Shear Viscosity of a Unitary Fermi Gas

Gabriel Wlazłowski, Piotr Magierski, and Joaquín E. Drut

Phys. Rev. Lett. **109**, 020406 — Published 12 July 2012

DOI: 10.1103/PhysRevLett.109.020406

LA-UR-12-20206
NT@UW-12-04

Shear Viscosity of a Unitary Fermi Gas

Gabriel Wlazłowski[1,2], Piotr Magierski[1,2], and Joaquín E. Drut[3]
[1] *Faculty of Physics, Warsaw University of Technology, ulica Koszykowa 75, 00-662 Warsaw, POLAND*
[2] *Department of Physics, University of Washington, Seattle, WA 98195–1560, USA and*
[3] *Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545–0001, USA*
(Dated: May 22, 2012)

We present the first *ab initio* determination of the shear viscosity η of the Unitary Fermi Gas, based on finite temperature Quantum Monte Carlo calculations and the Kubo linear-response formalism. We determine the temperature dependence of the shear viscosity to entropy density ratio η/s. The minimum of η/s appears to be located above the critical temperature for the superfluid-to-normal phase transition with the most probable value being $(\eta/s)_{\rm min} \approx 0.2\,\hbar/k_B$, which is close the Kovtun-Son-Starinets (KSS) universal value $\hbar/(4\pi k_B)$.

PACS numbers: 03.75.Ss, 05.60.Gg, 51.20.+d, 05.30.Fk

The Unitary Fermi Gas (UFG) represents a dilute but strongly correlated system, where the s-wave scattering between fermions saturates the unitarity bound for the cross section $\sigma(k) \leq 4\pi/k^2$ (k being the relative wave vector of colliding particles). The system is therefore characterized by the absence of intrinsic scales, making it *universal*, i.e. independent of the details of the interaction. On the other hand, the effects of interaction have to be treated nonperturbatively because of the lack of any small parameter. The extraordinary progress in experimental methods over the last decade have brought about the physical realization of such a system in the form of an ultracold gas of fermionic atoms [1]. As a consequence, the UFG has provided a new paradigm for many strongly interacting Fermi systems, attracting attention of theoretical physicists in various areas, including string theory, the quark-gluon plasma, neutron stars, nuclei, and to a certain extent high-T_c superconductivity [2].

Over the last few years, an impressive effort has been underway, both experimentally and theoretically, to establish the physical properties of the UFG and reveal its strongly correlated nature. One of the most prominent manifestations of such strong correlations is the observation of nearly ideal hydrodynamic behavior [3–5]. Studies of the transport properties of these systems are largely inspired by a conjecture formulated by Kovtun, Son, and Starinets (KSS) of the existence of a lower bound $\eta/s \geqslant \hbar/(4\pi k_B)$ on the ratio of the shear viscosity η to the entropy density s for any system [6]. As the bound is saturated for the case of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, it is expected that strongly correlated quantum systems are close to this bound. Indeed, very different physical systems known to be strongly interacting appear to be very close to the KSS bound: i) the quark-gluon plasma created in heavy ion collisions at the RHIC obey $\eta/s \leqslant 0.4\hbar/k_B$, ii) ultracold atomic gases at unitarity display $\eta/s \leqslant 0.5\hbar/k_B$, see [7] and references therein for an extensive overview. It has also been predicted that low-energy electrons in graphene monolayers are characterized by a low value of η/s, of the same order as that of the quark-gluon plasma and ultracold atomic gases [8].

In general, viscous (non-superfluid) hydrodynamics is characterized by two viscosity coefficients: the shear viscosity η and the bulk viscosity ζ. Contrary to the quark-gluon plasma, where the bulk viscosity is non-zero and can be a significant source of dissipation (especially near a phase transition), the bulk viscosity of the UFG vanishes as a result of scale invariance [9–11]. The UFG is therefore an excellent candidate for a perfect fluid, defined as the one with the lowest transport coefficients η and ζ allowed by quantum mechanics.

