Spontaneous Emission by Rotating Objects: A Scattering Approach

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We study the quantum electrodynamics (QED) vacuum in the presence of a body rotating along its axis of symmetry and show that the object spontaneously emits energy if it is lossy. The radiated power is expressed as a general trace formula solely in terms of the scattering matrix, making an explicit connection to the conjecture of Zel'dovich [JETP Lett. **14**, 180 (1971)] on rotating objects. We further show that a rotating body drags along nearby objects while making them spin parallel to its own rotation axis.

Quantum zero-point fluctuations of the electromagnetic field in vacuum lead to macroscopic manifestations such as the Casimir attraction between neutral conductors [1]. When objects are set in motion, they may pull out real photons from the fluctuating QED vacuum. In fact, accelerating boundaries radiate energy, and thus experience friction, through the dynamical Casimir effect [2] (see Ref. [3] for a recent review). Even two parallel plates moving laterally at a constant speed experience a non-contact frictional force [4, 5]. While a constant translational motion requires at least two bodies (otherwise, trivial due to Lorentz symmetry), a single spinning object can experience friction. In a recent work [6], Manjavacas and García studied such rotational friction by expressing the polarization fluctuations of a small spinning particle via the fluctuation-dissipation theorem, and obtained a frictional force even at zero temperature. In fact, this problem is closely related to a classical phenomenon known as *superradiance* due to Zel'dovich [7]. He argues that a rotating object amplifies certain incident waves, and further conjectures that, when quantum mechanics is considered, the object should spontaneously emit radiation only for these so-called *superradiating* modes. Indeed this is shown to be the case for a rotating (Kerr) black hole by Unruh [8]. This phenomenon, however, is different in nature from Hawking radiation [9]. One can also find similar effects for a superfluid [10].

In this letter, we treat the vacuum fluctuations in the presence of a rotating object exactly, except for the assumption of small velocities to avoid complications of relativity. By incorporating scattering techniques into the Rytov formalism [11], we find a general trace formula for the spontaneous emission by an arbitrary spinning object, solely in terms of its scattering matrix. We reproduce the results in the literature and find an expression for the radiation by a rotating cylinder. Finally, we study the interaction of a rotating body with a *test* object nearby and show that the rotating body drags along nearby objects while making them rotate parallel to its own rotation axis.

Our starting point is the Rytov formalism [11] which relates fluctuations of the electromagnetic (EM) field to fluctuating *sources* within the material bodies, and in turn to the material's dispersive properties, via the fluctuation-dissipation theorem. The EM fields are governed by the Maxwell equation

$$\left(\nabla \times \nabla \times -\frac{\omega^2}{c^2}\,\epsilon(\omega, \mathbf{x})\,\mathbb{I} \right) \mathbf{E} = \frac{\omega^2}{c^2}\mathbf{K}, \qquad (1)$$

where a linear, and nonmagnetic medium is assumed. Rytov postulates that the sources undergo fluctuations, which are related to the imaginary part of the local dielectric response $\epsilon(\omega, \mathbf{x})$ by

$$\langle \mathbf{K}(\omega, \mathbf{x}) \otimes \mathbf{K}^*(\omega, \mathbf{y}) \rangle = a_T(\omega)\,\mathrm{Im}\,\epsilon(\omega, \mathbf{x})\delta(\mathbf{x} - \mathbf{y})\,\mathbb{I}. \quad (2)$$

Here, $a_T(\omega)) = 2\hbar(n_T(\omega) + 1/2)$ where $n_T(\omega) = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the Bose-Einstein occupation number, and T is the temperature. Equations (1) and (2) define the fluctuations of the EM-fields in the presence of a static object at temperature T. For bodies in uniform motion, they are applied in the rest frame of the object, and then transformed to describe the EM-field fluctuations in the appropriate laboratory frame. With all contributions of the field correlation functions in a single frame, one can then compute various physical quantities of interest, such as forces, or energy transfer from one object to another, or to the vacuum. For non-uniform motion, we assume that the same equations apply locally to the instantaneous rest frame of the body [12]. This assumption should be valid as long as the rate of acceleration is less than typical internal frequencies characterizing the object, which are normally quite large. The modified Maxwell equations are easier to derive from a Lagrangian, $\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\epsilon' \mathbf{E}'^2 - \frac{1}{2}\mathbf{B}'^2$, with the primed fields defined with respect to the co-moving reference frame. To incorporate the fluctuating sources, we must add

