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**Parametric Amplification of Laser-Driven Electron Acceleration in Underdense Plasma**

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A new mechanism is reported that increases electron energy gain from a laser beam of ultra-relativistic intensity in underdense plasma. The increase occurs when the laser produces an ion channel that confines accelerated electrons. The frequency of electron oscillations across the channel is strongly modulated by the laser beam, which causes parametric amplification of the oscillations and enhances the electron energy gain. This mechanism has a threshold determined by a product of beam intensity and ion density.

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of motion,
\[
\frac{dr}{dt} = \frac{c}{\gamma} P, \tag{1}
\]
\[
\frac{dP}{dt} = -\frac{|e| E}{m_e c} - \frac{|e|}{\gamma m_e c} P \times B, \tag{2}
\]
where \(r\) is the electron position, \(t\) is the time in the laboratory (ion) frame of reference, \(e\) and \(m_e\) are the electron charge and mass, \(c\) is the speed of light, \(P\) is the dimensionless electron momentum normalized to \(m_e c\), and \(\gamma = \sqrt{1 + P^2}\) is the relativistic factor.

The electric field, \(E = E_{\text{ion}} + E_{\text{wave}}\), is a sum of a static field of the ion space-charge, \(E_{\text{ion}}\), and an oscillating field of the wave, \(E_{\text{wave}}\). The static component in the uniform channel is \(E_{\text{ion}} = e_y \omega_{pl}^2 m_e y / |e|\), where \(e_y\) is a unit vector, \(\omega_{pl} = \sqrt{4\pi n_0 e^2 / m_e}\) is the plasma frequency, and \(n_0\) is the density of the singly charged ions in the channel. The magnetic field, \(B = B_{\text{wave}}\), is only due to the wave. It is convenient to express the wave field in terms of a dimensionless vector potential \(a\):
\[
E_{\text{wave}} = -\frac{m_e c}{|e|} \frac{\partial a}{\partial t}, \quad B_{\text{wave}} = \frac{m_e c^2}{|e|} \nabla \times a. \tag{3}
\]
The vector potential \(a\) depends on \(t\) and \(z\) only via a phase variable
\[
\xi \equiv (ct - z) / \lambda, \tag{4}
\]
where \(\lambda\) is the laser wavelength. We choose \(a = a(\xi) [e_x \cos \theta + e_y \sin \theta]\), where \(e_x\) and \(e_y\) are unit vectors, \(\theta\) is a polarization angle, and \(a(\xi)\) is a sinusoidal function with a slowly varying envelope. To be specific, we use the following expression for \(a(\xi)\) in our subsequent numerical calculations: \(a(\xi) = 0\) for \(\xi \leq 0\), \(a(\xi) = 0.5 \left[ 1 - \cos (\pi \lambda \xi / cT) \right] a_0 \sin (2\pi \xi / \psi)\) for \(0 < \xi < cT / \lambda\), and \(a(\xi) = a_0 \sin (2\pi \xi / \psi)\) for \(\xi \geq cT / \lambda\), where \(a_0\) is the maximum amplitude, \(\psi\) is an initial phase, and \(T \gg \lambda / c\) is the pulse rise time.

The equations of motion simplify when dimensionless proper time \(\tau\) (proper time normalized to \(\lambda / c\)) is used instead of \(t\). The relation between the two is
\[
dt/d\tau = \gamma \lambda / c. \tag{5}
\]
We also introduce the dimensionless displacement \(u\) across the ion channel and a dimensionless parameter \(K\), which is a ratio of the wavelength to the electron skin-depth:
\[
u \equiv \omega_{pl} y / c, \quad K \equiv \omega_{pl} \lambda / c. \tag{6}
\]
Note that \(K \ll 2\pi\) in a significantly underdense plasma, which is the case of our primary interest here. Substituting the expressions for \(\mathbf{E}\) and \(\mathbf{B}\) into Eq. (2), we obtain the following closed set of equations:
\[
\frac{d}{d\tau} [P_y - a \sin \theta] = -\gamma K u, \tag{7}
\]
\[
\frac{dP_z}{d\tau} = [a \cos^2 \theta + P_y \sin \theta] \frac{da}{d\xi}, \tag{8}
\]
\[
\frac{d\xi}{d\tau} = \gamma - P_z, \tag{9}
\]
\[
\frac{du}{d\tau} = KP_y, \tag{10}
\]
where \(\gamma = \sqrt{1 + P_y^2 + P_z^2 + a^2 \cos^2 \theta}\). Equation (9) is a derivative of Eq. (4) with respect to \(\tau\), with Eq. (5) taken into account. It follows from Eqs. (7) - (10) that there is an integral of motion for the electron:
\[
\gamma - P_z + a^2 / 2 = C, \tag{11}
\]
where \(C\) is a constant determined by initial conditions. This integral of motion sets an upper limit for the amplitude of electron oscillations across the ion channel, which is \(u_{\text{max}} = \sqrt{2 + u(0)^2}\) for an electron that is initially at rest \((C = 1)\). The amplitude approaches \(u_{\text{max}}\) at very large values of \(P_z\) when \(\gamma - P_z \to 0\).