A large class of theoretical methods has been used to determine the transport coefficients of the UFG for homogeneous and trapped systems [12–20]. Here, the first *ab initio* calculation of the shear viscosity of the UFG is presented within the framework of the Path Integral Monte Carlo (PIMC) approach [21], which has been successfully used to compute other properties of the UFG [23–25]. To our knowledge, this is also the first *ab initio* calculation of the viscosity of a system with dynamical fermions, as previous *ab initio* calculations have involved the "quenched" approximation, in which the fermion determinant is set to unity [26, 27]. The fact that such a fully dynamical calculation is at all possible is not *a priori* obvious and should be regarded as one of our most important results. While statistical errors are explicitly under control, we provide only a limited assessment of systematic effects (finite density and volume). From our results it is clear that those effects *can* be controlled. While we focus our study on the shear viscosity, we have preliminary indications that the bulk viscosity vanishes at all temperatures, in agreement with the scale invariance arguments mentioned above. However we defer more careful determinations of both viscosities as well better control of systematic errors to future work.

Transport coefficients can be theoretically determined using linear response theory via the Kubo relations [11, 28]. In order to apply such relations within the framework of PIMC, we followed the method based on

the stress-tensor correlators [26, 27, 29]. Within this approach, the frequency-dependent shear viscosity is given by (in units such that $\hbar = k_B = m = 1$)

$$\eta(\omega) = \pi \frac{\rho_{xy,xy}(\boldsymbol{q}=0,\omega)}{\omega}, \tag{1}$$

while the static viscosity is defined in the limit of zero frequency: $\eta = \lim_{\omega\to 0^+}\eta(\omega)$. The spectral density $\rho_{ij,kl}(\boldsymbol{q},\omega)$ is related to the imaginary-time (Euclidean) stress-tensor correlator $G_{ij,kl}(\boldsymbol{q},\tau)$ by inversion of the relation

$$G_{ij,kl}(\boldsymbol{q},\tau) = \int_0^\infty \rho_{ij,kl}(\boldsymbol{q},\omega)\,\frac{\cosh\left(\omega(\tau-\beta/2)\right)}{\sinh\left(\omega\beta/2\right)}\,d\omega, \tag{2}$$

where $\beta = 1/T$ is the inverse temperature. In turn, the stress-tensor correlator has the form

$$G_{ij,kl}(\boldsymbol{q},\tau) = \int d^3\boldsymbol{r} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\langle\hat{\Pi}_{ij}(\boldsymbol{r},\tau)\hat{\Pi}_{kl}(\boldsymbol{0},0)\rangle, \tag{3}$$

where the average is performed over the grand canonical ensemble, $\hat{O}(\tau) = e^{\tau(\hat{H}-\mu\hat{N})}\hat{O}e^{-\tau(\hat{H}-\mu\hat{N})}$, $\hat{H}$ is the Hamiltonian of the system, μ is the chemical potential and $\hat{N}$ is the particle number operator. The stress-tensor operator $\hat{\Pi}_{ij}(\boldsymbol{r})$ is defined via the operator version of the Euler equation (summation over doubled index is assumed):

$$i[\hat{j}_k(\boldsymbol{r}),\hat{H}] = \partial_l\hat{\Pi}_{kl}(\boldsymbol{r}), \tag{4}$$

where $\hat{j}_k$ is the current operator. Since the current operator commutes neither with the kinetic-energy nor with the potential-energy parts of the Hamiltonian, it is convenient to split the stress tensor into two parts: $\hat{\Pi}_{kl} = \hat{\Pi}^{(T)}_{kl} + \hat{\Pi}^{(V)}_{kl}$. The kinetic-energy part $\hat{\Pi}^{(T)}_{kl}$ is well established and is the only contribution to the shear viscosity for a zero-range potential (see for example [15]). The potential-energy part $\hat{\Pi}^{(V)}_{kl}$ is more complicated, as defining the diagonal of the stress tensor is not trivial due to scale invariance, which is violated in our lattice calculations. Nevertheless, if we proceed with the stress tensor which on the lattice does not respect the sum rule $\int d^3\boldsymbol{r}\hat{\Pi}_{ii}(\boldsymbol{r}) = 2\hat{H}$ imposed by the scale invariance [10], we obtain results consistent with $\zeta = 0$. This matter is under further investigation.

