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Oleg A. Godin*

CIRES, University of Colorado and NOAA Earth System Research Laboratory, Physical Sciences Division, Boulder, Colorado 80305, USA

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We consider linear waves in compressible fluids in a uniform potential field, such as a gravity field, and demonstrate that a particular type of wave motion, in which pressure remains constant in each fluid parcel, is supported by inhomogeneous fluids occupying bounded or unbounded domains. We present elementary, exact solutions of linearized hydrodynamics equations, which describe the new type of waves in the coupled ocean-atmosphere system. The solutions provide an extension of surface gravity waves in an incompressible fluid half-space with a free boundary to waves in compressible, three-dimensionally inhomogeneous, rotating fluids.

Waves in compressible fluids subject to external potential forces are encountered in many physical systems, ranging from trapped quantum gases [1–4] to stars and planet atmospheres [5, 6]. Rather detailed studies have been done in the geophysical context, where external forces are due to the Earth's gravity field. In the ocean and atmosphere, mechanical waves occur at scales from less than a centimeter to thousands of kilometers [7–9], play a key role in the transfer of energy and momentum within and between the ocean and the atmosphere [7, 8, 10], and to a large degree control the weather and climate [11–13]. Exact solutions of idealized hydrodynamic problems, known as the Rossby, Kelvin, Lamb, and Poincaré waves, elucidate the effects of the fluid's buoyancy and compressibility, the Earth's rotation, as well as topography and bathymetry, on the wave processes, and provide much of the conceptual foundation of modern geophysical fluid dynamics [7–10, 14]. However, these solutions do not encompass the actual diversity of wave motions. An additional, distinct wave type is discussed in this paper. Here we present new, elementary, exact solutions of linearized hydrodynamics equations in a compressible fluid in a uniform gravity field. The solutions describe waves, in which pressure remains constant in each moving fluid parcel, in three-dimensionally inhomogeneous fluids in bounded or unbounded domains with or without rotation. In addition to the Earth's atmosphere and oceans, the incompressible wave motion is likely to be a component of wave fields in the atmosphere and interior of stars [5, 6], as well as in planetary atmospheres and, on smaller scales, in trapped gases [1–4]. Identification of the new wave type advances physical intuition about acoustic-gravity waves and the dynamics of the coupled ocean-atmosphere system and furnishes new benchmark problems to verify numerical models of geophysical fluid dynamics.

Consider continuous small-amplitude waves in a fluid with background (i.e., unperturbed by waves) pressure p_0 and density ρ_0 in a uniform gravity field with acceleration $\mathbf{g}$. The fluid is stationary and motionless in the absence of waves, and the background pressure and density are related by $\nabla p_0 = \rho_0 \mathbf{g}$. Linearization of the Euler, continuity, and state equations with respect to wave amplitude leads to the following set of equations [15, 16] governing wave fields:

$$\nabla p - \omega^2 \rho_0 \mathbf{w} + \left(\mathbf{w} \cdot \nabla \rho_0\right) \mathbf{g} - c_0^{-2} \left(p + \mathbf{w} \cdot \nabla p_0\right) \mathbf{g} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{w} + \left(p + \mathbf{w} \cdot \nabla p_0\right) / \rho_0 c_0^2 = 0, \tag{2}$$

where p and $\mathbf{w}$ are the pressure perturbation and fluid particle displacement due to the wave, ω is wave frequency, and c_0 is the sound speed. Fluid velocity $\mathbf{v} = -i\omega\mathbf{w}$. Time dependence $\exp(-i\omega t)$ of the wave field is assumed and suppressed. In Eqs. (1) and (2), we assume wave propagation to be an adiabatic thermodynamic process and disregard irreversible processes associated with viscosity, thermal conductivity, and diffusion of admixtures such as salt in sea water and water vapor in atmospheric air. This is the standard framework for analysis of acoustic-gravity waves in ocean and atmosphere [7–9, 14].

