Nonlinear Hydrodynamics and Fractionally Quantized Solitons at the Fractional Quantum Hall Edge

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Non-Linear hydrodynamics and Fractionally Quantized Solitons at Fractional Quantum Hall Edge

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We argue that dynamics of gapless Fractional Quantum Hall Edge states is essentially non-linear and that it features fractionally quantized solitons with charges $-\nu e$ propagating along the edge. Observation of solitons would be a direct evidence of fractional charges. We show that the non-linear dynamics of the Laughlin's FQH state is governed by the quantum Benjamin-Ono equation. Non-linear dynamics of gapless edge states is determined by gapped modes in the bulk of FQH liquid and is traced to the double boundary layer (*overshoot*) of FQH states. The dipole moment of the layer $\eta = \frac{1-\nu}{4\pi}$ is obtained in paper. Quantum hydrodynamics of FQH liquid is outlined.

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1. Introduction and Results. In a Fractional Quantum Hall state electrons collectively constitute an incompressible liquid almost free from dissipation. Excitations in this liquid are gapped by an energy Δ_ν determined by the Coulomb interaction[1, 2]. The gap is large (typically $\Delta_{1/3} \sim 10K$) compared with mK temperature, but small compared with the cyclotron energy $\sim 25meV$). The only low energy current carrying states in a FQH liquid are *edge states* localized on the boundary [3]. Edge states provide a valuable tool to probe FQH states.

In most FQH states the excitation gap is large compared to the energy of long wave edge states, and for that reasons is commonly neglected. The standard approach to the theory of Edge states starts from the Chern-Simons action in the bulk [4]. This action has no scale. It neglects gapped bulk modes but focuses on braiding properties of FQH states. The Hamiltonian of this theory differs from zero only by a confining potential. The boundary states are long-wave chiral bosons with the algebra

$$[f(x),\, f^H(x')] = \frac{2\nu}{\pi}\nabla_x \delta(x-x'), \tag{1}$$

propagating according to the linear wave equation

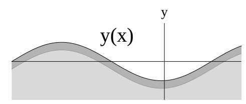
$$\dot f - c_0 \nabla_x f = 0. \tag{2}$$

Here $f^H = \frac{1}{\pi}\int \frac{f(x')-f(x)}{x'-x}dx'$ is the Hilbert transform of $f(x)$, the unperturbed droplet occupies the half plane $y<0$, x is a coordinate along the boundary. The sound velocity $c_0 = \hbar^{-1}\ell_B^2|\nabla_y U|$ is determined by the slope of the confining potential $U(y)$, $\ell_B = \sqrt{\Phi_0/2\pi B}$ is the magnetic length and $\Phi_0 = hc/e$.

This theory assumes that a FQH state does not change toward the boundary and neglects electric polarization caused by edge waves. This happens if the curvature of the potential is small compared to the gap $\ell_B^2\nabla_y^2 U \ll \Delta_\nu$, but a slope is larger than electric field $\ell_B^2\nabla_y U \gg e^2$. We accept these conditions.

Physics missed by this otherwise successful theory can be seen in the following setting. Let us suddenly perturb the edge by a classical instrument, say RF-source (whose spatial extent is larger than magnetic length ℓ_B) and then release the system. A smooth semiclassical density profile $f_0(x)$ will occur, Fig. 1. How does it propagate along the edge? The wave equation (2) suggests that the initial profile translates as $f(x,t) = f_0(x-c_0t)$ without changes. This may be true shortly after the perturbation, but at time of order of $\hbar/\Delta_\nu$ the profile is expected to change. This time is short. In typical $\nu = 1/3$ samples it is short, about $\sim 1ps$ [1], much less than time scales of dissipation. We show that at that already at that time the wave equation fails, giving rise to new important effects.