Using the PIMC method, the stress-tensor correlator (3) was evaluated at $\boldsymbol{q} = 0$ for 51 points in imaginary time τ, uniformly distributed in the interval $[0,\beta]$ on a spatial lattice of 8^3 points. Increasing the number of τ points did not affect the final results. A statistical ensemble of 5000 uncorrelated samples was generated at each temperature, thus reducing the statistical errors to a few percent (depending on the temperature and value of τ). To estimate the size of discretization errors, exploratory calculations on a 10^3 lattice were performed. All the calculations presented here were performed with an average particle number density $n = N/V \approx 0.09$. The systematic errors associated with the stress-tensor correlator, related to finite volume effects as well as effective-range corrections, are likely $\sim 10-15\%$ [21, 22]. For a more detailed discussion see Ref. [30].

To determine η, one has to solve Eq. (2) numerically, which is an ill-posed inversion problem, as there exist an infinite number of solutions which reproduce the correlator within its error bars. Therefore, estimating the shear viscosity requires additional information. Besides the non-negativity of the viscosity $\eta(\omega) \geqslant 0$, the sum rule and the asymptotic tail behavior (see [11] with subsequent corrections [15, 31]) have been used as *a priori* information. In the unitary limit these conditions read

$$\frac{1}{\pi}\int_0^\infty d\omega\left[\eta(\omega) - \frac{C}{15\pi\sqrt{\omega}}\right] = \frac{\varepsilon}{3}, \tag{5}$$

where C is Tan contact density [32] and ε is the energy density. The energy density is obtained directly from PIMC calculations, while the contact density is taken from Ref. [33]. Based on the results for the noninteracting Fermi gas, where $\eta_{\mathrm{FG}}(\omega) \propto \delta(\omega)$, and those obtained within the T-matrix approach [15] or kinetic theory [17], the shear viscosity $\eta(\omega)$ is expected to be a continuous function with Gaussian-like structure at low frequencies, smoothly evolving into the asymptotic tail behavior $\eta(\omega\to\infty) \backsimeq \frac{C}{15\pi\sqrt{m\omega}}$. Moreover, we assume that there is no sharp structure in the spectral density in low frequency limit (associated for example with well defined quasi-particles), which could be overlooked during the inversion process. We used these assumptions to construct the model used in the inversion procedure.

To perform the inversion we applied a methodology based on two complementary methods: Singular Value Decomposition (SVD) and Maximum Entropy Method (MEM), both described in Ref. [34]. Since these methods are based on completely different approaches, a solution that is in agreement simultaneously with both of them is regarded as the most favorable scenario. In order to estimate the stability of the combined methods with respect to the algorithm parameters, the "bootstrap" strategy was applied. Namely, about 200 reconstructions were performed, with randomly generated initial parameters (within some reasonably chosen interval). The collected set of samples was subsequently used to evaluate the average value of the shear viscosity and the standard deviation (see [30] for details).

In Fig. 1, the dimensionless static shear viscosity η/n is shown as a function of T/ε_F, where $\varepsilon_F = (3\pi^2 n)^{2/3}/2m$ is the Fermi energy of the noninteracting gas. The shear viscosity monotonically decreases with decreasing temperature. No drastic suppression of the viscosity below the critical temperature of the superfluid-normal phase transition $T_c \simeq 0.15\varepsilon_F$ is observed. However, note that below T_c the coefficient η describes the viscosity of the