The governing equations (1) and (2) are supplemented by boundary conditions. On a fluid-fluid interface, the linearized boundary conditions [15, 16] consist in continuity of the normal displacement and the quantity $p + \mathbf{w} \cdot \nabla p_0$. The latter has the meaning of the Lagrangian pressure perturbation, i.e., the pressure perturbation in a moving fluid particle, as opposed to the Eulerian pressure perturbation p at a fixed point in space. Only one boundary condition is imposed on a free surface:

$$p + \mathbf{w} \cdot \nabla p_0 = 0. \tag{3}$$

The physical meaning of the boundary condition (3) is that the total pressure remains constant in the fluid particles located on the free surface [16].

Without making any additional assumptions about the propagation medium, let us consider a special kind of fluid motion, in which there are no pressure perturbations in any fluid particles, i.e., Eq. (3) holds throughout the fluid. For waves of this kind, conditions on free boundaries, if any, are met automatically. At fluid-fluid interfaces, only the kinematic condition of the normal displacement continuity needs to be imposed. The governing equations (1) and (2) become

$$\nabla p - \omega^2 \rho_0 \mathbf{w} + (\mathbf{w} \cdot \nabla) \nabla p_0 = 0, \quad \nabla \cdot \mathbf{w} = 0. \tag{4}$$

According to Eq. (4), divergence of the particle displacement and, hence, velocity, equals zero, i.e., we are dealing with an incompressible motion of a compressible fluid. This, of course, is expected as there are no pressure changes in fluid particles.

Introduce a Cartesian coordinate system with horizontal coordinates x and y and vertical coordinate z increasing upward. Then $\mathbf{g} = (0, 0, -g)$. Application of the differential operator curl to both sides of the static equilibrium equation $\nabla p_0 = \rho_0 \mathbf{g}$ shows that the background density is horizontally stratified: $\rho = \rho(z)$, while the sound speed c can be a function of x, y, and z. From Eqs. (3) and (4) we find

$$p = \rho_0 g w_3, \;\; \mathbf{w}_h \equiv (w_1, w_2, 0) = k^{-1} \nabla_h w_3, \;\; w_3 = W(x, y)\exp(kz), \tag{5}$$

where $\nabla_h = (\partial/\partial x, \partial/\partial y, 0)$ $k = \omega^2/g$, and W is a solution of the two-dimensional (2-D) Helmholtz equation:

$$\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 + k^2 W = 0. \tag{6}$$

Note that the fluid motion described by Eq. (5) is irrotational, with $-ip/\rho\omega$ being the velocity potential. Lamb[8] considered linear acoustic-gravity waves in a vertically stratified perfect gas and concluded, erroneously, that irrotational motion is impossible unless $dc_0^2/dz = -(\gamma - 1)g$, where γ is the constant ratio of specific heats (see pp. 547–548 in Ref. 14). Lamb failed to recognize that a trivial solution of his Eq. (11) can correspond to a non-trivial wave motion, i.e., that a non-trivial wave motion can be simultaneously irrotational (curl $\mathbf{w} = 0$) and incompressible ($\nabla \cdot \mathbf{w} = 0$). It is straightforward to check that our solution (5) satisfies Lamb's Eq. (11) with arbitrary stratification of the sound speed. Using the terminology employed in the theory of mechanical waves in elastic media (i.e., solids), the wave (5) corresponds to the deformation of pure shear in the medium, and the wave should be called a "shear wave." The "shear wave" (5) exists despite the absence of shear rigidity in the inviscid fluids we consider. Instead, the restoring force is provided by the gravity.

The sound speed does not enter Eqs. (5) and (6), and the density can be an arbitrary piecewise continuous function of z. This should be compared to the other known analytic solution for acoustic-gravity waves in inhomogeneous fluids, the Lamb wave, which assumes constant sound speed and exponential stratification of the density [7, 14].