FIG. 1: Boundary waves: the boundary layer is highlighted



Corrections to linear waves (2) come from few sources: the curvature of the confining potential, mixing with higher Landau levels, disorder, and, more interestingly, the interaction between the gapless edge and gapped bulk modes. The latter is the leading effect in FQHE. It is the subject of the paper. We start by listing major results.

(i) We argue that the wave equation receives important corrections proportional to the scale $\kappa \sim \Delta_\nu \ell_B^2/\hbar$ of gapped excitations omitted in the Chern-Simons action. In this paper we focus only on the simplest FQH state with a single branch of excitations (Laughlin's state). We show that the quantum equation for edge modes in the Laughlin state is Quantum Benjamin-Ono equation

$$\dot f - c_0\nabla_x f - \kappa\nabla_x\left(\frac{f^2}{2} - \eta\cdot\nabla_x f^H\right) = 0,\ \eta = \frac{1-\nu}{4\pi} \tag{3}$$

The new terms in brackets cause new phenomena. One is particular interesting is *fractionally charged edge solitons.*

The meaning of the chiral boson is seen from its value on coherent state with an electronic density $\rho(x,y)$. It is a boundary density $f(x) = \int_0^{y(x)}\rho(x,y)dy$, where $y(x)$ is a boundary displacement counted from an unperturbed boundary $y=0$. The bosons act in the chiral Fock space, where $\mathrm{i}\hbar\pi f^H + \nu\nabla_x\pi_f{=}0$. Here $\pi_f \equiv -\mathrm{i}\hbar\frac{\delta}{\delta f}$ is a momentum of f. The constraint yields the algebra (1).

(ii) The most interesting and important term in this equation is the dispersion $\eta \cdot \nabla_x f^H$. This term reflects the *double boundary layer* or *overshoot* of the equilibrium Laughlin's state. That is: a difference of one-particle density ρ of the Laughlin state and the density ρ_I of integer QH state $\nu = 1$ of the same number of particles with charges νe is singular on the boundary

$$\rho(y) \approx \rho_I(y) + \eta\delta'(y). \quad (4)$$

The double layer is illustrated on the Fig. 2. The coefficient η is the dipole moment of the droplet. We show that

$$\eta = \int y(\rho - \rho_I)\mathrm{d}y = \frac{1-\nu}{4\pi}. \quad (5)$$

(iii) The non-linear edge theory follows from *quantum hydrodynamics of the bulk of FQH liquid.* We defer a detailed discussion of quantum hydrodynamics in the bulk to more extended publications. In this letter we outline only basic points and consider only potential flow.

The non-linear equation (3) previously appeared in two domains of physics. Its classical version has been derived by Benjamin [5] in 1967 for inner waves in a deep stratified incompressible fluid with a rapidly changing density or shear [6]. It is called the Benjamin-Ono equation.

This relation is not accidental: a FQH state is an incompressible quantum fluid with a rapidly changing density and shear at the boundary layer, Eq.(4), Fig. 2.

The quantum Benjamin-Ono equation (qBO) identical to (2,3) describes the chiral sector of Calogero model [7, 8]. This relation is not accidental also. Both FQH and Calogero liquids feature excitations with a fractional charge. One can treat the result of this paper as a proof of an anticipated equivalence between Calogero liquid and FQH Edge states.

The qBO has an intrinsic relation to a boundary Conformal Field Theory with the central charge $c = 1 - 6\nu^{-1}(\nu-1)^2$ situated in the exterior of the droplet. Conformal symmetry emerges with respect to deformations of the boundary of a FQH droplet. Also, qBO has a natural extension to non-Abelian FQH states. We will discuss these aspects elsewhere. In this paper we mention a few major features of qBO [7, 8] and focus on its derivation.

