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General Relationship Between the Entanglement Spectrum and the Edge State Spectrum of Topological Quantum States

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We consider (2+1)-dimensional topological quantum states which possess edge states described by a chiral (1+1)-dimensional Conformal Field Theory (CFT), such as e.g. a general quantum Hall state. We demonstrate that for such states the reduced density matrix of a finite spatial region of the gapped topological state is a thermal density matrix of the chiral edge state CFT which would appear at the spatial boundary of that region. We obtain this result by applying a physical instantaneous cut to the gapped system, and by viewing the cutting process as a sudden "quantum quench" into a CFT, using the tools of boundary conformal field theory. We thus provide a demonstration of the observation made by Li and Haldane about the relationship between the entanglement spectrum and the spectrum of a physical edge state.

Topological phases of matter are gapped quantum states which cannot be adiabatically deformed into a completely 'trivial' gapped system such as a trivial band insulator, without crossing a quantum phase transition. They are not characterized by symmetry breaking, but instead by certain global topological properties such as the presence of (topologically) protected edge states and/or a ground state degeneracy which depends on the topology of the surface on which the state resides[1]. Topological states of matter of this kind which have been discovered in nature include the integer and fractional quantum Hall states[2], and the recently discovered time-reversal invariant topological insulators[3–6].

Quantum entanglement is a purely quantum mechanical phenomenon which has no classical analog. For any pure quantum state (typically the ground state) of a system consisting of two disjoint subsystems A and B, complete information about the bipartite entanglement between the two subsystems is described by the reduced density matrix. Quantum entanglement provides an alternative characterization of the properties of the many-body system[7]. For example, as discovered by Levin and Wen[8], and by Kitaev and Preskill[9], the entanglement entropy of a topologically ordered state in a region of linear size l in two-dimensional position space contains a universal l-independent term, the 'topological entanglement entropy' (TEE), which is a characteristic of the topological order of the state. However, the TEE does not provide a complete description of a topological state of matter, since distinct topologically ordered states can have the same TEE. More complete information about a topological state of matter can be obtained from the eigenvalue spectrum of the reduced density matrix, often referred to as the entanglement spectrum[10]. In general, the density matrix ρ_A describing the entanglement between a subsystem A and the rest of the system can be written in the form of $\rho_A = e^{-H_{\rm E}}$, with $H_{\rm E}$ a Hermitian operator. One important physical feature of the so-defined *entanglement Hamiltonian* H_E is that its low-energy eigenstates correspond to those states in A, appearing in the Schmidt-decomposition[11] of the initial pure state, which are most entangled with the rest of the system. In general, $H_{\rm E}$ is different from the *physical Hamiltonian* of the system.

The focus of the present article is a remarkable observation made recently by Li and Haldane[10], and in subsequent works, for topological phases whose physical Hamiltonian possesses low energy states at an open boundary ('edge states'). This includes fractional quantum Hall states[10, 12–14], non-interacting topological insulators[15, 16] and the Kitaev honeycomb model[17]. It was found that for those systems the low-energy 'edge' states of the physical Hamiltonian at an actual open boundary of system A are in one-to-one correspondence with the low-lying eigenstates of entanglement Hamiltonian $H_{\rm E}$ (*i.e.* with the most entangled states). However, except for systems which can be reduced to non-interacting fermion problems[15–17], such a correspondence between the entanglement spectrum and the edge state spectrum of the physical Hamiltonian has only been supported by numerical evidence. No general argument for the validity of such a correspondence has been presented so far[40]. It is the purpose of the present Letter to demonstrate the general validity of this correspondence.

