm{W}(3, t/t_3)$ is used for $\phi_s(t)$ with $t_3 = 2t_0$. Note the narrow resonance at $\delta t_3/2\pi = 2$ corresponds to a trajectory where the phase flips occur when the motional wavepackets are not at the origin.

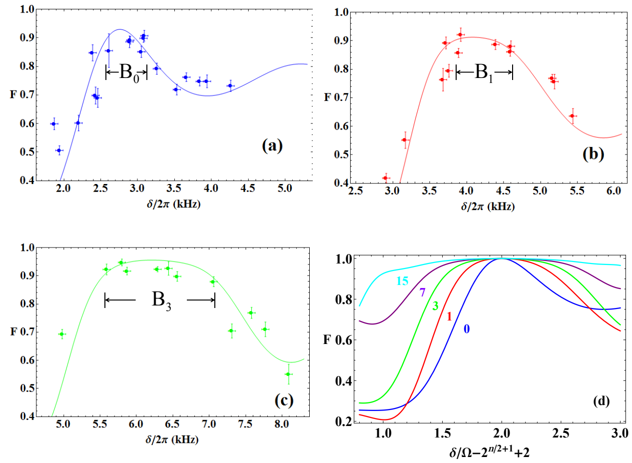

FIG. 3. The state fidelity of a two ion MS gate as a function of the detuning δ is compared for the first three Walsh functions being used for ϕ_s. In all three data sets, the ions were sideband cooled to the motional ground state before implementing the gate. In figures (a), (b) and (c) the measured state fidelity as a function of the detuning δ is compared with the theoretical curves. (d) shows the ideal fidelity curves for W(0)(blue), W(1)(red), W(3)(green), W(7)(purple) and W(15)(cyan). The curves are shifted in frequency in order to facilitate comparison and highlight the increasingly large regions of high fidelity for the higher order sequences. As a guide to the eye, the ideal curves shown in (d) are fit to the data by varying Ω and including an overall scale factor to account for additional experimental imperfections. The three different fit values of Ω agree with each other to within 10%. The estimates for the characteristic widths of the high fidelity regions, B_k, are shown to increase rapidly with the higher order sequences.

ing the techniques described in [26]. These states, $\{|F=0, m_F=0\rangle \equiv |\downarrow\rangle, |F=1, m_F=0\rangle \equiv |\uparrow\rangle\}$, have a splitting of 12.6428 GHz and are coupled to each other using stimulated Raman transitions. As described in [27], the Raman transition induced spin-dependent forces are created by the beat notes between two optical frequency combs that are generated by a 355nm mode-locked pulsed laser. The UV pulses have a duration of ~ 10 psec at a repetition rate of 80.57 MHz and have a nearly optimal center wavelength for minimizing off-resonant scattering from the excited P states in Yb^+[29]. At the position of the ion, the two overlapped beams are cross-polarized and mutually orthogonal to a magnetic field of 5 G with a geometry such that the momentum transfer only excites the transverse modes of motion which have a resonance frequency of 1.5 MHz. The two Raman beams are frequency shifted with AOMs to set up the appropriate beat notes in the interference field at the location of the ions. As in [27], driving one AOM with a single frequency and the other with two frequencies generates the bichromatic beat note that gives rise to the MS interaction. The red and blue phases $\phi_{r/b}$ are therefore defined by the phases of these RF drive frequencies. The composite pulse is implemented by splitting the operation into segments, between which the phases ϕ_r and ϕ_b are shifted. In this setup, symmetric detuning errors are the result of fluctuating RF trap voltages which manifests itself as noise on the oscillation frequency.