2. Quantum Benjamin-Ono Equation. The qBO is a Hamiltonian equation. The dispersion term $\eta\nabla_x^2 f^H$ has the dimension of viscosity but contrary to real viscosity does not produce dissipation. It gives a non-analytic correction to the dispersion of linear waves

$$\hbar\left(\omega(k) - c_0 k\right) = \kappa\eta k|k|. \quad (6)$$

A noticeable feature of the qBO is that the "dissipationless viscosity" η depends only on the filling fraction ν and, in this sense, is universal. Contrary, in classical liquids [5, 6] the coefficient η depends on the equation of state and is not quantized.. This term is similar in origin to the "Hall viscosity" [12], but is not directly related. It is inherent to FQHE, η vanishes for the IQH.

Quantum BO equation has been studied in connection with Calogero model in [7–9]. We list few major facts. The equation is integrable despite of being non-local. It features *solitons*. There are two branches of solitons: one is *ultrasonic*, another is *subsonic*. Quite remarkably, both carry quantized electron charges. An ultrasonic soliton carries an integer of electron charge $q = +e$. It is a bump on the edge - a coherent state of an electron. A subsonic soliton is a coherent state of holes - a dent on the edge. It carries integer of a fractional charge $q = -\nu e$ of an opposite sign. Shapes of the elementary solitons are especially simple:

$$f_q(x + c_0 t) = \frac{q}{\pi}\frac{a}{a^2(x - v_q t)^2 + 1}, \quad q = 1,\ -\nu. \quad (7)$$

Velocity of a soliton (relative to the sound) is $v_q = q\kappa\eta a$ is inversely proportional to the magnetic field. It is proportional to its amplitude a and its charge q. The amplitude $a > 0$ is arbitrary, but the charge is quantized.

Benjamin-Ono equation receives corrections from effects related to electrostatic forces. However fractionalized solitons are protected as long as FQH state exists.

There two ways to observe fractional solitons. One is to generate individual solitons by applying a time dependent voltage protocol $eV(t) = \hbar c_0 f_q(c_0 t)$ through a quantum dot connected to the conductor via a tunnel barrier [10] and measure a time dependent current at a distance of from the point contact. Similar measurements have been done in nanoelectronic devices (see, e.g.,[11]).

Alternative way to observe solitons is through the *soliton train*. When an RF-source creates a large dent of a size $L \gg \ell_B$ in the boundary density which involves a large number n of electrons, the dent collapses through a shock wave to oscillatory features which further separate to a stable a sequence of pulses carried a fraction $-\nu e$ of an electronic charge (see e.g.,[7]). This happens very fast at the time $(\hbar/\Delta_\nu)L/(n\ell_B) \sim$ 1-10ps. Pulses can be seen through the time dependence of the edge current.

3. Phenomenological Hamiltonian is the starting point of the analysis. A space where the Hamiltonian acts is the result of a projection on the first Landau level enforced by the condition $\hbar\omega_c \gg \Delta_\nu$. It is the set of states obtained by a deformation of the Laughlin ground state ψ_0 by holomorphic polynomials. In a radial gauge suitable for a central-symmetric confining potential *coherent states* of this space are

$$\psi_V = Z_V^{-\frac{1}{2}} e^{\frac{1}{\hbar}\sum_i^N V(z_i)}\psi_0, \quad \psi_0 = \Delta^\beta e^{-\sum_i^N \frac{|z_i|^2}{4\ell_B^2}}, \quad (8)$$

where $\Delta = \prod_{i>j}^N (z_i - z_j)$ and Z_V is a normalization.

A complex potential $V(z)$ is analytic at infinity and such that $4\pi\sigma = -\Delta V$ is real. A meaning of σ is a density of "holes", or vortices, cf. (15). Also a set of permissible operators is spanned by a product of holomorphic and anti-holomorphic operators. We denoted the averages of symmetric operators in a given V-state as $\langle\mathcal{O}\rangle_V = \int \psi_V^* \mathcal{O}\psi_V \prod_i \mathrm{d}^2 r_i$.