General setup– In this Letter, we show that for a generic (2+1)-dimensional topological state which possesses edge states described by a conformal field theory, the entanglement Hamiltonian $H_{\rm E}$ is proportional to the Hamiltonian H_L of a physical chiral (say L-moving) edge state appearing an actual spatial boundary of subsystem A, in the long-wavelength limit and in any fixed topological sector. For example, our conclusion applies to all the Abelian and non-Abelian quantum Hall states described by Chern-Simons effective field theories in the bulk[18–

21]. In order to relate the entanglement spectrum and the spectrum of the physical edge state Hamiltonian, we consider a bipartition of the toplogical state on a cylinder into two parts A and B as shown in Fig. 1 (a). The (physical) Hamiltonian H can be written in the form

$$H = H_A + H_B + H_{AB} \tag{1}$$

where H_A and H_B denote the Hamiltonians in disconnected regions A and B, respectively, each of which has (two) open boundaries. The term H_{AB} couples regions A and B across their joint boundary. For example, for a 2D gapped tight-binding model $H = \sum_{\langle ij \rangle} c_i^\dagger t_{ij} c_j$ realizing[22] the integer quantum Hall effect, the term H_{AB} contains all the electron hopping terms across the boundary between A and B.

Now we consider a deformed Hamiltonian containing a parameter $\lambda \in [0, 1]$ (similar to Ref. [23]):

$$H(\lambda) = H_A + H_B + \lambda H_{AB} \tag{2}$$

By construction, $H(\lambda = 0)$ is the Hamiltonian of the two decoupled cylinders A and B, and $H(\lambda = 1)$ is the Hamiltonian of the whole cylinder $A \cup B$. Since we are interested in such topological states which possess chiral edge states, the Hamiltonian $H(\lambda = 0)$ will have chiral and anti-chiral edge states propagating at the boundary between regions A and B, as shown in Fig. 1 (b). When $\lambda \neq 0$, the term λH_{AB} introduces a coupling between the regions A and B. Denote the bulk gap of the Hamiltonian $H = H(\lambda = 1)$ by $E_{\rm g}$. When the coupling λ is small enough such that the energy scale of the coupling term λH_{AB} is much smaller than the bulk gap $E_{\rm g}$, the gapped bulk states described by H_A and H_B are almost entirely unaffected by the coupling term λH_{AB}, whose main effect is then to induce an inter-edge coupling between the chiral and anti-chiral edge states. Since each individual edge state is described by a chiral conformal field theory (CFT), the theory of the two edges between regions A and B is described by a non-chiral CFT. Thus at low-energy the coupling term λH_{AB} between regions A and B is reduced to a local interaction in the CFT. For simplicity, we assume that this interaction is a relevant perturbation of the CFT in the renormalization group (RG) sense, so that the two counterpropagating edges will be gapped for arbitrarily small coupling λ. Thus we expect that in this case the system described by the Hamiltonian $H(\lambda = 1)$ to be adiabatically connected to that described by $H(\lambda)$ for a small but non-vanishing value of λ. In this case, the entanglement properties of $H(\lambda = 1)$ are expected to be the same as those of $H(\lambda)$ with a small λ. The latter describes the entanglement between the left- and the right-movers of the edge state CFT. It turns out that our result still holds when λ is an (RG-) irrelevant coupling: a more detailed discussion of this situation will be given below, as well as in the supplementary material[24].

Below we will solve this entanglement problem for the edge state CFT, by mapping it to a problem of a quantum quench. We then solve the latter (quantum quench) problem in the standard manner by using the work of Calabrese and Cardy[25, 26] which employs the methods of boundary conformal field theory (BCFT)[27].

Reduced density matrix of the edge CFT– Next we study the entanglement properties of the Hamiltonian $H(\lambda)$ for small values of λ, which, as explained above, amounts to the study of the $(1+1)$ dimensional problem of coupled edge states,

$$H_{\rm edge}(\lambda) = H_L + H_R + \lambda H_{\rm int} \tag{3}$$

Here, H_L and H_R denote the Hamiltonians of left-moving (L) and right-moving (R) edge states, and $\lambda H_{\rm int}$ a (RG-) relevant inter-edge coupling. The L- and R- moving edge states are the low-energy excitations of the subsystem in regions A and B, respectively. Again, the entanglement properties between the subsystems A and B are reduced to those between left and right moving $(1+1)$-dimensional edge states. If we denote the ground state of the Hamiltonian $H_{\rm edge}(\lambda)$ from Eq.(3) by $|G\rangle$, then our goal is to obtain the density matrix of the L-moving edge state subsystem defined by