Horizontal and vertical components of the displacement $\mathbf{w}$ (5) share the same exponential dependence on the vertical coordinate. Despite the exponential increase in the displacement amplitude with z, the pressure (5) as well as the power flux $p\mathbf{v}$ and wave energy densities [16] decrease with z when ρ_0 decreases with z sufficiently rapidly. The solution (5), (6) applies equally to unbounded fluid as well as to fluid limited from above and/or from below by pressure-release surface(s). In a fluid with piecewise continuous parameters and horizontal (in the absence of the wave) interfaces, horizontal $\mathbf{w}_h$ and vertical w_3 components of the particle displacement are continuous and still given by Eq. (5); pressure perturbations p are discontinuous at interfaces where density is discontinuous (Fig. 1). According to Eq. (5), surfaces of constant pressure coincide with surfaces of constant density in the wave. It should be emphasized that, as follows from Eqs. (3) and (5), waves with no pressure variations in fluid particles do not exist when a medium has a horizontal boundary other than a free surface.

Equation (6) is satisfied by a superposition of 2-D plane waves $W(x, y) = \exp(i\mathbf{q} \cdot \mathbf{r})$, $\mathbf{q} = k(\cos\varphi, \sin\varphi, 0)$ with arbitrary angular spectrum. In addition to homogeneous plane waves, for which the horizontal wave vector $\mathbf{q}$ is real, Eq. (6) is satisfied by inhomogeneous plane waves, for which $\mathbf{q} = \mathbf{q}_r + i\mathbf{q}_i$ has the real $\mathbf{q}_r$ and imaginary $\mathbf{q}_i$ parts. Dispersion relations for homogeneous and inhomogeneous plane waves are, respectively, $gq = \omega^2$ and

$$\mathbf{q}_r \cdot \mathbf{q}_i = 0, \quad q_r^2 - q_i^2 = k^2. \tag{7}$$

In plane waves, $\mathbf{w}_h = k^{-1}(i\mathbf{q}_r - \mathbf{q}_i)w_3$. Hence, particles move along circles, which lie in planes parallel to $\mathbf{q}_r$ and making an angle arctan(q_i /k) with the vertical plane; radii of the circles increase exponentially with z (Fig. 2). Horizontal displacements along $\mathbf{q}_i$ are in phase, and those along the perpendicular direction $\mathbf{q}_r$ are a quarter-period out of phase with the vertical displacement, while the Eulerian pressure variations p are always in phase with w_3. In a particular case, when W is a plane wave and fluid with the sound speed $c \to\infty$ occupies half-space $z < 0$ with a pressure-release boundary, Eq. (5) reduces to the known solution (see Sec. 40.1 in Ref. 9) for surface gravity waves in a stratified incompressible fluid.

When fluid occupies a bounded domain, vertical boundaries and interfaces have no effect on the vertical distribution of the field and impose conditions only on the function $W(x, y)$, which is a solution of Eq. (6). A number of boundary-value problems for Eq. (6) have been considered in the literature in an acoustic context [16] as well as in the context of gravity waves in incompressible fluid [8], including reflection from boundaries and interfaces as well as guided propagation in the horizontal plane. Explicit solutions of the boundary-value problem can be readily found for various types of boundary conditions and for various geometries (in the horizontal plane) of the boundaries and/or interfaces.

Consider waves in a fluid half-space $y < z \tan\theta$ with a plane rigid boundary $y = z\tan\theta,\ 0<\theta<\pi/2$ (Fig. 3). The vertical extent of the fluid can be either infinite or bounded from above (and/or from below) by a horizontal free surface. Since the normal displacement of fluid particles vanishes on a rigid surface, we have $w_2 = w_3 \tan\theta$ at the boundary. Solving this Eqs. (5) and (6), we find

$$W(x,y) = \left[B_1 \exp(ikx/\cos\theta) + B_2 \exp(-ikx/\cos\theta)\right]\exp(ky\tan\theta), \qquad (8)$$

