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Haruki Watanabe and Hitoshi Murayama

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Unified description of Nambu–Goldstone bosons without Lorentz invariance

Haruki Watanabe[1, 2, *] and Hitoshi Murayama[1, 3, 4, †]

[1] *Department of Physics, University of California, Berkeley, California 94720, USA*
[2] *Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan*
[3] *Theoretical Physics Group, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
[4] *Kavli Institute for the Physics and Mathematics of the Universe (WPI), Todai Institutes for Advanced Study, University of Tokyo, Kashiwa 277-8583, Japan*

Using the effective Lagrangian approach, we clarify general issues about Nambu–Goldstone bosons without Lorentz invariance. We show how to count their number and study their dispersion relations. Their number is less than the number of broken generators when some of them form canonically conjugate pairs. The pairing occurs when the generators have a nonzero expectation value of their commutator. For non-semi-simple algebras, central extensions are possible. Underlying geometry of the coset space in general is *partially symplectic*.

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Introduction. —Spontaneous symmetry breaking (SSB) is ubiquitous in nature. The examples include magnets, superfluids, phonons, Bose–Einstein condensates (BEC), and neutron stars. When continuous and global symmetries are spontaneously broken, the Nambu–Goldstone theorem [1–3] ensures the existence of gapless excitation modes, *i.e.*, Nambu–Goldstone bosons (NGBs). Since the long-distance behavior of systems with SSB is dominated by NGBs, it is clearly important to have general theorems on their number of degrees of freedom (*d.o.f.*) and dispersion relations.

In Lorentz-invariant systems, the number of NGBs $n_{\rm NGB}$ is always equal to the number of broken generators $n_{\rm BG}$. All of them have the identical linear dispersion $\omega = c|k|$. However, once we discard the Lorentz invariance, the situation varies from one system to another.

Until recently, systematic studies on NGBs without Lorentz invariance have been limited. (See Ref. [4] for a recent review.) Nielsen and Chadha [5] classified NGBs into two types: Type-I (II) NGBs have dispersion relations proportional to odd (even) powers of their momenta in the long-wavelength limit. They proved $n_{\rm I} + 2n_{\rm II} \geq n_{\rm BG}$, where $n_{\rm I}$ ($n_{\rm II}$) is the number of Type-I (II) NGBs. Schäfer *et al.* [6] showed that $n_{\rm NGB}$ is exactly equal to $n_{\rm BG}$ if $\langle 0|[Q_i, Q_j]|0\rangle$ vanishes for all pairs of the symmetry generators Q_i. Similar observation is given in Ref. [7]. Given these results, Brauner and one of us (HW) [8] conjectured

$$n_{\rm BG} - n_{\rm NGB} = \frac{1}{2}\operatorname{rank}\rho, \tag{1}$$

$$\rho_{ij} \equiv \lim_{\Omega\to\infty} \frac{-i}{\Omega}\langle 0|[Q_i, Q_j]|0\rangle, \tag{2}$$

where Ω is the spatial volume of the system.

In this Letter, we clarify these long-standing questions about the NGBs in Lorentz-non-invariant systems by proving the conjecture and showing the equality in Nielsen-Chadha theorem with an improved definition using effective Lagrangians $\mathcal{L}_{\rm eff}$. We also clarify how the central extension of the Lie algebra makes a contribution to ρ [9].

Coset Space. —When a symmetry group G is spontaneously broken to its subgroup H, the space of ground states form the coset space G/H where two elements of G are identified if $g_1 = g_2 h$ for $^\exists h \in H$. Every point on this space is equivalent under the action of G, and we pick one as the origin. The unbroken group H leaves the origin fixed, while the broken symmetries move the origin to any another point. The infinitesimal action of G is given in terms of vector fields $\mathbf{h}_i = h_i{}^a \partial_a$ ($i = 1, \cdots, \dim G$) on G/H, where $\partial_a = \frac{\partial}{\partial \pi^a}$ with the local coordinate system $\{\pi^a\}$ ($a = 1, \cdots, n_{\rm BG} = \dim G - \dim H$) around the origin. The infinitesimal transformations $\mathbf{h}_i$ satisfy the Lie algebra $[\mathbf{h}_i, \mathbf{h}_j] = f^k{}_{ij}\mathbf{h}_k$. We can always pick the coordinate system *s.t.* π^as transform linearly under H, namely that $\mathbf{h}_i = \pi^b R^p(T_i)_b{}^a \partial_a$, where $R^p(T_i)$ is a representation of H [10]. On the other hand, the broken generators are realized non-linearly, $\mathbf{h}_b = h_b{}^a(\pi)\partial_a$ with $h_b{}^a(0) \equiv X_b{}^a$. Since broken generators form a basis of the tangent space at the origin, the matrix X must be full-rank and hence invertible.