$$\rho_L = \mathrm{Tr}_R \left(|G\rangle \langle G| \right), \tag{4}$$

where Tr_R denotes the trace over the R-moving edge state degrees of freedom. In general, the ground state $|G\rangle$ will depend on all the details of the coupling between the R- and the L- moving edges states. However, due to the gapless nature of $H_{\rm edge}(\lambda = 0)$ describing the *decoupled* edges, certain universal properties can be inferred in the long-wavelength limit without reference to any detailed features of this coupling.

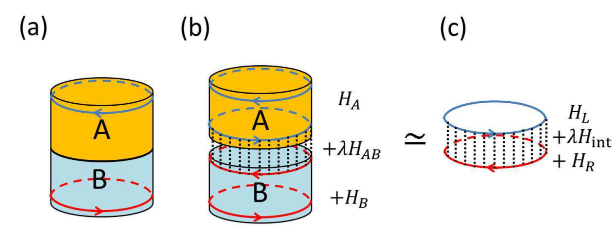

FIG. 1: (a) A topological state on a cylinder with a bipartition into two regions A and B. (b) The deformed system (see text) with the coupling between A and B regions weighted by a factor $\lambda \in [0, 1]$. The system can be understood as two cylinders A and B, with edge states propagating along the boundary between A and B, coupled by an inter-edge coupling. (c) For small enough λ, the coupling between the gapped bulk states can be neglected, and the problem can be reduced to an inter-edge coupling problem described by a $(1+1)$-dimensional conformal field theory with a (RG-) relevant coupling $\lambda H_{\rm int}$.

In order to understand the entanglement properties of the state $|G\rangle$, we relate them to another problem –

the "quantum quench" problem. Consider a "quantum quench" of the system composed of the coupled edges, Eq. (3). For all times $t < 0$ the system is in the ground state $|G\rangle$ of the Hamiltonian $H_{\text{edge}}(\lambda_0)$ with non-vanishing coupling $\lambda_0 \neq 0$ between the edges. At time $t = 0$ the coupling λ_0 between the edges is suddenly switched off, so that $\lambda = 0$ for $t \geq 0$. After the quantum quench, the left and right moving edge states evolve independently with the Hamiltonian $H_{\text{edge}}(\lambda = 0) = H_L + H_R$ of the decoupled edges. Space- and time-dependent correlation functions after a sudden quench, as above, have been studied extensively by Calabrese and Cardy[25, 26], who applied BCFT to obtain general properties of such correlation function in the long-time and long-wavelength regime. This is relevant for our purpose because the density matrix ρ_L is uniquely determined by the set of all equal-time correlation functions of operators with support solely on the L-moving edge,

$$\begin{aligned} &C(t, \{x_i\}) = \\ &\langle G| e^{it(H_L+H_R)} \hat{O}_{L,1}(x_1)...\hat{O}_{L,n}(x_n) e^{-it(H_L+H_R)} |G\rangle \equiv \\ &\equiv \mathrm{Tr}_L \left[e^{-itH_L} \rho_L e^{itH_L} \hat{O}_{L,1}(x_1)...\hat{O}_{L,n}(x_n) \right]. \end{aligned} \tag{5}$$