where $B_{1,2}$ are arbitrary constants. Equations (5) and (8) give the particle displacement $\mathbf{w}$ as a superposition of two inhomogeneous, three-dimensional plane waves, which propagate horizontally with the phase speed $(g/\omega)\cos\theta$ and the group speed $(g/2\omega)\cos\theta$ in directions parallel to the sloping boundary. The relation between the real part of the horizontal wave vector and frequency is given by $\omega^2 = gq_r\cos\theta$ in agreement with Eq. (7). The amplitude of the displacement vector (and of the vertical displacement w_3) decreases exponentially, when an observation point moves downward parallel to the boundary, and remains constant, when the observation point moves in the direction normal to the boundary. The spatial distribution of pressure p (5) depends on the density stratification and generally is not a superposition of two three-dimensional plane waves. In every horizontal plane, we have two plane waves propagating along the boundary and exponentially attenuating with distance from the boundary.

Note that in a fluid with a sloping boundary, unlike unbounded fluid or fluid with a vertical boundary, the waves (5) cannot propagate in an arbitrary horizontal direction. In an incompressible fluid of constant density with a horizontal free surface, our result (5), (8) reduces to Stokes' solution for an edge wave along a sloping beach [17].

Now, consider waves in a compressible fluid rotating along a vertical axis with angular velocity $\mathbf{\Omega} = (0, 0, f/2)$. f is referred to as the Coriolis parameter. To account for the Coriolis force acting on moving fluid particles in a rotating reference frame, the term $-2i\omega\mathbf{\Omega}\times\mathbf{w}$ should be added in the left side of Eq. (1). Then, for waves in which there are no Lagrangian pressure perturbations, from Eqs. (1)–(3) we find

$$p = \rho_0 g w_3,\ \ \mathbf{w}_h = \frac{\exp(kz)}{k\left(1 - f^2/\omega^2\right)}\left(\nabla_h W - \frac{2i}{\omega}\Omega\times\nabla_h W\right),\ \ w_3 = W(x,y)\exp(kz), \qquad (9)$$

where $W(x, y)$ is a solution of the Helmholtz equation (6) with k^2 replaced by $k^2(1–f^2/\omega^2)$. It is satisfied by an arbitrary superposition of homogeneous and inhomogeneous, 2-D plane waves $\exp(i\mathbf{q}\cdot\mathbf{r})$ provided

$$\mathbf{q}_r \cdot \mathbf{q}_i = 0, \quad q_r^2 - q_i^2 = \omega^2\left(\omega^2 - f^2\right)/g^2. \tag{10}$$

Real and imaginary parts of the wave vector (and, hence, the directions of the fastest variations of wave amplitude and phase in the horizontal plane) are orthogonal. Homogeneous plane waves () exist only when wave frequency exceeds the Coriolis parameter. Phase and group velocities of the wave are parallel to the horizontal wave vector **q** and have magnitudes $c_{ph} = g/\sqrt{\omega^2 - f^2}$ and $c_{gr} = g\sqrt{\omega^2 - f^2}/\left(2\omega^2 - f^2\right)$, respectively. When frequency increases from $|f|$ to infinity, the phase speed steadily decreases from infinity to zero; the group speed tends to zero, when frequency tends to f or infinity, and has a maximum $c_{gr} = 2^{-3/2} g/|f|$ at $\omega = |f|\sqrt{3/2}$. Unlike the homogeneous plane waves, the inhomogeneous waves exist at all frequencies. At $\omega > |f|$, the phase speed of the inhomogeneous waves is smaller than that of the homogeneous waves.

When $f \neq 0$, according to Eq. (9), motion in the wave remains incompressible $(\nabla \cdot \mathbf{w} = 0)$ but is no longer irrotational. In particular, the vertical component of the vorticity vector curl**v** is $-ikfw_3$. Equation (9) shows that the effects of fluid rotation on wave motion become negligible at frequencies $\omega >> |f|$, as expected.