The long-distance excitations are described by the NGB fields $\pi^a(x)$ that map the space-time into G/H. We now write down its $\mathcal{L}_{\rm eff}$ in a systematic expansion in powers of derivatives, because higher derivative terms are less important at long distances.

Effective Lagrangians without Lorentz invariance. — We discuss the $\mathcal{L}_{\rm eff}$ for the NGB *d.o.f.* following Refs. [11, 12]. Under *global* symmetry G, the NGBs transform as $\delta\pi^a = \theta^i h_i{}^a$ where θ^i are infinitesimal parameters. However, we do not make θ^i local (gauge) unlike in these papers because it puts unnecessary restrictions on possible types of symmetries and their realizations, as we will see below.

It is well-known that a symmetry transformation can change the Lagrangian density by a total derivative. The examples include space-time translations, supersymmetry, and gauge symmetry in the Chern–Simons theory [13]. We allow for this possibility in the $\mathcal{L}_{\rm eff}$ of the NGB fields. We assume spatial translational invariance and rotational invariance at sufficiently long distances in the continuum limit, while we can still discuss their SSB.

If Lorentz invariant, the $\mathcal{L}_{\rm eff}$ is highly constrained,

$$\mathcal{L}_{\rm eff} = \frac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b + O(\partial_\mu^4). \tag{3}$$

The invariance of the Lagrangian under G requires that g_{ab} is a G-invariant metric on G/H, namely $\partial_c g_{ab} h_i{}^c + g_{ac}\partial_b h_i{}^c + g_{cb}\partial_a h_i{}^c = 0$. When the coordinates π^a are reducible under H, the metric g is a direct sum of irreducible components $g_{ab} = \sum_p F_p^2 \delta_{ab}^p$ where δ_{ab}^p vanishes outside the irreducible representation p with arbitrary constants F_p for each of them.

On the other hand, once we drop Lorentz invariance, the general $\mathcal{L}_{\rm eff}$ has substantially more freedom,

$$\begin{aligned}\mathcal{L}_{\rm eff} = {} & c_a(\pi)\dot{\pi}^a + \frac{1}{2}\bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2} g_{ab}(\pi)\partial_r \pi^a \partial_r \pi^b \\ & + O(\partial_t^3, \partial_t\partial_r^2, \partial_r^4),\end{aligned} \tag{4}$$

where $\bar{g}_{ab}$ is also G-invariant. Here and hereafter, $r = 1, \cdots, d$ refers to spatial directions.

Note that the spatial isotropy does not allow terms with first derivatives in space in the $\mathcal{L}_{\rm eff}$. Therefore, the spatial derivatives always start with at least the second power $O(\partial_r^2)$. (Actually, it is not critical for us whether there are terms of $O(\partial_r^2)$; it may as well start at $O(\partial_r^4)$ without affecting our results, as we will see below.)

The Lagrangian density changes by a total derivative under the infinitesimal transformation $\delta\pi^a = \theta^i h_i{}^a$ *iff*

$$(\partial_b c_a - \partial_a c_b)\, h_i{}^b = \partial_a e_i. \tag{5}$$

The functions $e_i(\pi)$ introduced in this way are actually related to the charge densities of the system. By paying attention to the variation of the Lagrangian by the surface term

$$\delta\mathcal{L}_{\rm eff} = \theta^i \partial_t (c_a h_i{}^a + e_i), \tag{6}$$

we can derive the Noether current for the global symmetry $j_i^0 = e_i - \bar{g}_{ab} h_i{}^a \dot{\pi}^b$. For the ground state is time independent $\dot{\pi}^b = 0$,

$$e_i(0) = \langle 0 | j_i^0(x) | 0 \rangle. \tag{7}$$

It must vanish in the Lorentz-invariant case, which can be understood as the special situation where c_a and e_i vanish, and $g_{ab} = c^2 \bar{g}_{ab}$.