These states have been studied in [15]. We mention only an important sum rule elementary followed from the value of the dilatation operator $\sum_i \langle r_i\cdot(\nabla_i+\nabla_i^\dagger)\rangle_V = -2N$

$$\frac{1}{N}\sum_i \langle \frac{r_i^2}{2\ell_B^2} - N\beta - \frac{1}{\hbar} r_i\cdot\nabla_i \mathrm{ReV}\rangle_{\mathrm{V}} = 1 - \frac{\beta}{2}. \tag{9}$$

We construct the Hamiltonian based on a few defining properties: (i) Lauglin's w.f. is the ground state; (ii) action of the Hamiltonian preserves the projected space; (iii) long waves of a FQH liquid are Galilean invariant; (iv) on closed manifolds all states are gapped [16].

Under these assumptions the zero Hamiltonian of the Chern-Simons theory is replaced by the Bernoulli energy

$$H = \int \frac{m_\nu}{2} \hat{\mathrm{v}}^\dagger \hat{\rho} \hat{\mathrm{v}}\, \mathrm{d}^2 r, \tag{10}$$

where $m_\nu = \frac{\pi\beta\hbar}{\kappa} \sim \frac{\hbar^2\ell_B^2}{\Delta_\nu}$ is an effective mass obtained from the value of a gap, $\hat{\rho}(r) = \sum_i \delta(r - r_i)$ is the density, and $\hat{\mathrm{v}} = \hat{v}_x - i\hat{v}_y$ is the velocity. Velocity of a particle is

$$\frac{\mathrm{i}}{2\hbar} m_\nu \hat{\mathrm{v}}_i = \partial_{z_i} - \frac{e}{2c} A(z_i) - \sum_{j\neq i} \frac{\beta}{z_i - z_j}, \quad \beta = \frac{1}{\nu}. \tag{11}$$

Here $A = A_x - iA_y$ is an external e.m. potential , $z_i = x_i + \mathrm{i} y_i$ is the complex coordinate. We use $\beta = 1/\nu$.

A subtle point of this Hamiltonian is the definition of the velocity operator (11). It differs from the velocity of individual electrons but corresponds to the velocity of "composite particles" - electrons with an attached flux converting them to bosons. This velocity changes slowly in long-wave excited states. It enters hydrodynamics.

Another noticeable feature is a normal ordering of velocities entered (11).It ensures that $H\psi_0 = 0$.

The Hamiltonian (10) can be viewed as a quantized version of "effective" Hamiltonians proposed in [13].

4. Quantum hydrodynamics of the FQHE liquid describes dynamics of velocity and density *fields* when the number of particles is large. The fields are defined as operators acting on averages $\langle\mathcal{O}\rangle_V$. In particular, the velocity field is defined as $\rho(r)\mathrm{v}(r)\langle\mathcal{O}\rangle_V = \langle\sum_j \delta(r - r_j)\mathrm{v}_j \mathcal{O}\rangle_V$, where $\rho(r) = \langle\hat{\rho}(r)\rangle_V$. In this representation

$$m_\nu \mathrm{v} = 2\partial_z(\pi_\rho - \mathrm{i}V), \quad \pi_\rho = -\mathrm{i}\hbar\frac{\delta}{\delta\rho}. \tag{12}$$

In the restricted space (8) matrix elements of π_ρ are imaginary, ΔV is real, hence the liquid is incompressible

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0. \tag{13}$$

It is customary describe the incompressible flow in terms of the *stream function*

$$\vec{\mathrm{v}} = \vec{\nabla}\times\Psi, \quad m_\nu\Psi = \mathrm{i}\pi_\rho + \mathrm{Re}V. \tag{14}$$

Eq. (14) gives an interpretation to the deformation potential V. Its real part is the diagonal part of the stream function. Diagonal parts of vorticity and energy are

$$\langle H\rangle_V = \frac{1}{2m_\nu}\int |\nabla V|^2 \rho \mathrm{d}^2 r, \;\; m_\nu \langle\vec{\nabla}\times\vec{\mathrm{v}}\rangle_V = -\Delta V \tag{15}$$