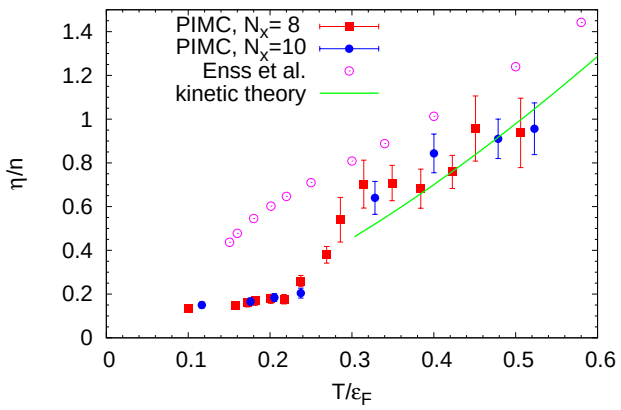

FIG. 1: (Color online) The dimensionless static shear viscosity η/n as a function of T/ε_F for an 8^3 lattice (red) squares and 10^3 lattice (blue) circles. The error bars only represent the stability of the combined (SVD and MEM) inversion procedure with respect to changes in the algorithm parameters. The (green) line depicts the prediction of kinetic theory [12]. For comparison, recent results of the T-matrix theory produced by Enss *et al.,* are plotted as open (purple) circles [15].

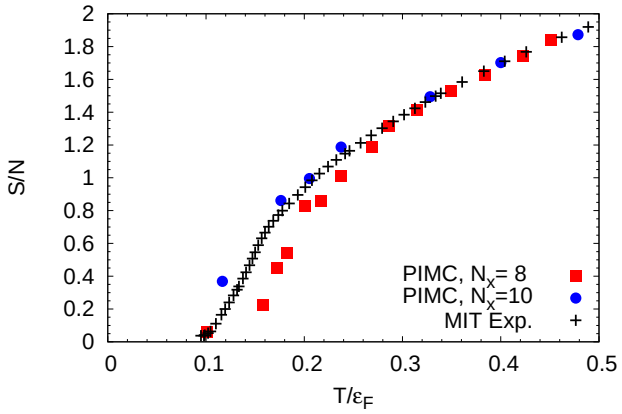

FIG. 2: (Color online) Entropy per particle as a function of T/ε_F for the 8^3 lattice in (red) squares and 10^3 lattice in (blue) circles. The entropy per particle extracted from the recent MIT measurement [35] is plotted with (black) crosses.

normal fluid component only. The results on 8^3 and 10^3 lattices exhibit satisfactory agreement. Surprisingly, our results approach the predictions of kinetic theory already at $T \gtrsim 0.3\varepsilon_F$ [12]. Note that the PIMC results are significantly below all known results in the vicinity of T_c.

In Fig. 2, the value of the entropy obtained from PIMC calculations is shown (extracted as in Ref. [21]), together with the results extracted from the recent high-precision MIT measurement [35]. For temperatures $T > 0.25\varepsilon_F$, both lattices reproduce experimental data reasonably well. At low temperatures $T < 0.25\varepsilon_F$ the 8^3-lattice results deviate from the measurements, producing systematically lower values. On the other hand, the 10^3-lattice results reproduce correctly the temperature dependence of the entropy, yet slightly overestimating the experimental values. These discrepancies are attributed to systematic errors that are known to be present at low temperatures even for larger lattices [25]. Consequently, we expect the ratio η/s to be significantly affected by uncertainties related to the entropy at low temperatures.

In Fig. 3 the ratio η/s is presented as a function of temperature. The PIMC calculations reveal the existence of a deep and rather narrow minimum in η/s at temperatures around $0.20-0.25\varepsilon_F$, which is above T_c. Again, the ratio η/s is located around the kinetic theory predictions already at $T \gtrsim 0.3\varepsilon_F$ [12]. The estimation of the η/s-ratio reveals $(\eta/s)_{\rm min} \approx 0.2$ as the most probable value for the minimum. This result is about 2.5 times higher than the KSS bound $\eta/s \geqslant 1/4\pi \approx 0.08$. Such a low value has been reported only for pure gluons as a result of lattice calculations [26, 27].