Let a fluid occupy a half-space $y < 0$ with a rigid vertical boundary at $y = 0$. For a plane wave $W(x, y) = \exp(iq_1x + iq_2y)$ that satisfies the boundary condition at $y = 0$, according to Eq. (9) we have $q_2 = i\omega^{-1} f q_1$. Of physical interest are waves that remain finite in each horizontal plane. From Eqs. (9) and (10) we find the only solution of this kind:

$$w_3 = \text{const}.\exp\left(-ikx\,\mathrm{sgn}\,f + k|f|\,y/\omega + kz\right). \tag{11}$$

It describes a boundary (edge) wave that propagates along the vertical rigid wall in a direction that is determined uniquely by the geometry of the wall and the direction of rotation. A change of sign of the Coriolis parameter f (as occurs when moving from the Northern to the Southern Hemisphere) reverses the direction of propagation of the edge wave along the Ox axis. Note also that the edge wave propagates in opposite directions along western and eastern walls. The phase speed of the edge wave is independent of f and equals the phase speed of the free wave in the absence of vertical boundaries and rotation.

In many respects (such as dependence of the direction of propagation on the sign of the Coriolis parameter and geometry of the boundary, and the relation between the phase speeds of the edge and a respective free wave), the edge wave (11) is similar to the Kelvin waves. The Kelvin wave is an edge (boundary) wave propagating along a vertical rigid wall in shallow water (i.e., in a finite layer of incompressible fluid of constant density between horizontal free and rigid boundaries) [8]. The edge wave (11) is a "deep-water" counterpart of the Kelvin wave. It is unaffected by fluid compressibility, density stratification, and presence of horizontal free surface(s).

Consider a fluid half-space $y < z\tan\theta$ with a plane rigid boundary (Fig. 3). The vertical extent of the fluid can be either infinite or bounded from above (and/or from below) by a horizontal free surface. For a plane-wave solution $W(x, y) = \exp(iq_1x + iq_2y)$ that satisfies the condition $w_2 = w_3\tan\theta$ on the boundary, from Eqs. (9) and (10), we find

$$q_1 = \frac{k}{\cos\theta}\left(\pm 1 - \frac{f}{\omega}\sin\theta\right), \quad q_2 = \frac{-ik}{\cos\theta}\left(\sin\theta \mp \frac{f}{\omega}\right). \tag{12}$$

The dispersion equation of the edge waves (12) can be written as $gq_r \cos\theta = \omega|\omega \mp f \sin\theta|$. For the wave amplitude to remain finite in the horizontal plane, there should be Im $q_2 \le 0$. When $|f| < \omega \sin\theta$, both solutions (12) satisfy this requirement, and we have two distinct edge waves, which, as in the case of the sloping rigid boundary in a non-rotating fluid, propagate along the Ox coordinate axis in opposite directions along the boundary. When $|f| > \omega \sin\theta$, only one of horizontal wave vectors (12), namely,

$$q_1 = -\frac{k \operatorname{sgn} f}{\cos\theta}\left(1 + \frac{|f|}{\omega}\sin\theta\right), \quad q_2 = \frac{-ik}{\cos\theta}\left(\sin\theta + \frac{|f|}{\omega}\right), \tag{13}$$

satisfies the inequality Im $q_2 \le 0$. Then, there exists only one edge wave, with its direction of propagation determined by the direction of the fluid rotation. This is similar to what was found in the case of a vertical rigid wall in rotating fluid. In fact, at $\theta \to 0$, the solution described by Eq. (13) reduces to the solution (11) we obtained for the vertical wall.

In summary, incompressible wave motion, in which pressure and density remain constant in each moving fluid parcel, is found to be supported by inhomogeneous compressible fluids occupying either unbounded domains or domains with horizontal pressure-release surfaces and sloping rigid boundaries. Gravity is the restoring force in the incompressible wave motion. The waves are described by simple, exact solutions of linearized equations of hydrodynamics of inhomogeneous, compressible fluid in a uniform gravity field. The exact solutions are valid under surprisingly general assumptions about the environment and reduce to some classical wave types in appropriate limiting cases. Allowance for three-dimensional variation of the sound speed and for arbitrary density stratification, including density discontinuities, makes the exact solutions an attractive model of waves in a coupled ocean-atmosphere system.