$$\Delta\mathcal{L} = \mathbf{K}' \cdot \mathbf{E}' \qquad (3)$$

to the Lagrangian, where $\mathbf{K}'$ is defined in the object's frame. To make contact with the EM-field in the vacuum, we should recast all fields in the stationary (lab) frame.

In this letter we assume that the velocity of the object is small such that, to the lowest order in v/c, the electromagnetic fields transform as

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}.$$

We discuss below the limits that this assumption places on the generality of our analysis. In the lab frame the EM equation is then given by

$$\left[\nabla \times \nabla \times - \frac{\omega^2}{c^2}\mathbb{I} - \frac{\omega^2}{c^2}\tilde{\mathbb{D}}(\epsilon'-1)\mathbb{D} \right] \mathbf{E} = \frac{\omega^2}{c^2}\tilde{\mathbb{D}}\mathbf{K}', \quad (4)$$

where $\mathbb{D} = \mathbb{I} + \frac{1}{i\omega}\mathbf{v}\times\nabla\times$ and $\tilde{\mathbb{D}} = \mathbb{I} + \frac{1}{i\omega}\nabla\times\mathbf{v}\times$. Also ϵ' is the response function defined in the object's rest frame, now written in terms of laboratory coordinates. Equation (4) describes the EM-field in the lab frame in terms of sources defined in the object's frame, $\langle\mathbf{K}'(\omega',\mathbf{x}')\otimes\mathbf{K}'^*(\omega',\mathbf{y}')\rangle = a_T(\omega')\,\mathrm{Im}\,\epsilon(\omega',\mathbf{x}')\delta(\mathbf{x}'-\mathbf{y}')\,\mathbb{I}$, where ω' and $\mathbf{x}'$ are understood as the frequency and position in the moving frame, which should be transformed to those in the lab frame.

Next we turn to the main point of this study, namely a solid of revolution spinning with angular frequency Ω along its axis of symmetry. We choose time and polar coordinates (t, r, ϕ, z) and (t', r', ϕ', z') in the lab and the object reference frames respectively, and take the rotation along the z axis. The two coordinate systems are related by

$$t' = t, \quad r' = r, \quad \phi' = \phi - \Omega t, \quad z' = z. \quad (5)$$

Consider a fluctuation of the source characterized by frequency ω' and azimuthal index m', $\mathbf{K}'_{\omega',m'}(t',\mathbf{x}') = e^{-i\omega't'+im'\phi'}\mathbf{f}_{\omega',m'}(r',z')$. Note that $\mathbf{f}$ depends only on the coordinates r' and z'. As a matter of notation, we *define* $\mathbf{K}(t,\mathbf{x}) \equiv \mathbf{K}'(t',\mathbf{x}')$, *i.e.* we drop the prime when the function is expressed in the lab-frame coordinates. The coordinate transformation in Eq. (5) then implies that the frequency as seen by the rotating object is shifted by Ωm,

$$\mathbf{K}_{\omega,m}(t,\mathbf{x}) = \mathbf{K}'_{\omega-\Omega m,m}(t',\mathbf{x}').$$

This equation, in turn, recasts the source fluctuations into the lab coordinates:

$$\langle\mathbf{K}_m(\omega,\mathbf{x})\otimes\mathbf{K}_m^*(\omega,\mathbf{y})\rangle = a_T(\omega-\Omega m)\times$$
$$\times\,\mathrm{Im}\,\epsilon(\omega-\Omega m, r, z)\frac{\delta(r_\mathbf{x}-r_\mathbf{y})\delta(z_\mathbf{x}-z_\mathbf{y})}{2\pi r}\,\mathbb{I}. \quad (6)$$

Here, we used the constraint that the object is rotationally symmetric in the ϕ direction. Equation (4) together with the above equation describes the electromagnetic field fluctuations in the presence of a rotating body. Note that Eqs. (2) and (6) include both positive and negative frequencies; since $a_T(\omega)$ and $\mathrm{Im}\,\epsilon(\omega)$ are odd functions, rest frame fluctuations are identical for opposite signs.