Before presenting the proof, we explain the advantage in not gauging the symmetry. A tedious calculation verifies $\partial_c(h_i{}^a \partial_a e_j - f^k{}_{ij} e_k) = 0$, with a general solution,

$$h_i{}^a \partial_a e_j = f^k{}_{ij} e_k + c_{ij}. \tag{8}$$

Therefore, $e_i(\pi)$ transform as the adjoint representation under G, up to possible integration constants $c_{ij} = -c_{ji}$. These constants play important roles as seen below.

In presence of such constants, the global symmetry cannot be gauged [11]. This is reminiscent of the Wess–Zumino term that also changes by a surface term under a global symmetry and produces an anomaly upon gauging [14, 15]. It is known that the constants can be chosen to vanish with suitable definitions of e_i for semi-simple Lie algebras, while a non-trivial second cohomology of the Lie algebra presents an obstruction [16].

Proof of the conjecture. —The basic point to show is that when $\rho_{ij} \neq 0$, the NGB fields for the generators i and j are *canonically conjugate* to each other.

From Eq. (7) and the assumed translational symmetry, the formula for ρ in Eq. (2) is reduced to

$$\rho_{ij} = -i\langle 0 | [Q_i, j_j^0] | 0 \rangle = h_i{}^a \partial_a e_j \big|_{\pi=0}. \tag{9}$$

Obviously, this must vanish for unbroken generators by definition. Combining this with Eq. (5), we have

$$h_i{}^a h_j{}^b (\partial_b c_a - \partial_a c_b) \big|_{\pi=0} = \rho_{ij}. \tag{10}$$

We now solve this differential equation around the origin. The Taylor expansion of $c_a(\pi)$ can be written as $c_a(\pi) = c_a(0) + (S_{ab} + A_{ab})\pi^b + O(\pi^2)$, where S_{ab} and A_{ab} stand for the symmetric and antisymmetric part of the derivative $\partial_b c_a|_{\pi=0}$. Obviously $c_a(0)$ and S_{ab} lead to only total derivative terms in the $\mathcal{L}_{\rm eff}$ thus will be dropped later:

$$c_a(\pi)\dot{\pi}^a = A_{ab}\dot{\pi}^a\pi^b + \partial_t\left[c_a(0)\pi^a + \frac{1}{2}S_{ab}\pi^a\pi^b\right] + O(\pi^3). \tag{11}$$

The equation for the antisymmetric part $2X_c{}^a X_d{}^b A_{ab} = \rho_{cd}$ has an unique solution which gives

$$c_a(\pi)\dot{\pi}^a = \frac{1}{2}\rho_{ab}\dot{\tilde{\pi}}^a\tilde{\pi}^b + O(\tilde{\pi}^3), \tag{12}$$

where $\tilde{\pi}^a \equiv \pi^b (X^{-1})_b{}^a$. Since the matrix ρ is real and antisymmetric, we can always transform it into the following form by a suitable orthogonal transformation $\tilde{Q}_i = O_{ij} Q_j$:

$$\rho = \begin{pmatrix} M_1 & & & & & \\ & \ddots & & & & \\ & & M_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad M_\alpha = \begin{pmatrix} 0 & \lambda_\alpha \\ -\lambda_\alpha & 0 \end{pmatrix}. \tag{13}$$

Here, $\lambda_\alpha \neq 0$ for $\alpha = 1, \cdots, m = \frac{1}{2}\mathrm{rank}\,\rho$, while the remaining elements identically vanish.