Similar to the other analytical solutions employed in geophysical hydrodynamics such as the Rossby, Kelvin, Lamb, Poincaré, and Stokes waves, the body and edge waves described by Eqs. (5), (8), (9), and (12) are exact solutions of idealized problems, which only approximately represent the real ocean and atmosphere. Further research is required to investigate the effects of dissipation, nonlinearity, finite ocean depth, background currents and winds, variation of the Coriolis parameter, etc. on the waves discussed in this paper.

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* Electronic address: Oleg.Godin@noaa.gov

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Figure Captions

FIG. 1. (Color online) Vertical profiles of the background density ρ and amplitudes of wave-induced perturbations in the Eulerian pressure p and the vertical displacement w_3 of fluid parcels.

FIG. 2. (Color online) Surfaces of constant pressure and constant density in a homogeneous plane wave propagating along the x axis (solid lines) and in the absence of waves (dashed lines). Also shown are circular trajectories of fluid particles and the vector of their velocity at different phases of the wave. Large arrow shows the direction of wave propagation.

FIG. 3. (Color online) Sketch showing geometry of the edge wave problem. Arrow 1 shows the direction of propagation of an edge wave which exists at any frequency and for the arbitrary slope of the plane rigid surface. Arrow 2 shows the direction of propagation of an additional edge wave, which exists at $\omega > | f | \sin\theta$ in a rotating fluid and at any frequency in a non-rotating ($f = 0$) fluid. The propagation directions are shown for the Northern Hemisphere ($f > 0$) and are reversed in the Southern Hemisphere. Arrow 3 shows the direction of the exponential decrease of the amplitude of the fluid velocity. All three arrows are parallel to the rigid boundary.

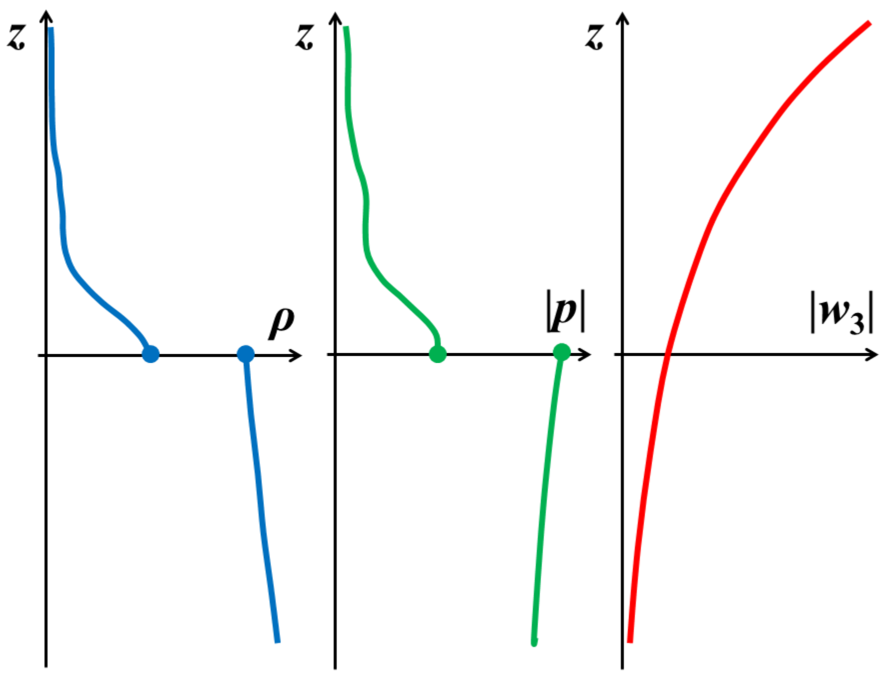

Figure 1.