(All the coordinates $x_1, x_2, ...x_n$ reside entirely on the L-moving edge.) In the quantum quench problem, the ground state $|G\rangle$ of the coupled edge Hamiltonian $H_{\text{edge}}(\lambda_0 \neq 0)$ represents an initial condition at time $t = 0$ for the evolution with the gapless (critical) decoupled edge system Hamiltonian $H_L + H_R$ at subsequent times $t > 0$. This initial state can be viewed[25, 26, 28] as a boundary condition on the gapless theory of the right and left moving edges.[41] It can thus be described using the methods of boundary critical phenomena[29]. Moreover, in the present case of a one-dimensional edge, the resulting boundary condition can be analyzed by using the powerful tools of BCFT[27]. The key result that we will use from the theory of boundary critical phenomena is that an *arbitrary* boundary condition on a gapless bulk theory will *always* renormalize at long distances into a scale invariant boundary condition[25, 30, 31]. In a (1+1) conformal bulk theory, such as the one describing the one-dimensional edges, any scale invariant boundary condition must be one of a known list of conformally invariant boundary conditions[27]. Consequently, as emphasized in [25], in the long wavelength limit the correlation functions at a general boundary condition described by a general state $|G\rangle$ are equal to those at a conformally invariant boundary condition described by a state $|G_*\rangle$ which represents a (boundary) fixed point to which the boundary state $|G\rangle$ flows under the renormalization group (RG). The difference between $|G\rangle$ and $|G_*\rangle$ can be represented by an imaginary time evolution operator,

$$|G\rangle \simeq Z^{-1/2} e^{-\tau_0(H_L+H_R)} |G_*\rangle \tag{6}$$

where $\tau_0 > 0$ is the so-called extrapolation length[25, 29] and stands for the RG "distance" of the general boundary state $|G\rangle$ to the conformal boundary state $|G_*\rangle$. $Z = \langle G_*| e^{-2\tau_0(H_L+H_R)} |G_*\rangle$ is a nomalization factor. Physically, the energy scale $1/\tau_0$ is determined by the energy gap $E(\lambda_0)$ induced by the coupling term $\lambda_0 H_{\text{int}}$ between the edges.

In a so-called rational CFT[32] such as the one under consideration, all conformal invariant boundary states $|G_*\rangle$ are known[27] to be *finite* linear combinations of so-called Ishibashi states[33] which have the form

$$|G_{*,a}\rangle = \sum_{n=0}^{\infty} \sum_{j=1}^{d_a(n)} |k(a,n), j; a\rangle_L \otimes |-k(a,n), j; \bar{a}\rangle_R. \tag{7}$$

Here a denotes a topological sector in the underlying topological theory, *i.e.* a topological flux threading the cylinder in Fig. (1(a)), which is represented in the CFT describing the edges by a primary state of a corresponding conformal symmetry algebra (Virasoro or other) of conformal weight h_a. ($\bar{a}$ denotes the conjugate sector and state of conformal weight $h_{\bar{a}} = h_a$.)

The label a runs over all possible particle types of the topological state [32]. Here $k(a,n) = 2\pi(h_a + n)/l$ denotes the momentum, where l is the circumference of the cylinder; $j = 1, 2, ..., d_a(n)$ labels the elements of an orthonormal basis in the subspace of fixed momentum $k(a,n)$. Notice that the L- (R-) moving edge system only contains excitations with positive (negative) momentum. We note that the state in Eq. (7) is an example of a so-called *maximally entangled* state. The explicit form, Eq. (7), of the Ishibashi states $|G_{*,a}\rangle$, resulting from conformal invariance, is of great help in determining the form of the reduced density matrix ρ_L for the L-moving edge. Upon directly combining Eqs. (6) with (7) one obtains

$$\begin{aligned} &|G_a\rangle \simeq \\ &\simeq \sum_{n=0}^{\infty} \frac{e^{-2\tau_0 vk(a,n)}}{Z_a^{1/2}} \sum_{j=1}^{d_a(n)} |k(a,n), j; a\rangle_L \otimes |-k(a,n), j; \bar{a}\rangle_R \end{aligned}$$

which yields the following form of the density matrix of the L-moving edge upon tracing out the R-moving edge,

$$\begin{aligned} \rho_{La} &= \mathrm{Tr}_R \left(|G_a\rangle \langle G_a| \right) \simeq \\ &\simeq \sum_{n=0}^{\infty} \frac{e^{-4\tau_0 vk(a,n)}}{Z_a} \sum_{j=1}^{d_a(n)} |k(a,n), j; a\rangle_L \langle k(a,n), j; a|_L \\ &= Z_a^{-1} \hat{P}_a \; e^{-4\tau_0 H_L} \hat{P}_a \end{aligned} \tag{8}$$