The most important step in the proof is to write down the explicit expression of the $\mathcal{L}_{\rm eff}$ in Eq. (4),

$$c_a(\pi)\dot{\pi}^a = \sum_{\alpha=1}^m \frac{1}{2}\lambda_\alpha(\tilde{\pi}^{2\alpha}\dot{\tilde{\pi}}^{2\alpha-1} - \dot{\tilde{\pi}}^{2\alpha}\tilde{\pi}^{2\alpha-1}), \tag{14}$$

which is in the familiar form of the Lagrangian on the phase space $L = p_i\dot{q}^i - H$ [17]. Namely, $\tilde{\pi}^{2\alpha-1}$ and $\tilde{\pi}^{2\alpha}$ are canonically conjugate variables, and they together represent *one d.o.f.* rather than two. Hereafter we call the first set of $\tilde{\pi}^a$s ($a = 1, \cdots 2m$) Type-B, while the rest Type-A. Hence, $n_{\rm A} + 2n_{\rm B} = n_{\rm BG}$ with $n_{\rm A} = n_{\rm BG} - 2m$ and $n_{\rm B} = m$. Thus we proved the conjecture Eq. (1).

The definition of a *d.o.f.* here is the conventional one in physics, *i.e.*, one needs to specify both the instantaneous value and its time derivative for each degree of freedom as initial conditions. This definition does not depend on the terms with spatial derivatives in the Lagrangian.

Now we are in the position to prove that the equality is satisfied in the Nielsen–Chadha theorem if the term with two spatial derivatives exists with a non-degenerate metric g_{ab}. Then Eq. (4) implies that the Type-A NGB fields have linear dispersion relations $\omega_a \propto k$, while the Type-B NGB fields quadratic dispersions $\omega_\alpha \propto k^2$. In this case, our Type-A (B) coincides with their Type-I (II) respectively, and the Nielsen–Chadha inequality is saturated.

On the other hand, if we allow the second-order term $O(\partial_r^2)$ to vanish accidentally but the fourth-order term $O(\partial_r^4)$ to exist, the unpaired (Type-A) NGBs happen to have a quadratic dispersion ($\omega^2 \propto k^4$, and hence Type-II) yet count as independent *d.o.f.* each [8]. Therefore, the Nielsen–Chadha theorem is still an inequality in general. In contrast, our distinction between Type-A and Type-B NGBs is clearly determined by the first two time derivatives, and defines the number of *d.o.f.* unambiguously. Therefore, the classification between odd and even powers in the dispersion relation is not an essential one, and our theorem is stronger than that by Nielsen–Chadha.

Note that the Lagrangian formalism is mandatory in our discussion, because the presence of the first-order derivatives in time essentially affects the definition of the canonical momentum, while a Hamiltonian is written with a fixed definition of the canonical momentum.

Examples. —The simplest and most famous example of a Type-B NGB is the Heisenberg ferromagnet $H = -J\sum_{\langle i,j\rangle} \boldsymbol{s}_i \cdot \boldsymbol{s}_j$ with $J > 0$ on a d-dimensional square lattice ($d > 1$). In this case, the original symmetry group O(3) is spontaneously broken down to the subgroup O(2). The coset space is $\mathrm{O}(3)/\mathrm{O}(2) \cong S^2$. We assume that the ground state has all spins lined up along the positive z direction without a lack of generality. Even though there are two broken generators, there is only one NGB with the quadratic dispersion relation $\omega \propto k^2$.

The coset space can be parametrized as $(n_x, n_y, n_z) = \left(\pi^1, \pi^2, \sqrt{1-(\pi^1)^2-(\pi^2)^2}\right)$. The O(3) transformation ${h_i}^a = \epsilon_{iaj}n_j$ ($i,j = x,y,z$; $a = 1,2$) is realized linearly for unbroken generator ${h_z}^a(\pi) = \epsilon_{ab}\pi^b$, while nonlinearly for broken ones ${X_a}^b = \epsilon_{ab}$. One can show that the $\mathcal{L}_{\rm eff}$ consistent with the O(3) symmetry up to $O(\partial^2)$ is

$$\mathcal{L}_{\rm eff} = m\frac{n_y\dot{n}_x - n_x\dot{n}_y}{1+n_z} + \frac{1}{2}\bar{F}^2\dot{\boldsymbol{n}}^2 - \frac{1}{2}F^2\partial_r\boldsymbol{n}\partial_r\boldsymbol{n}. \quad (15)$$

Comparing to definitions Eqs. (4) and (5), we can read off c_a and e_i as $c_1 = \frac{mn_y}{1+n_z}$, $c_2 = -\frac{mn_x}{1+n_z}$ and $e_i = mn_i$. Hence $m = \langle j_z^0\rangle$ represents the magnetization of the ground state. It is clear that there is only one Type-B NGB because π^1 and π^2 are canonically conjugate to each other, with a quadratic dispersion $\omega \propto k^2$.