Equations (7) - (10) reproduce a well-known result that, in the absence of ions, the maximum \(\gamma\)-factor for an electron that is initially at rest is
\[
\gamma_{\text{vac}} = 1 + a_0^2 / 2. \tag{12}
\]
The electron moves axially slower than the wave and is therefore subject to dephasing, which limits electron energy gain from the wave. In the absence of ions, the dimensionless dephasing rate, \(\gamma - P_z\), is constant and equal to unity. The ion channel can reduce this dephasing rate significantly by means of electron oscillations in the static field across the channel. Indeed, consider an electron that is initially at rest, but slightly displaced from the axis of the channel. The constant \(C\) in Eq. (11)
driving force. However, the conditions for instability become much more favorable at ultra-relativistic intensities.

In the limit of low densities \((K \ll 1)\) and ultra-relativistic intensities \((a_0 \gg 1)\), the \(K^2\)-terms that do not involve \(a^2\) can be neglected in Eq. (13). We also set \(a(\xi) = a_0 \sin(2\pi \xi + \psi)\), implying that \(\xi > cT/\lambda\), which simplifies Eq. (13) to an equation,

\[
u'' - \frac{uu'''}{2(C - u^2/2)} + \frac{K^2 u}{2(C - u^2/2)^3} + \frac{K^2 u}{2(C - u^2/2)} + \frac{K^2 a^2 \cos^2 \theta u}{2(C - u^2/2)^3} = \frac{\sin \theta}{C - u^2/2} K \alpha',
\]

where the prime denotes \(d/d\xi\). This equation generalizes Eq. (11) of Ref. [14] to the case of arbitrary polarization. The two equations are identical when \(\theta = \pi/2\), except for notations. We use phase \(\xi\) rather than time \(t\) as an independent variable, which helps to distinguish between resonant excitation of oscillations by an external force (betatron resonance) and parametric amplification of oscillations.

For small displacements, Eq. (13) reduces to a linear equation for a driven oscillator with a modulated eigenfrequency, \(u'' + K^2 (1 + \cos^2 \theta a^2/2) u = \sin \theta K \alpha'\). In the case of a non-relativistic wave \((a_0 \ll 1)\), the oscillations across the channel are stable in an underdense plasma \((K \ll 1)\). The reason is that the eigenfrequency, which is close to \(K\) for \(a_0 \ll 1\), is significantly lower than the frequency of the modulations and the frequency of the driving force. However, the conditions for instability become much more favorable at ultra-relativistic intensities.

In what follows, we set \(C = 1\) in Eq. (14), which implies that \(P_2(0) = 0\). The results for \(C = 1\) can be rescaled to the case of an arbitrary \(C\) by replacing \(u\) with \(C^{-1/2} u\) and \(\kappa\) with \(C^{-3/2} \kappa\). This rescaling generalizes the results to the case of an arbitrary initial \(P_2\) via Eq. (11).

The case of small polarization angles offers further simplifications, because the driving term on the right hand side of Eq. (14) is small. It is instructive to consider a linearized version of Eq. (14),

\[
u'' + \frac{1}{2} \kappa^2 \sin^2 (2\pi \xi + \psi) u = 2 \pi \theta \kappa \cos (2\pi \xi + \psi).
\]

In contrast to the non-relativistic case, the eigenfrequency is now strongly modulated and, as a result, its maxima increase with \(a_0\) for a given ion density (fixed \(K\)). Solutions of Eq. (16) are stable for \(\kappa < 1\), but become unstable when \(\kappa\) is increased. The instability is caused by eigenfrequency modulations because it develops even at \(\theta = 0\) when there is no driving force. We shall therefore use the term parametric to refer to this instability, with the changing parameter being the eigenfrequency. At \(\theta = 0\), Eq. (16) becomes a Mathieu differential equation that has exponentially growing solutions when \(\kappa\) exceeds a certain threshold value \(\kappa_\ast\). We find that \(\kappa_\ast \approx 10.2\). The nonlinearity retained in Eq. (14) plays a destabilizing role and it formally allows the amplitude to reach \(u_{\text{max}}\), causing unlimited energy gain. This observation suggests that the energy gain in low-density plasma depends on \(K\) as well as on \(\kappa\) and that it needs to be determined from Eqs. (7) - (10).