Here we have used the linear dispersion $H_L |k, j; a\rangle_L = vk |k, j; a\rangle_L$, $H_R |-k, j; \bar{a}\rangle_L = vk |-k, j; \bar{a}\rangle_R$ where v is the edge state velocity and k stands for $k(a,n)$. The label a indicates that ρ_{La} is an operator defined in the topological sector corresponding to topological flux a threading the cylinder, and $\hat{P}_a$ is the projection operator onto that sector of the Hilbert space of the CFT. In cylinder geometry there is no entanglement between different topological sectors (denoted by different labels a).

Eq. (8) is the central result of this work, which demonstrates that the entanglement between L-moving and R-moving edge states in a CFT induced by a relevant coupling is always characterized by a "thermal" density matrix within a fixed topological sector (or primary state in CFT). In other words, in each topological sector the "entanglement Hamiltonian" $H_{\rm E} = -\log\rho_L = 4\tau_0 H_L + \log Z$ is proportional to the Hamiltonian H_L of a physical edge up to a possible shift of the ground state energy in that sector which ensures the proper normalization of the density matrix as a probability distribution. Our result demonstrates not only that the excitation energies of the entanglement spectrum are the same as those of the spectrum of the Hamiltonian of the edge state of the topological system appearing (by assumption) at a physical boundary of region A, in the long-wavelength limit modulo a global rescaling, but also that the most entangled states are in one-to-one correspondence with the low-energy edge states which occur at this boundary.

Example: Free fermions– A simple example in which the general notions, developed in the preceeding part of this article, can also be illustrated using elementary many-body techniques is that of the 2D integer quantum Hall (IQH) state. This state can be described by a free fermion theory, the entanglement properties of which have been studied extensively in the literature[15, 16, 34]. However, it is still helpful to present the results here as an illustration, in the language of the much more general formulation obtained above. The edge states of an IQH state with integer filling fraction $\nu = N$ consist of N flavors of non-interacting chiral fermions. For simplicity, we consider an IQH state with filling fraction $N = 1$, whose edge state dynamics is governed by the Hamiltonian

$$H_L = \sum_k vkc_k^\dagger c_k,\ \ H_R = -\sum_k vkd_k^\dagger d_k \tag{9}$$

The simplest inter-edge coupling term is a single-particle inter-edge tunneling

$$H_{\rm int} = E_{\rm g}\sum_k\left(c_k^\dagger d_k + d_k^\dagger c_k\right) \tag{10}$$

with $E_{\rm g}$ the bulk gap which acts as a high-energy cutoff scale for the edge theory. The coupled Hamiltonian $H_L + H_R + \lambda H_{\rm int}$ is a free Fermion Hamiltonian which can be diagonalized by a unitary transformation to $H_L + H_R + H_{\rm int} = \sum_{k,s=\pm1} E_k\gamma_{ks}^\dagger\gamma_{ks}$ with the gapful energy dispersion $E_k = \sqrt{v^2k^2 + E_{\rm g}^2}$. Here $\gamma_{k,i}$ $(i = 1, 2)$ are quasiparticle annihilation operators. The ground state $|G\rangle$ of this gapped system is determined by the conditions $\gamma_{k,i}|G\rangle = 0$ $(i = 1, 2)$. One obtains[24] the following explicit expression for $|G\rangle$ (unnormalized):