The electromagnetic fields outside the object receive contributions both from the fluctuating sources within the object and from fluctuations (both zero-point and at finite temperature, thermal) in the vacuum outside the object. The source fluctuations in the vacuum are given by Eq. (2) as described below.

First we consider the fluctuations inside the object and find the corresponding field correlation functions. Equation (4) together with free Maxwell equation in the vacuum ($\epsilon = 1$, $\mathbf{K}' = 0$) gives the electric field via the Green's function

$$\mathbf{E}_{\text{in-fluc}}(\omega,\mathbf{x}) = \frac{\omega^2}{c^2}\int_{\text{in}}d\mathbf{z}\,\mathbb{G}(\omega,\mathbf{x},\mathbf{z})\cdot\tilde{\mathbb{D}}\mathbf{K}(\omega,\mathbf{z}), \quad (7)$$

where we have rewritten $\mathbf{K}'$ in terms of lab-frame coordinates. The subscript on $\mathbf{E}$ indicates that the electric field is due to the inside fluctuations (but is possibly computed outside the object). The required Green's function (with one point inside and the other outside the object) can be formally expanded as

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = \frac{i}{2}\sum_{\alpha_m}\mathbf{E}^{\text{out}}_{\bar{\alpha}_m}(\omega,\mathbf{x})\otimes\mathbf{F}_{\alpha_m}(\omega,\mathbf{z}). \quad (8)$$

The index α_m denotes a set of quantum numbers including m, the eigenvalue of the angular momentum along the z direction (in units of $\hbar$). Also $\bar{\alpha}$ indicates the time reversal of the partial wave α. For example, for spherical waves $\alpha_m = (P, l, m)$, where P is the polarization and l is the eigenvalue of the total angular momentum. Here, $\mathbf{E}^{\text{out(in)}}$ is the usual outgoing (incoming) wave, while $\mathbf{F}$ is a solution to the homogeneous EM equation inside the object, *i.e.* Eq. (4) with the RHS set to zero. The important constraints are the continuity equations these functions satisfy on the surface of the object

$$\left(\mathbf{F}_{\alpha_m}\right)_\parallel = \left(\mathbf{E}^{\text{in}}_{\alpha_m} + S_{\bar{\alpha}_m}\mathbf{E}^{\text{out}}_{\alpha_m}\right)_\parallel,$$

where S is the scattering matrix. A similar equation holds for the curl acting on the fields. It turns out that the knowledge of the surface values of $\mathbf{F}$, which in turn can be expressed in terms of the scattering matrix S_{α_m}, is sufficient for computing the correlation functions. Equations (7) and (8) along with the source fluctuations of Eq. (6), and a few integrations by parts yield

$$\langle\mathbf{E}(\omega,\mathbf{x})\otimes\mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{in-fluc}} = \frac{\omega^2}{4c^2}\sum_{\alpha_m}a_T(\omega-\Omega m)$$
$$\times\left(1-|S_{\alpha_m}|^2\right)\mathbf{E}^{\text{out}}_{\alpha_m}(\omega,\mathbf{x})\otimes\mathbf{E}^{\text{out}*}_{\alpha_m}(\omega,\mathbf{y}). \quad (9)$$