5. Holomorphic fields, potential incompressible flow and Edge states. In a system without boundaries all modes are gapped [1, 14]. If there is a boundary, gapless edge states emerge. In this paper we focus only on edge states, deferring discussion of hydrodynamics of the bulk. For this purposes it is sufficient to consider only a potential flow where a stream function is harmonic

$$\vec{\nabla}\times\vec{\mathrm{v}} = 0, \quad \Delta\Psi = 0. \tag{16}$$

Potential flow corresponds to deformations of the w.f. (8) by analytic functions inside the domain occupied by the liquid. All singularities of V are outside of the domain (analytic functions do not exist on closed manifolds, so as gapless modes). It has been shown in [15] that a holomorphic deformation of Laughlin's state changes only the shape of the droplet, leaving the density and the area unchanged. In the leading $1/N$ order, the bulk density is uniform $\bar{\rho} = \nu B/\Phi_0$. In a radial potential the droplet in the ground state is a disk with a radius $R = \sqrt{N/\pi\bar{\rho}}$.

Incompressible potential flow with a free boundary and a constant density is a standard subject in classical hydrodynamics [17]. Its extension to the quantum case is straightforward. Use the Green formula and the Cauchy-Riemann condition to express the Hamiltionian only through the boundary value of the fluid potential

$$H = \frac{m_\nu\bar{\rho}}{2}\oint \Psi\partial_n\Psi ds, \quad \langle H\rangle_V = \frac{\bar{\rho}}{2\mathrm{i}m_\nu}\oint \bar{V}\mathrm{d}V. \tag{17}$$

The bulk Hamiltonian vanishes. Then the only governing equation is the *kinematic boundary condition* [17]

$$\dot{y} + v_x\nabla_x y + v_y = 0, \tag{18}$$

where v_x, v_y are velocities of the inner layer tangential and normal to the unperturbed boundary. In the rest of the paper we derive the relation between velocity and boundary elevation. The result is

$$v_x = c_0 - \kappa\bar{\rho}y(x), \quad v_y = \kappa\eta\cdot y_{xx}^H \tag{19}$$

We obtain the qBO (3) by using (18) and $f(x) = \bar{\rho}y(x)$.

6. Chiral constraint A relation between potential V, hence velocity, and the shape of the boundary has been obtained in [15]. We re-derive it invoking Dyson's arguments [18] and refine the results of [15]. Dyson's arguments are somewhat heuristic but transparent and short.

Let us express an expectation value $\langle\mathcal{O}\rangle_V$ as a path integral over the density field $\langle\mathcal{O}\rangle_V = \int \mathcal{O}[\hat{\rho}]e^{-\beta F_V[\hat{\rho}]}\mathcal{D}\hat{\rho}$. The chiral constraint (aka *loop equation* [15]) is the saddle point condition ensured by a large number of particles

$$\Big(\frac{\delta}{\delta\rho} - \beta\frac{\delta}{\delta\rho}F_V[\rho]\Big)\mathcal{O}[\rho] = 0. \tag{20}$$

The functional $F_V[\rho]$ can be treated as the free energy of 2D-Coulomb plasma. It consists of energy and entropy. $-\beta F_V = \log|\psi_V|^2 - \int \rho\log\rho \mathrm{d}^2 r$. The entropy is the Jacobian of passing from integration over particle coordinates to a path integral over the density field.

In order to find the energy of the plasma we write $\sum_{i,j\neq i}\log|r_i - r_j| = \sum_{i,j}\log|r_i - r_j + \ell\delta_{ij}| - \sum_i \log\ell(r_i)$, where $\ell(r)$ is the mean distance between particles. Exclusion of "self-interaction" allows to replace sums by integrals: $\sum_{j\neq i}\log|r_i - r_j| = -\frac{1}{2}\varphi(r_i) - \log\ell(r_i)$, where $\varphi(r) = -2\int\log|r-r'|\rho\,\mathrm{d}^2 r$ is the Coulomb potential of the plasma. This gives

$$\beta F_V = \int[\frac{\beta}{2}(\varphi - \bar\varphi) + \beta\log\ell + \log\rho - \frac{2}{\hbar}\mathrm{Re}V]\rho\mathrm{d}^2 r, \quad (21)$$

where $\bar\varphi = -2\int_0^R \log|r-r'|\bar\rho\mathrm{d}^2 r' = \pi\bar\rho r^2$ is the potential of a neutralizing uniform background charge $\bar\rho$.