$$\begin{aligned}|G\rangle &= e^{-H_{\rm e}}|G_*\rangle \qquad (11)\\ |G_*\rangle &= \exp\left\{-\sum_{k>0}\left(c_k^\dagger d_k + d_{-k}^\dagger c_{-k}\right)\right\}|G_L\rangle\otimes|G_R\rangle\,.\end{aligned}$$

and $H_e \simeq \frac{1}{2E_{\rm g}}(H_L + H_R)$ in the long wavelength limit. The operators $c_k^\dagger d_k$ and $d_{-k}^\dagger c_{-k}$ with $k > 0$ create quasiparticle excitations of the system of the two edges, so that $|G_*\rangle$ is an equal-weight superposition of all quasiparticle excitation states in the massless theory; this is nothing but the Ishibashi state for the Free fermion CFT (in the sector without topological flux). Thus, with this form of H_e, we recover correctly (in the long wavelength limit) the general relation (6); the extrapolation length is $\tau_0 = 1/2E_{\rm g}$. As expected, the energy scale $1/\tau_0$ is determined by the energy gap $2E_{\rm g}$ of particle-hole excitations.

*Discussion.–*We now briefly discuss the situation with λ an (RG-) irrelevant coupling by considering the example of a Laughlin $1/m$ state whose edge theory is described by a Luttinger liquid[35],

$$\begin{aligned}\mathcal{L} &= \frac{m}{2\pi}\left(\partial_t - v\partial_x\right)\phi_L\partial_x\phi_L + \frac{m}{2\pi}\left(-\partial_t - v\partial_x\right)\phi_R\partial_x\phi_R\\ &\quad + \lambda\cos\left[\frac{1}{R}(\phi_L - \phi_R)\right] \qquad (12)\end{aligned}$$

with λ the inter-edge tunneling. The electron tunneling corresponds to $R = 1$ which is irrelevant. Therefore an infinitesimal λ does not open a gap. A gapped state can be induced by a sufficiently large $\lambda > \lambda_c$. However, as is well-known, a marginal coupling term $g\partial_\mu\phi_L\partial^\mu\phi_R$ can be added which can tune the scaling dimension of the electron tunneling λ until it becomes relevant at some g_c. Our earlier argument applies to $g > g_c$, in which case the entanglement spectrum was shown to be that of a chiral Luttinger liquid. Since tuning of g preserves the gap of the pair of (1+1)D edge states, the entanglement spectrum at $g = 0$ must be adiabatically connected to that at $g > g_c$, which means that it must also be a chiral Luttinger liquid. More details are discussed in the supplementary material[24].

Our analysis also applies to other systems described by coupled CFTs, besides topological states. In particular, it provides an explanation of the recent numerical and analytical results on the entanglement spectrum of coupled spin chains[36]. Moreover, it may be interesting to try to apply our approach to the quantum quench problem and the dynamics of $S_{\rm topo}$ in topologically ordered systems[37]. Finally, since the relationship between a general boundary state and a scale invariant boundary condition which is the endpoint of the RG flow also holds for higher dimensional scale invariant bulk theories[29], we expect that our result will generalize to higher dimensional topological states, such as $(3+1)$ dimensional topological insulators, and especially the fractional topological insulators[38] which cannot be analyzed using free fermion methods[15, 16]. Details of this generalization will be left for future work.

In closing, we would like to note that the reduced density matrix (8) in the topological sector "a" yields an entanglement entropy of the form $S = -{\rm Tr}\left(\rho_L\log\rho_L\right) =$

$\alpha L - S_{\rm topo}$, with $S_{\rm topo} = \log\left(D/d_a\right)$ the topological entanglement entropy[8, 9]. Here d_a is the quantum dimension of the quasi-particle of type "a", and $D = \sqrt{\sum_a d_a^2}$ is the total quantum dimension. This relation to the topological entropy has been noticed in Ref. [9], though in that work the form of the density matrix as in our Eq. (8) was taken as an assumption.[42] The present paper proves this assumption.

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[1] X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).

[2] *The Quantum Hall Effect.* edited by R. E. Prange and S. M. Girvin (Springer-Verlag, 1987).