The effect of Walsh modulation on the spin-dependent force can be plainly seen with a single ion. In the case of a single ion, the phase $\Phi(t)$ is global and the only operation that results in a pure spin state is one that restores the initial spin state. The disentanglement of the spin and motion and consequential revival of the initial spin state should occur when $\delta t_g/2\pi = 2^n j$ where $2^n - 1 = k$ is the Walsh function index. If the spin is initialized to $|\downarrow\rangle$ in the $\hat{\sigma}_z$ basis and the joint spin-motion state after the operation is $\hat{\rho}$, then the state-fidelity is $F_1 = \mathrm{Tr}\left[|\downarrow\rangle\langle\downarrow|\hat{\rho}\right]$. Ignoring heating effects and assuming an initial thermal state of motion, the fidelity is $F_1 = \frac{1}{2}\left(1 + \exp\left[-(\bar{n}+1/2)\left|2\alpha_k(t_g)\right|^2\right]\right)$ where $\bar{n}$ is the average excitation number of the harmonic oscillator and $\alpha_k(t_g) = \frac{\Omega}{2}\int_0^{t_g} dt \mathrm{W}\left(k, t/t_g\right) e^{-i(\delta+\Delta)t}$. Because the lowest order term for the infidelity is $\mathcal{O}(|\alpha_k|^2)$, the infidelity of the Walsh modulated operation at $\delta t_g/2\pi = 2^n j$ is $\mathcal{O}(\Delta^{2n+2})$. This effect is clearly seen in Fig. 2 where the higher order Walsh sequences exhibit spin revivals of high purity over a much larger range of detunings in the neighborhood of $\delta = 2\pi/t_g$, where the gate is optimized for speed.

The effect of the Walsh modulation on a two-qubit gate is more complicated than that of a single qubit operation since the term in (2) proportional to $\hat{S}_N^2$ must be taken into account. The Walsh modulation of ϕ_s changes the evolution of $\Phi(t)$ in general, but not in the case where $\delta = 2^{n+1}\pi/t_g$ since the evolution is a series of closed circles in phase space. In this case, $\Phi(t_g) = \Omega^2 t_g/\delta$ and a fully entangling operation is achieved when $\Phi(t_g) = \pi/2$. This implies that in order to use $\mathrm{W}\left(2^n - 1, t/t_g\right)$, the gate time must be at least $t_g = 2^{n/2}\pi/\Omega$. While the exponential nature of this composite gate becomes daunting for large n, small errors can easily be corrected with a modest increase in the gate time. In the case of two ions, the maximally entangling operation ideally implements the transformation $|\downarrow\downarrow\rangle \Rightarrow |\downarrow\downarrow\rangle + e^{i\theta}|\uparrow\uparrow\rangle$ where the phase θ is determined by the phase of the drive field. With this target state, the fidelity is $F_2 = 1/4\left|e^{-(\bar{n}+1/2)|2\alpha_k(t_g)|^2} + ie^{-i\Omega^2\Phi_k(t_g)}\right|^2$ and is measured in the same manner as described in [27]. The phase $\Phi_k(t_g) = \sum_{i>j=0}^{k} \mathrm{Im}\left[\varphi_i^*\varphi_j\right] - \frac{1}{\delta}\left(t_g - \frac{1}{\delta}\sum_{i=0}^{k}\sin(\delta t_i)\right)$ is written here in terms of sums over the different parts of a pulse sequence. The parameters t_i refer to the duration of the $(i+1)^{th}$ segment of a sequence and the parameters $\varphi_i = (-)^i \int_{t_{i-1}}^{t_i} dte^{-i\delta t}$ with $t_{-1} = 0$ and $t_k = t_g$. The state fidelity measurement for the MS gate [27] is compared for the different pulse sequences in Fig. 3. The increased robustness to detuning errors can be quantified by defining a characteristic width of the high fidelity region, which we refer to as the passband B_k. Since the smallest infidelity in the data sets is of order 0.1, we choose to define the passband as the range of detunings where the infidelity is always observed to be lower than 0.2 and estimate $B_0 \approx 0.5$ kHz, $B_1 \approx 0.7$ kHz and $B_3 \approx 1.5$ kHz, demonstrating the composite sequences' tendency to suppress symmetric detuning errors. The maximum fidelities observed for the sequences $k = \{0, 1, 3\}$ are respectively $F_2 = \{0.91 \pm 0.02, 0.92 \pm 0.02, 0.95 \pm 0.1\}$. The relative modest increase in the maximum fidelity achieved indicates that trapping frequency is not changing significantly, (1kHz), on a time scale of less than the time between recalibrations of the detuning δ, which is approximately 5 min. We observed trap frequency fluctuations of 1 kHz on a time scale of about 1 hour. The combined effects of state preparation and detection errors contribute $\sim 2 \times 10^{-2}$ to the infidelity, with the remaining error being dominating by intensity fluctuations due to beam pointing instabilities.