The solution of Eqs. (7) - (10) for \(\kappa = 12\) is shown in Fig. 1. The instability that develops in this case \((\kappa > \kappa_\ast)\) causes the oscillations to grow significantly. The increase in their amplitude leads to increased axial acceleration, such that the maximum \(\gamma\)-factor \((\gamma_{\text{max}})\) is greater than the maximum value achieved in the absence of ions \(\gamma_{\text{vac}}\) of Eq. (12)]. Figure 2 shows \(\gamma_{\text{max}}\) as a function of \(a_0\) and \(\kappa\). At \(a_0 > 10\), the stability boundary is indeed determined only by \(\kappa\) since the threshold coincides with the
caused by a strong modulation of the electron oscillation netic field of the wave. The instability develops when
\[ \delta u \]
damping (the coefficients are determined by the stable unper-
turbated solution just below the threshold. The eigenfrequency is
again strongly modulated. We deliberately neglect the force term in the linearized equation and we find that a slight variation of coefficients in the linearized equation (the coefficients are determined by the stable unperturbed solution) leads to exponential growth of \( \delta u \). Such behavior is a signature of a parametric instability and it clearly distinguishes the parametric instability caused by modulations of the eigenfrequency from a resonance with the driving force. Figure 3 shows the unstable oscillations and the resulting maximum value of \( \gamma \) for \( \theta = \pi / 2 \) and \( \kappa = 5 > \kappa_\ast (\pi / 2) \). The maximum value of \( \gamma \) is almost 8 times greater than \( \gamma_{\text{vac}} \) in this strongly nonlinear regime.

In summary, we have shown how parametric amplification of electron oscillations in the ion channel can enhance electron energy gain from a laser beam of ultra-relativistic intensity. The effect results from instability caused by a strong modulation of the electron oscillation frequency by the laser field. The origin of the modulation is the oscillations of the relativistic \( \gamma \)-factor caused by ultra-relativistic electron motion in the electromagnetic field of the wave. The instability develops when the laser intensity \( I \) exceeds a threshold value
\[ I \left[ \text{W/cm}^2 \right] \lambda^2 [\mu \text{m}] = 1.37 \times 10^{18} (\kappa_\ast / 2 \pi)^2 n_c / n_0, \] (17)
where \( n_c \) is the critical density, \( n_0 \) is the ion density, and \( \kappa_\ast \) is the threshold parameter determined by the laser polarization (see the inset of Fig. 2). Our single particle model can also be applicable if the channel contains an underdense electron population in addition to the accelerated electron beam. Our analysis is not very restrictive with regard to the beam density because mutual repul-
sion of ultra-relativistic electrons can be much weaker than the focusing force from the ions even at substantial electron densities [15]. The effect of the electrons on the wave is negligible at sufficiently low electron densities. A self-consistent nonlinear calculation of the electron re-
sponse to the laser field is needed to quantify the upper limit on electron density, since the electron motion in the channel is ultrarelativistic.

The presented mechanism could be employed in laser-target experiments to generate hot electrons, provided that appropriate applicability conditions are satisfied. The thickness of the preplasma layer needs to be comparable to the interaction length required for the enhanced acceleration, \( l \approx \gamma_{\text{max}} \lambda \). The width of the laser beam must exceed \( \sqrt{a_0 c / \omega_{\text{ph}}} \) to ensure that ultra-relativistic electrons with \( \gamma \geq a_0 \) are not expelled by the transverse gradient of the ponderomotive pressure. This condition also guarantees that the electron remains within the beam during its transverse oscillations. It is also plausible that this mechanism is involved in formation of energetic electron tails observed in Ref. [2, 7, 10, 12], because the laser intensity in the experiments was in the range of the threshold intensities defined by Eq. (17). A realistic modeling of the experiment is required for a more conclusive assessment. Finally, the amplification of electron oscillations via the presented mechanism may also be beneficial in those applications that employ transverse electron oscillations in a confining quasi-static field to generate x- and gamma-rays [12, 16], provided that the mechanism is robust with respect to spectral broadening of the laser beam [17].

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