A subtle point of this approach is the value of the mean distance between particles. It is different in the bulk and close to the boundary. In the bulk the mean distance is isotropic and $\rho\ell^2 \sim 1$. The short distance term and the entropy in (21) sum up to $(\frac{\beta}{2} - 1)\log\rho$.

Close to the boundary, ℓ entering (21) is the distance in the direction normal to the boundary. Since the mean distance along the boundary is constant $\sim \ell_B$ we have $\ell_B\ell \sim 1/\rho$. In this case the short distance term and the entropy term sum up to $(\beta-1)\log\rho$. Then Eq.(20) gives $2\pi\kappa^{-1}\Psi = \varphi - \bar\varphi - 2(1-\nu)\log\rho$.

Unfortunately, Dyson's arguments miss exponential corrections important at the boundary. Notice that in the case of the IQHE when $\beta = 1$ the term $(\beta-1)\log\rho$ vanishes. The Dyson's arguments give $2\pi\kappa^{-1}\Psi = \varphi - \bar\varphi$. This equation treats the density as a step-function. Instead, the exact density of the IQHE is

$$\rho_I = \frac{1}{2}\bar\rho\,\mathrm{erfc}(\frac{y}{\sqrt{2}\ell_B}) \approx \bar\rho\Big(\Theta(-y) + \frac{\ell_B}{y\sqrt{2\pi}}e^{-\frac{y^2}{2\ell_B^2}}\dots\Big),$$

where $(-y)$ is the distance to the boundary.

This failure can be "repaired" by replacing the plasma Coulomb potential $\bar\varphi = \pi\bar\rho r^2$ of a neutralizing uniform charge $\bar\rho$ in (21) by the potential of the charge ρ_I: $\varphi_I = -2\int\log|r-r'|\rho_I(r')\mathrm{d}^2 r' \approx \pi\bar\rho r^2(1+O(e^{-y^2/2\ell_B^2}))$. Then

$$2\pi\kappa^{-1}\Psi = \varphi - \varphi_I - 2(1-\nu)\log\rho. \quad (22)$$

This *ad hoc* procedure reflects a discreteness of particles.

So far we did not assume that the flow is potential. If the flow is potential the Laplace operator nulls the l.h.s. of (22). We obtain a Liouville-type equation

$$\rho - \rho_I + \eta\Delta\log\rho = 0. \quad (23)$$

We wish to have a more satisfactory mathematical justification of this equation. To support the Liouville equation we comment that it can be can be checked against the sum rule (9), and that its numerical solution obtained by A. Abanov appears to be undistinguishable from a numerical *ab initio* simulation of the Laughlin's wave function (8) presented in Fig. 2 [20, 21].

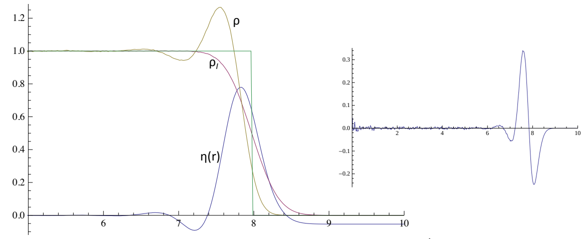


FIG. 2: Boundary double layer of $\nu = \frac{1}{3}$ state computed for 200 particles [21]: Left (in decreasing order): overshoot of the density ρ, density ρ_I (at $\nu = 1$), the dipole moment $-\eta(r) = \int^r y'(\rho_I - \rho)\mathrm{d}y'$. The value of $\eta|_\infty = \frac{1-\nu}{4\pi} \approx 0.053$ is clearly seen. Right: $\rho - \rho_I$ illustrates the double layer (4).