Walsh functions have long been known by the electrical engineering, astronomy and radio communications communities to have useful error correcting properties [25]. While the Walsh functions are not the only option for choosing how to modulate the drive field of the spin-dependent force gate, we hope their introduction in the context of quantum control provides a useful tool for the further development of dynamical decoupling and related areas. In the formalism of dynamical decoupling, the function $\alpha_k(t_g)$ can be viewed as an optimized filter function designed to suppress the effects of a noise source centered at $\delta/2\pi$ [28]. In comparison to the Uhrig sequence which is optimal in the number of pulses used for a given order of noise suppression [18], Walsh modulation is optimal in the number of elementary sequences, (see supplemental material for the definition of an elementary sequence), which allows for both a simple mathematical construction and simple synthesis using integrated circuits. It is worth noting that, unlike Walsh filters, the Uhrig sequence is designed for low frequency noise and is not an effective filter for noise centered at a finite frequency.

The supression of symmetric detuning errors comes at the cost of an increase in the gate time for a fixed coupling strength Ω, meaning that increasingly complex gate sequences will eventually perform worse than simpler ones as the gate becomes sensitive to other noise sources. As described in [22], some heating effects might also be supressed through the technique described in this Letter, but the influence of other errors such as fluctuations in Ω requires further investigation. It is worth noting that the extra time needed for performing the modulated sequence might be partially offset by the decreased sensitivity to the initial temperature of the oscillator thereby reducing the amount of resource intensive cooling that may be needed. As quantum information experiments progress, this technique of coherent error suppression in quantum bus operations might prove to be an important ingredient in scaling toward larger systems and more complex algorithms.

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[1] C. A. Sackett et al. Nature **404**, 256 (2000).
[2] G. Kirchmair et al. New J. Phys. **11**, 023002 (2009).
[3] P. J. Lee et al. Opt. Lett. **28**, 1582 (2003).
[4] J. P. Home et al. New J. Phys. **8**, 188 (2006).
[5] J. Benhelm et al. Nature Physics **4**, 463 (2008).
[6] K. Mølmer and A. Sørensen. Phys. Rev. Lett. **82**, 1835 (1999).
[7] E. Solano et al. Phys. Rev. A **59**, R2539 (1999).
[8] G. J. Milburn et al. Fortschr. Phys. **48**, 801 (2000).
[9] R. Blatt and D. Wineland. Nature **453**, 1008 (2008).

[10] T. Monz et al. arxiv 1009.6126v1 (2010).
[11] M. A. Rowe et al. Nature **409**, 791 (2001).
[12] G. Kirchmair et al. Nature **460**, 494 (2009).
[13] K. Kim et al. Nature **465**, 590 (2010).
[14] A. Friedenauer et al. Nature Physics **4**, 757 (2008).
[15] R. Gerritsma et al. Nature **463**, 68 (2009).
[16] H. Y. Carr and E. M. Purcell. Phys. Rev. **94**, 630 (1954).
[17] L. Viola and S. Lloyd. Phys. Rev. A **58**, 2733 (1998).
[18] G. S. Uhrig. Phys. Rev. Lett. **98**, 100504 (2007).
[19] M. J. Biercuk et al. Phys. Rev. A **79**, 062324 (2009).
[20] S.-L. Zhu and Z. D. Wang. Phys. Rev. Lett. **91**, 187902 (2003).
[21] J. D. Jost et al. Nature **459**, 683 (2009).
[22] C.-Y. Chen and M. Feng. Phys. Rev. A **77**, 012325 (2008).
[23] H. K. Cummins et al. Phys. Rev. A **67**, 042308 (2003).
[24] Y.-D. Wang, A. Kemp, and K. Semba. Phys. Rev. B **79**, 024502 (2009).
[25] K. G. Beauchamp. *Applications of Walsh and Related Functions* (Academic Press Inc. 1984).
[26] S. Olmschenk et al. Phys. Rev. A **76**, 052314 (2007).
[27] D. Hayes et al. Phys. Rev. Lett. **104**, 140501 (2010).
[28] L. Cywinski et al. Phys. Rev. B **77**, 174509 (2008).
[29] W. C. Campbell et al. Phys. Rev. Lett. **105**, 090502 (2010).