However for an anti-ferromagnet, $J < 0$, the overall magnetization cancels between sublattices, and therefore $e_i = 0$, which in turn requires $c_a = 0$. As a consequence, the lowest order term in the time derivative expansion has two powers, and we find that both π^1 and π^2 represent independent Type-A NGBs with linear dispersions $\omega \propto |k|$. The generalization to the ferrimagnetic case is straightforward.

Another example is the spontaneously broken translational invariance that leads to acoustic phonons in an isotropic medium [18]. The displacement vector $\boldsymbol{u}(x)$ represents the NGBs under the spatial translation $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{\theta}$, hence $G = \mathbb{R}^3$ and $H = 0$. Then with O(3) symmetry of spatial rotations, the most general form of the continuum $\mathcal{L}_{\rm eff}$ is

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\dot{\boldsymbol{u}}^2 - \frac{c_\ell^2}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2 - \frac{c_t^2}{2}(\boldsymbol{\nabla}\times\boldsymbol{u})^2. \quad (16)$$

We recover the usual result of one longitudinal and two transverse phonons with linear dispersions $\omega = c_\ell k$ and $\omega = c_t k$, respectively (Type-A). When the O(3) symmetry is reduced to $\mathrm{SO}(2)\times\mathbb{Z}_2$ for rotation in the xy-plane and the reflection $z \to -z$, there are considerably more terms one can write down. Using the notation $\psi = u_x + iu_y$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, and $\bar{\psi}$ and $\bar{\partial}$ for their complex conjugates, the most general $\mathcal{L}_{\rm eff}$ is

$$\mathcal{L}_{\rm eff} = \frac{ic_{xy}}{2}\bar{\psi}\dot{\psi} + \frac{1}{2}\dot{u}_z^2 + \dot{\bar{\psi}}\dot{\psi} - F_0^2(\bar{\partial}\psi)(\partial\bar{\psi}) - \frac{1}{2}F_1^2(\partial_z u_z)^2 \\ -(\bar{\partial}u_z, \partial_z\psi)\begin{pmatrix} F_2^2 & F_3^2 \\ F_3^2 & F_4^2 \end{pmatrix}\begin{pmatrix} \partial u_z \\ \partial_z\bar{\psi} \end{pmatrix} - \frac{1}{2}(F_5^2(\partial\psi)^2 + c.c.). \quad (17)$$

With $c_{xy} \neq 0$, we find there is one Type-A NGB with a linear dispersion, and one Type-B NGB with a quadratic dispersion. The first term $\frac{ic_{xy}}{2}\bar{\psi}\dot{\psi} = \frac{1}{2}c_{xy}(u_y\dot{u}_x - u_x\dot{u}_y)$ implies

$$\rho_{xy} = -i\langle 0|[P_x, j_y^0]|0\rangle = c_{xy} \neq 0. \quad (18)$$

Namely, this Lie algebra is a *central extension* of the abelian algebra of the translation generators, *i.e.* $[P_i, P_j] = c_{ij}\Omega$. As pointed out in Ref. [19], when the medium is electrically charged, an external magnetic field along the z-axis precisely leads to this behavior with $c_{xy} = 2\omega_c$ (the cyclotron frequency), because the gauge-invariant translations in a magnetic field are generated by

$P_i = -i\hbar\partial_i - \frac{e}{c}A_i$, which satisfy $\langle 0|[P_x, P_y]|0\rangle = i\frac{\hbar e}{c}B_z N$ with the number of the particles N. This would not be possible with the gauged $\mathcal{L}_{\rm eff}$ in Ref. [11] that does not allow for the central extension.