The minimum value for the ratio $(\eta/s)_{\rm min} \approx 0.2$, is significantly lower than predictions of all current calculations, which yield a minimum $\simeq 0.5$. However, these methods are in principle unreliable when applied to the UFG at $T \simeq T_c$, where the minimum appears. Moreover, the η/s ratio calculated from PIMC is also significantly lower than the experimental measurements [3–5], which also give the value $\simeq 0.5$. Note, however that these measurements are performed in trapped systems. The trap-averaged viscosity $\langle \eta/n \rangle = \frac{1}{N\hbar} \int \eta(\boldsymbol{r})\, d^3\boldsymbol{r}$ may affect the determination of the minimum value. To solve this puzzle one should apply an averaging procedure to the uniform case results, using e.g. Local Density Approximation. It is well known that this procedure leads to a divergence due to the violation of the hydrodynamic description at the edges of the cloud [36]. To perform a reliable averaging procedure the collisionless edges should be treated using kinetic theory. This, however, is a hard task that requires the knowledge of second-order transport coefficients like the relaxation time, which are currently poorly known.

Since our main result for the minimal value if η/s is significantly lower than other predictions as well as experimental results, we have performed exploratory calculations to estimate the size of systematic effects. We have checked the stability of the inversion procedure with respect to the default model as well the impact of the non-zero value of the effective range, see [30] for details. Our conservative estimation indicates that the minimal value of the η/s-ratio is lower than 0.45.

In summary, we have presented the first attempt to determine the shear viscosity of the UFG through an *ab initio* PIMC approach. The minimum value of the η/s

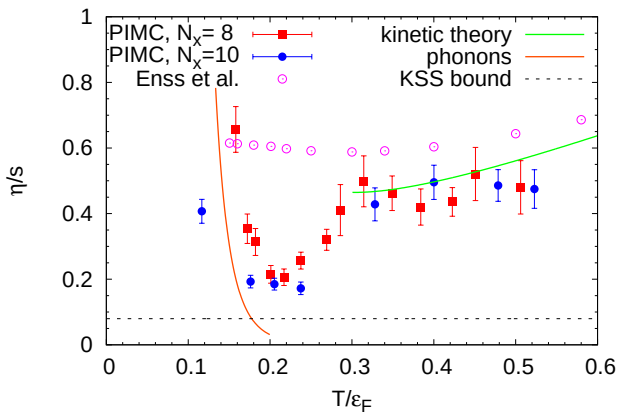

FIG. 3: (Color online) Ratio of the shear viscosity to entropy density η/s as a function of T/ε_F for an 8^3 lattice (red) squares and 10^3 lattice (blue) circles. The error bars only represent the stability of the combined (SVD and MEM) inversion procedure with respect to the change of algorithm parameters, and do not include systematic errors of the entropy determination. Results of the T-matrix theory are plotted by open (purple) circles [15]. In the high and low temperature regimes, known asymptotics are depicted: for $T > 0.3\varepsilon_F$ the prediction of kinetic theory [12] as a green line, and for $T < 0.2\varepsilon_F$ the contribution from phonon excitations [13] as a brown line. The KSS bound appears as a dashed black line.

ratio was estimated to be lower than 0.45 with the most probable value being $(\eta/s)_{\rm min} \approx 0.2$, located around $T \approx 0.20 - 0.25\varepsilon_F$. This value is close to the KSS bound and suggests that the unitary Fermi gas is the best candidate for the perfect fluid. As our results can be significantly affected by systematic errors, further and more precise investigations are called for.

We thank M. Zwierlein for making the MIT experiment data [35] available to us, and T. Enss for providing us with the T-matrix results [15]. We are indebted to A. Bulgac for instructive discussions and a careful reading of the manuscript. We acknowledge support under U.S. DOE Grant DE-FC02-07ER41457, and contract N N202 128439 of the Polish Ministry of Science. One of the authors (G.W.) acknowledges the Polish Ministry of Science for the support within the program "Mobility Plus - I edition" under contract No. 628/MOB/2011/0. Calculations reported here have been in part performed at the Interdisciplinary Centre for Mathematical and Computational Modelling (ICM) at Warsaw University and on the University of Washington Hyak cluster funded by the NSF MRI grant PHY-0922770.

[1] S Giorgini, L.P. Pitaevskii, S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008); I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).