Next we turn to fluctuations caused by outside (vacuum) sources. Equation (2) may appear to suggest that these fluctuations vanish because $\mathrm{Im}\,\epsilon = 0$ in the vacuum, whereas vacuum fluctuations exist even in the absence of any objects. The key is that an integral over infinite space leads to $1/\mathrm{Im}\,\epsilon$ and thus the limit $\mathrm{Im}\,\epsilon \to 0$ should be taken with care [14]. A careful analysis similar to that for the inside contribution yields the correlation function due to the outside sources as

$$\langle\mathbf{E}(\omega,\mathbf{x})\otimes\mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{out-fluc}} = \frac{\omega^2}{4c^2}a_{T_0}(\omega)\times$$
$$\times\sum_{\alpha_m}\left(\mathbf{E}^{\text{in}}_{\alpha_m}(\omega,\mathbf{x}) + S_{\alpha_m}\mathbf{E}^{\text{out}}_{\alpha_m}(\omega,\mathbf{x})\right)$$
$$\otimes\left(\mathbf{E}^{\text{in}*}_{\alpha_m}(\omega,\mathbf{y}) + S^*_{\alpha_m}\mathbf{E}^{\text{out}*}_{\alpha_m}(\omega,\mathbf{y})\right). \quad (10)$$

Note that the function a_{T_0} is defined at the environment temperature T_0 and depends only on ω, unlike the inside fluctuations which depend on the shifted frequency, $\omega - \Omega m$. Equation (10) has a rather intuitive form: each term in the dyadic expansion is a linear combination of the incoming wave plus the scattered outgoing wave.

Summing Eqs. (9) and (10), the total correlation function is given by

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{in-fluc}} + \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{out-fluc}}.$$

The above equation completely characterizes the vacuum fluctuations in the presence of a rotating body possibly at a different temperature from the vacuum. Interestingly, even at zero temperature, the rotating body spontaneously emits energy, as can be computed by averaging over the Poynting vector, $\langle \mathbf{E} \times \mathbf{B} \rangle$, and integrating over a surface enclosing the object. The sum of the in- and out-fluctuation contributions yields

$$P = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega \times$$
$$\sum_{\alpha_m} [n_T(\omega - \Omega m) - n_{T_0}(\omega)] \left(1 - |S_{\alpha_m}(\omega)|^2\right). \quad (11)$$

Note that the singularity of n_T at $\omega = \Omega m$ is removed since $1 - |S|^2$ vanishes there. Equation (11) can be written in a basis-independent form, where the radiation takes the form

$$P = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega \times$$
$$\text{Tr}\left[\left(n_T(\omega - \Omega \hat{l}_z) - n_{T_0}(\omega)\right)\left(\mathbb{I} - \mathbb{S}^\dagger(\omega)\mathbb{S}(\omega)\right)\right]. \quad (12)$$

Here $\hat{l}_z$ is the z-component of the angular momentum operator (in units of $\hbar$) and $\mathbb{S}$ is the (basis-independent) scattering matrix. Note that this equation reduces to the heat radiation from a static object in the limit of zero angular velocity [14, 15]. We are specifically interested in zero temperature. In this limit, the number of radiated photons in the partial wave α_m is

$$\frac{d\mathcal{N}_{\alpha_m}}{d\omega} = \Theta(\Omega m - \omega)\left(|S_{\alpha_m}(\omega)|^2 - 1\right), \quad (13)$$

where Θ is the Heaviside function. Hence, the radiation only comes from the frequency window $0 < \omega < \Omega m$ for the mth partial wave. In fact, Zel'dovich proposed *classical* superradiance for the same frequency regime [7]. He considered a rotating cylinder and argued that exactly for waves in the above frequency range, the amplitude of the scattered wave (in absolute value) is larger than one, *i.e.* $|S_{\alpha_m}(\omega)| > 1$; see Refs. [16, 17] for further discussion. He further conjectured that taking quantum mechanics into account would lead to spontaneous emission. Equations (11-13) are, to our knowledge, the first expressions for spontaneous emission by a rotating object given explicitly in terms of the scattering matrix,

which show that, at zero temperature, the radiation is generated exactly in the superradiating channels.