As a more nontrivial example, let us consider a spinor BEC with $F = 1$. The symmetry group is $G = \mathrm{SO}(3) \times \mathrm{U}(1)$, where SO(3) rotates three components of $F = 1$ states, while U(1) symmetry gives the number conservation. The Lagrangian is written using a three-component complex Schrödinger field ψ,

$$\mathcal{L} = i\hbar\psi^\dagger\dot{\psi} - \frac{\hbar^2}{2m}\partial_r\psi^\dagger\partial_r\psi + \mu\psi^\dagger\psi - \frac{\lambda}{4}(\psi^\dagger\psi)^2 - \frac{\kappa}{4}|\psi^T\psi|^2. \tag{19}$$

Since the potential reads $\frac{\lambda+\kappa}{2}\hat{n}^2 - \mu\hat{n} - \frac{\kappa}{2}\hat{\boldsymbol{S}}^2$ ($\hat{n} \equiv \psi^\dagger\psi$, $\hat{\boldsymbol{S}} \equiv \psi^\dagger\boldsymbol{S}\psi$ and $\boldsymbol{S}$ is the 3 by 3 spin matrix), we identify two possibilities for condensates

$$\psi = v_p(0,0,1)^T \quad \text{or} \quad v_f(1,i,0)^T \tag{20}$$

for "polar" ($-\lambda < \kappa < 0$) or "ferromagnetic" ($\kappa > 0$) states, where $v_p = \sqrt{\frac{2\mu}{\lambda+\kappa}}$ and $v_f = \sqrt{\frac{\mu}{\lambda}}$ [20]. The magnetization density is given by $e_i(0) = \langle j_i^0\rangle = -i\hbar\epsilon_{ijk}\psi_j^*\psi_k$. In the polar case, there is no net magnetization, and the symmetry is broken to $H = \mathrm{SO}(2) \subset \mathrm{SO}(3)$. For the ferromagnetic case, there is a net magnetization $e_z(0) = 2\hbar v_f^2$, and the symmetry is broken to the diagonal subgroup H of U(1) and $\mathrm{SO}(2) \subset \mathrm{SO}(3)$. Therefore, the unbroken symmetry is the same for both cases ($H = \mathrm{SO}(2) = \mathrm{U}(1)$), yet we see three Type-A NGBs for the polar case while one Type-A and one Type-B NGB for the ferromagnetic case as shown below.

For the polar case, we parameterize ψ as

$$\psi = (v_p + h)e^{i\theta}(\vec{n} + i\vec{\chi}), \quad \vec{n}^2 = 1, \quad \vec{\chi} \perp \vec{n}. \tag{21}$$

After integrating out the gapped modes h and $\vec{\chi}$, the Lagrangian (19) becomes

$$\mathcal{L}_{\rm eff} = \frac{\hbar^2}{\lambda+\kappa}\dot{\theta}^2 + \frac{\hbar^2}{|\kappa|}\dot{\vec{n}}^2 - \frac{\hbar^2 v_p^2}{2m}\left[(\partial_r\theta)^2 + (\partial_r\vec{n})^2\right]. \tag{22}$$

We do find three Type-A NGBs with linear dispersions.

For the ferromagnetic case, we parameterize ψ as

$$\psi = (v_f + h)\frac{e^{i\theta}}{1+z^*z}(1-z^2,\, i(1+z^2),\, 2z)^T. \tag{23}$$

After integrating out h, we find

$$\mathcal{L}_{\rm eff} = 2\hbar v_f^2 i\frac{z^*\dot{z} - \dot{z}^*z}{1+z^*z} + \frac{\hbar^2}{\lambda}\left(\dot{\theta} - i\frac{z^*\dot{z} - \dot{z}^*z}{1+z^*z}\right)^2 - \frac{\hbar^2 v_f^2}{m}\left[\left(\partial_r\theta - i\frac{z^*\partial_r z - \partial_r z^* z}{1+z^*z}\right)^2 + \frac{2\partial_r z^*\partial_r z}{(1+z^*z)^2}\right]. \tag{24}$$

Clearly, z and z^* are canonically conjugate to each other, representing one Type-B NGB with a quadratic dispersion, while θ one Type-A NGB with a linear dispersion.