[2] *The BCS-BEC crossover and the unitary Fermi Gas* Lecture Notes in Physics, Volume 836. Springer, Berlin, Heidelberg (2012), ed. W. Zwerger.

[3] A. Turlapov, J. Kinast, B. Clancy, L. Luo, J. Joseph, J.E. Thomas, J. Low Temp. Phys. **150**, 567 (2008).

[4] C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schaefer, J. E. Thomas, Science **331**, 58 (2011).

[5] C. Cao, E. Elliott, H. Wu and J.E. Thomas, New J. Phys. **13**, 075007 (2011).

[6] P. Kovtun, D.T. Son, A.O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).

[7] T. Schäfer and D. Teaney, Rep. Prog. Phys. **72**, 126001 (2009).

[8] M. Müller, J. Schmalian, and L. Fritz, Phys. Rev. Lett. **103**, 025301 (2009).

[9] D.T. Son, Phys. Rev. Lett. **98**, 020604 (2007).

[10] Y. Nishida and D.T. Son, Phys. Rev. D **76**, 086004 (2007).

[11] E. Taylor and M. Randeria, Phys. Rev. A **81**, 053610 (2010).

[12] G.M. Bruun and H. Smith, Phys. Rev. A **72**, 043605 (2005); Phys. Rev. A **75**, 043612 (2007).

[13] G. Rupak and T. Schäfer, Phys. Rev. A **76**, 053607 (2007).

[14] T. Schäfer, Phys. Rev. A **76**, 063618 (2007).

[15] T. Enss, R. Haussmann, W. Zwerger, Ann. Phys. **326**, 770 (2011).

[16] H. Guo, D. Wulin, C.-C. Chien, and K. Levin, Phys. Rev. Lett. **107**, 020403 (2011).

[17] M. Braby, J. Chao and T. Schäfer, New J. Phys. **13**, 035014 (2011).

[18] L. Salasnich, F. Toigo, J. Low Temp. Phys. **165**, 239 (2011).

[19] A. LeClair, New J. Phys. **13**, 055015 (2011).

[20] M. Mannarelli, C. Manuel, L. Tolos, arXiv:1201.4006v1.

[21] A. Bulgac, J.E. Drut, P. Magierski, Phys. Rev. A **78** 023625 (2008).

[22] J. E. Drut, arXiv:1203.2565.

[23] A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett. **96** 090404 (2006).

[24] P. Magierski, G. Wlazłowski, A. Bulgac and J.E. Drut, Phys. Rev. Lett. **103**, 210403 (2009); P. Magierski, G. Wlazłowski, and A. Bulgac, Phys. Rev. Lett. **107**, 145304 (2011).

[25] J.E. Drut, T.A. Lähde, G. Wlazłowski, P. Magierski, Phys. Rev. A **85**, 051601(R) (2012).

[26] A. Nakamura, S. Sakai, Phys. Rev. Lett. **94**, 072305 (2005).

[27] H.B. Meyer, Phys. Rev. D **76**, 101701(R) (2007); Phys. Rev. Lett. **100**, 162001 (2008).

[28] D.N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, Consultants Bureau, New York (1974).

[29] D. Teaney, Phys. Rev. D **74**, 045025 (2006).

[30] See Supplemental Material for technical details concerning the inversion procedure, and the discussion of systematic errors.

[31] W.D. Goldberger and Z.U. Khandker, Phys. Rev. A **85**, 013624 (2012).

[32] S. Tan, Ann. Phys. **323**, 2952 (2008).

[33] J.E. Drut, T.A. Lähde, T. Ten, Phys. Rev. Lett. **106**, 205302 (2011).

[34] P. Magierski, G. Wlazłowski, arXiv:1110.5441v1.

[35] M.J.H. Ku, A.T. Sommer, L.W. Cheuk, M.W. Zwierlein, Science **335**, 563 (2012).

[36] T. Schäfer and C. Chafin, *Scaling Flows and Dissipation in the Dilute Fermi Gas at Unitarity* Chap. 10 in *BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer, Berlin, 2012).