[3] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).

[4] X.-L. Qi and S.-C. Zhang, *Topological insulators and superconductors*, e-print arXiv:1008.2026 (2010).

[5] J. E. Moore, Nature Phys. **5**, 378 (2009).

[6] For more generic topological insulators, see also X. L. Qi, T. L. Hughes and S. C. Zhang, *Phys. Rev. B*, **78**, 195424 (2008); A. Schnyder *et al*, *ibid*, **78**, 195125 (2008); A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).

[7] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).

[8] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).

[9] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).

[10] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).

[11] A. Peres, Phys. Lett. A **202**, 16 (1995).

[12] A. M. Läuchli, E. J. Bergholtz, J. Suorsa, and M. Haque, Phys. Rev. Lett. **104**, 156404 (2010).

[13] N. Regnault, B. A. Bernevig, and F. D. M. Haldane, Phys. Rev. Lett. **103**, 016801 (2009).

[14] R. Thomale, A. Sterdyniak, N. Regnault, and B. A. Bernevig, Phys. Rev. Lett. **104**, 180502 (2010).

[15] A. M. Turner, Y. Zhang, and A. Vishwanath, Phys. Rev. B **82**, 241102 (2010).

[16] L. Fidkowski, Phys. Rev. Lett. **104**, 130502 (2010).

[17] H. Yao and X.-L. Qi, Phys. Rev. Lett. **105**, 080501 (2010).

[18] S.-C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[19] B. Blok and X. G. Wen, Phys. Rev. B **42**, 8133, *ibid.* 8145 (1990).

[20] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[21] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).

[22] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

[23] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B **74**, 045125 (2006).

[24] Supplementary on-line material of the current article.

[25] P. Calabrese and J. Cardy, Phys. Rev. Lett. **96**, 136801 (2006).

[26] P. Calabrese and J. Cardy, J. Stat. Mech. p. 06008 (2007).

[27] J. L. Cardy, Nucl. Phys. **B324**, 581 (1989).

[28] K. Symanzik, Nucl. Phys. B **190**, 1 (1981).

[29] H.W. Diehl, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1986), Vol. 10.

[30] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B **360**, 641 (1991).

[31] see e.g. A. W. W. Ludwig, Int. J. Mod. Phys. B **8**, 347 (1994).

[32] G. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).

[33] N. Ishibashi, Mod. Phys. Lett. **A4**, 251 (1989).

[34] I. Peschel, J. Phys. A **36**, L205 (2003).

[35] X.-G. Wen, Advances in Physics **44**, 405 (1995).

[36] D. Poilblanc, Phys. Rev. Lett. **105**, 077202 (2010).

[37] D. I. Tsomokos, A. Hamma, W. Zhang, S. Haas, and R. Fazio, Phys. Rev. A **80**, 060302 (2009).

[38] B. A. Bernevig and S. C. Zhang, *Phys. Rev. Lett.* **96**, 106802 (2006); M. Levin and A. Stern, *ibid.* **103**, 196803 (2009); J. Maciejko, *et al*, *ibid.* 105, 246809 (2010); B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, e-print arXiv:1005.1076 (2010); G. Y. Cho and J. E. Moore, e-print arXiv:1011.3485.

[39] I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991).

[40] After our work was completed and while it was being written up, we became aware of the preprint arXiv:1102.2218 in which general analytical arguments were presented for the mentioned correspondence in a large class of fractional quantum Hall states. The methods used in both works are entirely different.

[41] This can be achieved in any dimension of space by analytical continuation of the real-time Keldysh contour to imaginary time, and subsequent exchange of the roles of space and imaginary time (possible due to the underlying effective relativistic invariance of the low-energy edge state theory).

[42] As pointed out in Ref. [9], the topological entropy is closely related mathematically to the '*Boundary Entropy*' of Ref. [39], as both are determined in the same fashion by the 'high-energy' spectrum through a modular transformation.