Note that, having assumed non-relativistic velocities, we must keep only the leading contribution in $\Omega R/c$ to radiation from each partial wave. In fact, higher partial waves typically contribute in higher powers of this quantity. Therefore, to compute the total radiation at zero temperature, we must keep only the lowest partial wave, while at finite temperature higher partial waves can make a comparable, or even larger, contribution due to the Boltzmann weight.

Using the general expression for radiation, we now discuss some simple special cases, namely a sphere and a cylinder. To find the S-matrix of a rotating object we have to solve a complicated equation — Eq. (4) with the RHS set to zero, but the task is made easier by making a further assumption that the object's radius is small enough so that $|\sqrt{\epsilon}|\Omega R/c \ll 1$, which allows us to neglect the explicit dependence on the velocity ($\mathbb{D} \approx \mathbb{I}$). Also, for small Ω we need only consider the low-frequency response.

For a *sphere* the T-matrix is related to its (electric) polarizability α by: $T_{1m1m}^{EE}(\omega) = i\frac{2\omega^3}{3c^3}\alpha(\omega - \Omega m)$; the argument of α is $\omega - \Omega m$ because that of ϵ is shifted. The value of these functions for negative arguments can be obtained through their analytic properties. The scattering matrix is related to the T-matrix via $S = 1 + 2T$, and the energy radiation in the lowest partial wave can be computed by Eq. (11). The result is indeed in agreement with Ref. [6].

The T-matrix for the *cylinder* is rather complicated. Further, one must take into account all polarizations. Here, we quote the final result for a slowly rotating cylinder of radius R and length L, and at zero temperature everywhere,

$$P = \frac{2\hbar L R^2}{3\pi c^3}\int_0^\Omega d\omega\, \omega^4 \left|\text{Im}\frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 1}\right|. \quad (14)$$

This equation, valid for arbitrary ϵ, reproduces the result in Ref. [18] in the limit of small conductivity.

The radiation from a rotating object exerts pressure on nearby objects. Consider a spherical object with angular velocity Ω along the z direction and a second small spherical body— a *test object* — at rest, placed at a separation d on the x axis. One finds that, in addition to the force along the x axis, there is a tangential force in the y direction. Furthermore, a torque is exerted on the test object. To compute this effect, we break up the EM-field correlation function into radiation (due to propagating photons) and non-radiation (due to zero-point fluctuations) parts:

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{rad}} + \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{non-rad}}.$$

The non-radiation part gives the Casimir force which makes no contribution to the tangential force or the

torque. At zero temperature, the radiation comes only through the superradiating modes with $\omega < \Omega m$,

$$\langle \mathbf{E}(\omega, \mathbf{x}) \otimes \mathbf{E}^*(\omega, \mathbf{y}) \rangle_{\text{rad}} = \frac{\hbar \omega^2}{2c^2} \sum_{\alpha_m} \Theta(\Omega m - \omega)$$

$$\times \left(1 - |S_{\alpha_m}|^2 \right) \mathbf{E}_{\alpha_m}^{\text{out}}(\omega, \mathbf{x}) \otimes \mathbf{E}_{\alpha_m}^{\text{out}*}(\omega, \mathbf{y}). \quad (15)$$

We assume that d is large compared to the length scales of both the rotating body and the test object, and thus compute the first reflection of the radiation off of the test object [19]. First we transform the outgoing waves to regular waves about the origin of the second object via translation matrices [13]. These waves scatter on the second object giving

$$M = \frac{\hbar c^2}{8\pi d^2} \int_0^\Omega d\omega \, \frac{1}{\omega^2} (|S_{11E}|^2 - 1)(1 - |\mathfrak{S}_{11E}|^2), \quad (16)$$

for the torque, and

$$F_y = \frac{\hbar}{32\pi d} \int_0^\Omega d\omega \, (|S_{11E}|^2 - 1)(1 - \operatorname{Re}\mathfrak{S}_{11E}), \quad (17)$$

for the shear force on the test object, where S_{11E} and $\mathfrak{S}_{11E}$ are the scattering matrices of the rotating and the test object, respectively, in the lowest partial wave ($l = m = 1, P = E$). This partial wave gives the leading order in $1/d$ for small Ω. Note that the torque falls off as $1/d^2$ with separation while the force goes as $1/d$. Furthermore the signs are positive, *i.e.* a rotating body drags along objects nearby while making them rotate parallel to its own rotation axis.