Underlying Geometry. —Having demonstrated our theorem Eq. (1) at work in very different examples, we now study the underlying geometry. Usually, canonically conjugate pairs in mechanics (such as Type-B NGBs) imply a *symplectic structure* mathematically, which requires an even-dimensional manifold M, and if closed, a nontrivial second de Rham cohomology $H^2(M) \neq 0$. However, we have seen in the last two examples that Type-A and Type-B NGBs can coexist on an odd-dimensional M with $H^2(M) = 0$. This puzzle can be solved as follows.

The time integral of the first term in Eq. (4) defines a one-form $c = c_a \mathrm{d}\pi^a$ on the coset space, and its exterior derive a closed two-form $\sigma = \mathrm{d}c$. Using the coordinates in Eq. (14), $\sigma = \sum_{\alpha=1}^m \lambda_\alpha \mathrm{d}\tilde{\pi}^{2\alpha} \wedge \mathrm{d}\tilde{\pi}^{2\alpha-1}$ for m Type-B NGBs, which resembles a symplectic two-form. However, Type-A NGB fields for the remaining $n_{\rm BG} - 2m$ broken generators do not have terms with first order in time derivatives, and hence do not take part in σ. Therefore, σ has a constant rank but is degenerate, and hence is not a symplectic structure in the usual sense.

This kind of a partially symplectic (or *presymplectic* [16]) structure is possible on a coset space by considering the following fiber bundle, $F \hookrightarrow G/H \xrightarrow{\pi} B$, where the base space $B = G/(H \times F)$ is symplectic. The fiber F is a subgroup of G that commutes with H. The symplectic structure ω on B is pulled back to G/H as $\sigma = \pi^*\omega$. Since $\mathrm{d}\omega = 0$ on B implies $\mathrm{d}\sigma = 0$ on G/H, we can always find a one-form c such that $\mathrm{d}c = \sigma$ locally on G/H, which appears in $\mathcal{L}_{\rm eff}$. Type-B NGBs live on the symplectic base manifold B, whose coordinates form canonically conjugate pairs, while the Type-A NGBs live on the fiber F, each coordinate representing an independent NGB. The Type-A and Type-B NGBs can coexist on G/H in this fashion.

The Heisenberg ferromagnetic model has the coset space $S^2 = \mathbb{C}\mathrm{P}^1$ which is Kähler and hence symplectic, with one Type-B NGB. On the other hand, the spinor BEC example in its ferromagnetic state has $G/H = \mathbb{R}\mathrm{P}^3$ which is not symplectic. The last term in Eq. (24) is nothing but the Fubini–Study metric on $S^2 = \mathbb{C}\mathrm{P}^1$ which is Kähler and hence symplectic. The first term in Eq. (24) defines the one-form c whose exterior derivative $\mathrm{d}c$ gives precisely the Kähler form associated with the metric up to normalization. However θ is an orthogonal direction with no connection to the symplectic structure. We can define the projection $\pi : \mathbb{R}\mathrm{P}^3 \to S^2$ simply by eliminating the θ coordinate. It shows the structure of a fiber bundle $U(1) \hookrightarrow \mathbb{R}\mathrm{P}^3 \xrightarrow{\pi} \mathbb{C}\mathrm{P}^1$, which is the well-known Hopf fibration (the difference between S^3 and $\mathbb{R}\mathrm{P}^3 = S^3/\mathbb{Z}_2$ is not essential here). The phonons in the magnetic field also show a partially symplectic structure.

In fact, it is always possible to find such a symplectic manifold B if G is compact semi-simple, thanks to the Borel's theorem [21]. Generalizations to non-semi-simple groups would be an interesting future direction in mathematics.

Final Remarks. —In this Letter, we exclusively focused on true NGBs. We do not regard pseudo NGBs [22] as NGBs, since they do not correspond to the broken symmetries and tend to acquire mass corrections. Also, we assumed there are no gapless excitations other than NGBs; especially, this assumption fails when there is a Fermi surface on the ground state. Taking such *d.o.f.* into account would be another interesting future direction.

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* hwatanabe@berkeley.edu
† hitoshi@berkeley.edu, hitoshi.murayama@ipmu.jp

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