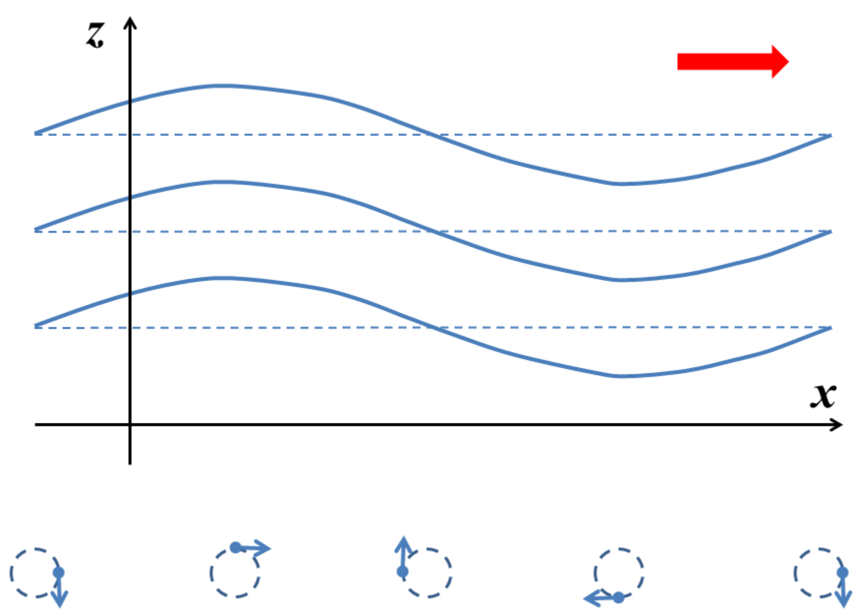

Figure 2.

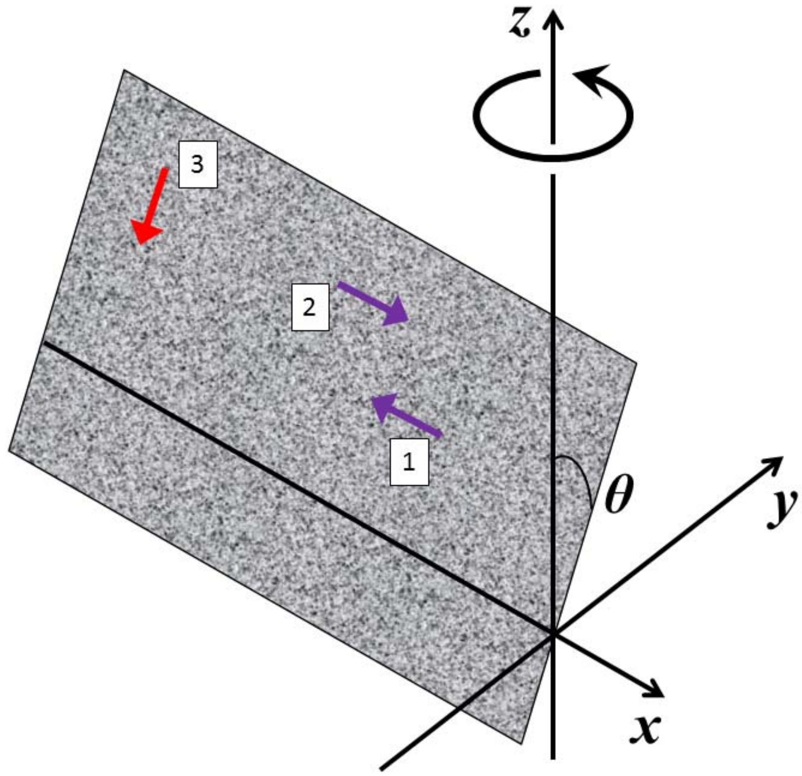

Figure 3.