To get an estimate for the magnitude of radiation effects, we consider a rapidly spinning nanotube of radius R and length L, and assume that $\Omega R/c$ is small. We then find that the rotation slows down by an order of magnitude over a time scale of $\tau \sim (I/\hbar) (c^3/LR^2\Omega^3)$. The moment of inertia of a nanotube can be as small as 10^{-33} in SI units [20] (compare with $\hbar \approx 10^{-34}$). So even at small velocities, τ can be of the order of a few hours.

We have derived a universal formula for the spontaneous emission by a rotating object in terms of the scattering matrix that makes an explicit connection to the physical principles behind superradiance. Furthermore, it allows one to circumvent assumptions about small size or small conductivity used in the literature. We also believe the current formalism naturally generalizes to relativistic motion. Finally, generalization of the technical and conceptual aspects of this work to the interaction of multiple (moving) objects would be worthwhile.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[2] G. T. Moore, J. Math. Phys. **11**, 2679 (1970); S. A. Fulling and P. C. W. Davies, P. Roy. Soc. Lond. A Mat. **348**, 393 (1976); M.-T. Jaekel and S. Reynaud, Phys. Lett. A **167**, 227 (1992); G. Calucci, J. Phys. A-Math. Gen. **25**, 3873 (1992); P. A. Maia Neto and S. Reynaud, Phys. Rev. A **47**, 1639 (1993); C. K. Law, *ibid.* **49**, 433 (1994); V. V. Dodonov, Phys. Lett. A **207**, 126 (1995); G. Barton and A. Calogeracos, Ann. Phys.-New York **238**, 227 (1995); O. Méplan and C. Gignoux, Phys. Rev. Lett. **76**, 408 (1996); M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233 (1999).
[3] D. A. R. Dalvit, P. A. Maia Neto, and F. D. Mazzitelli, in *Lecture Notes in Physics*, Vol. 834 (2011) p. 419.
[4] J. B. Pendry, J. Phys.-Condens. Mat. **9**, 10301 (1997).
[5] A. I. Volokitin and B. N. J. Persson, J. Phys.-Condens. Mat. **11**, 345 (1999); Rev. Mod. Phys. **79**, 1291 (2007).
[6] A. Manjavacas and F. J. García de Abajo, Phys. Rev. Lett. **105**, 113601 (2010); Phys. Rev. A **82**, 063827 (2010).
[7] Y. B. Zel'dovich, JETP Lett. **14**, 180 (1971).
[8] W. G. Unruh, Phys. Rev. D **10**, 3194 (1974).
[9] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[10] A. Calogeracos and G. E. Volovik, JETP Lett. **69**, 281 (1999).
[11] S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Priniciples of statistical radiophysics*, Vol. 3 (Springer, Berlin, 1989).
[12] J. van Bladel, P. IEEE **64**, 301 (1976).
[13] S. J. Rahi, T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, Phys. Rev. D **80**, 085021 (2009).
[14] M. Krüger, T. Emig, and M. Kardar, Phys. Rev. Lett. **106**, 210404 (2011).
[15] C. W. J. Beenakker, Phys. Rev. Lett. **81**, 1829 (1998).
[16] J. D. Bekenstein and M. Schiffer, Phys. Rev. D **58**, 064014 (1998).
[17] M. Richartz, S. Weinfurtner, A. J. Penner, and W. G. Unruh, Phys. Rev. D **80**, 124016 (2009).
[18] Y. B. Zel'Dovich, L. V. Rozhanskii, and A. A. Starobinskii, Radiophys. and Quantum Electronics **29**, 761 (1986).
[19] M. F. Maghrebi, Phys. Rev. D **83**, 045004 (2011).
[20] P. Král and H. R. Sadeghpour, Phys. Rev. B **65**, 161401 (2002).