7. Boundary double layer and its dipole moment Eq.(23) has a profound consequence. It shows that the density behavior is singular: on approaching the boundary the density oscillates and shoots up before falling down. A detailed structure of the overshoot is not well understood. However, we know that it is a double layer and as we see in a moment, only the boundary dipole moment of the layer enters the edge dynamics. The dipole moment is easy to compute. The overshoot was observed numerically in [19] but have not been given any particular significance. Independently it has been found analytically in [15]. The value of the dipole moment (5) follows from the sum rule (9) and also from the Liouville equation (23).

A detailed structure of the layer follows from (23). Iterations of (23) allow us to conjecture the asymptotic expansion for the density. In units of ℓ_B [20, 22]

$$\rho = \bar\rho\Big(1 + \frac{e^{-\xi^2/2}}{\nu\sqrt{2\pi}}(2\pi\eta\,\xi + O(\xi^{-1})\Big), \quad \ell_B\xi = y < 0. \quad (24)$$

At $\ell_B \to 0$ the leading term of (24) is a double layer presented earlier (4).

8. Transformation of velocities Now we are in a position to compute the velocity in terms of the boundary elevation to complete the governing equation (18). As in (18) we assume the density moves together with the boundary $\rho(x,y) = \rho_0(y - y(x))$, where $\rho_0(y)$ is the density of the ground state, and compute the boundary value of the Coulomb potential of the plasma φ entering the chiral constraint (22). Computing $\int\log|r-r'|\rho_0(y' - y(x'))dx'dy'$, we shift the variable $y' \to y' + y(x')$, subtract ρ_I and expand in $y(x)$. We obtain $\varphi(x,y) - \varphi_0(y - y(x)) \approx$

$$2\int dx'(y(x') - y(x))\int\frac{y-y'}{|r-r'|^2}[\rho_0(y') - \rho_I]dy' - 2\pi\bar\rho y y(x).$$

The integral over y' is localized inside the boundary layer. If we choose y to be on the inner boundary of the layer the range of $|y - y'| \sim \ell_B$. We can replace $|r - r'|^{-2}$ in the integral by $(x - x')^{-2}$ obtaining a transformation law of the stream function under a displacement of the boundary by $y(x)$ (valid for any shape of the boundary)[23]

$$\Psi(x,y) = \Psi_0(y - y(x)) - \kappa\left(\bar\rho y \cdot y(x) + \eta \cdot y_x^H\right). \quad (25)$$

For a flat boundary $\Psi_0 = c_0 y$. This prompts (19) and subsequently qBO.

The relation between V and $y(x)$ follows from (25,14)

$$(\pi\beta\hbar)^{-1}V' = \frac{\bar\rho}{4\pi}\int \frac{y(x')dx'}{z-x'} + \frac{\eta}{2\pi}\int \frac{y(x')dx'}{(z-x')^3}. \quad (26)$$

The Benjamin-Ono equation (3) and the current algebra (1) follow from the transformation law (25), and the value of the dipole moment (5).

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[20] This formula has been suggested by A. Abanov based on numerical data.
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[22] Contrary to the asymptotic expansion for ρ_I eq. (24) is valid only for $\xi < 0$. Asymptotic expansion for $\xi > 0$: $\rho \sim e^{-\xi^2/2}\xi^{-\beta}\Big(1+O(\xi^{-1})\Big)$ (A. Zabrodin, P. Wiegmann, unpublished). This asymptote would be affected by the confining potential.
[23] This exercise is equivalent to the Hadamard formula for a variation of solution of a boundary value problem upon a variation of the boundary J. Hadamard, J. Math. Pures Appl. 4, 27 (1898)