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Phys. Rev. Lett. **108**, 086404 — Published 23 February 2012

DOI: 10.1103/PhysRevLett.108.086404

Theory of a Fermi-Liquid-to-Non-Fermi-Liquid Quantum Phase Transition in Dimensions $d > 1$

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We develop a theory for a generic instability of a Fermi liquid in dimension $d > 1$ against the formation of a Luttinger-liquid-like state. The density of states at the Fermi level is the order parameter for the ensuing quantum phase transition, which is driven by the effective interaction strength. A scaling theory in conjunction with an effective field theoy for clean electrons is used to obtain the critical behavior of observables. In the Fermi-liquid phase the order-parameter susceptibility, which is measurable by tunneling, is predicted to diverge for $1 < d < 3$.

PACS numbers: 71.10.Hf; 71.10.Ay; 71.30.+h

Landau's Fermi-liquid theory provides a very successful paradigm in condensed matter physics. By mapping the low-lying excitations in interacting Fermi systems onto those of noninteracting ones [1] it explains many properties of electrons in solids, including the linear temperature (T) dependence of the specific heat, and the quadratic T-dependence of the electrical resistivity [2]. In a renormalization-group (RG) context it can be understood as the scaling behavior near a stable fixed point (FP) that governs the low-T behavior of the system [3]. Because of this success, deviations from Fermi-liquid (FL) behavior have attracted considerable attention [4]. Examples include parts of the normal phase of high-T_{c} superconductors [5], heavy-fermion systems [6], and the paramagnetic phase of the helimagnet MnSi at low T [7].

There are different sources for non-Fermi-liquid (NFL) behavior. One is the vicinity of a quantum critical point. Another is the existence of Goldstone modes due to a spontaneously broken symmetry and resulting long-range order. The coupling of electrons to critical soft modes and Goldstone modes has been proposed to explain the behavior of heavy-fermion systems [6] and MnSi [8], respectively. More generic mechanisms for NFL behavior, that do not rely on underlying long-range order, are hard to find. In one-dimensional (1-d) fermion systems an arbitrarily small repulsive interaction amplitude K_{s} (we will restrict ourselves to a point-like interaction in the spin-singlet particle-hole channel [9]) leads to an instability of the FL against a Luttinger liquid (LL) that has a vanishing density of states (DOS) at the Fermi level and is characterized by sound-like excitations [10]. A natural question is whether in dimensions $d > 1$ a similar instability will occur for K_{s} greater than some $K_{\text{s}}^{\text{c}} > 0$. Despite substantial efforts, to date no description of such an instability has been found.

There are, however, indications that an instability exists. Perturbation theory in the FL phase yields nonanalytic dependencies on T, or the wave number k, for, e.g., the spin susceptibility and the specific heat coefficient [11–13]. For generic d they take the form T^{d-1} or k^{d-1}, with multiplicative logarithms in odd d. For $d = 1$, the logarithmic divergencies coincide with the perturbative signatures of the LL [14]. This is reminiscent of disordered electrons, where perturbation theory generically yields a $T^{(d-2)/2}$ or k^{d-2} behavior, which turns into $\log T$ or $\log k$ in $d = 2$. These perturbative "weak-localization" effects signalize the instability of the disordered FL against an Anderson or Anderson-Mott insulator [15, 16]. In $d = 2$ this instability occurs at arbitrarily small values of the disorder, whereas in $d > 2$ a metal-insulator transition occurs at a nonzero critical value of the disorder. It is thus natural to speculate that a transition from a FL to a LL can occur in $d > 1$.

In this Letter we construct a theory that describes a quantum phase transition from a FL to a NFL state with a vanishing DOS at the Fermi level in $d > 1$. The DOS serves as the order parameter (OP) for the transition; the FL is the ordered phase. $d_{\text{c}}^{-} = 1$ is the lower critical dimension for the transition; fluctuations destroy the ordered phase for $d \leq d_{\text{c}}^{-}$. For $d = 1 + \epsilon$ ($\epsilon \ll 1$) the theory is controlled and the critical value of the interaction strength is $K_{\text{s}}^{\text{c}} = O(\epsilon^{1/2})$. For larger d the critical behavior is obtained from scaling considerations. In many respects our theory is analogous to the nonlinear sigma-model for the classical Heisenberg transition near $d = 2$ [17], and to the Anderson-Mott metal-insulator transition of disordered interacting electrons [16, 18], even though the latter is to a non-standard insulator, while we describe a transition to a non-standard metal [19].

To identify the DOS as the OP for the FL-to-NFL transition we consider a Ward identity that reflects the broken symmetry between retarded and advanced degrees of freedom in a FL. It relates a two-particle correlation function F_2 (schematically, $\langle\bar{\psi}\bar{\psi}\psi\psi\rangle$, with $\bar{\psi}$ and ψ fermion fields; a more explicit expression of F_2 will be given in Eq. (13)) to a single-particle function F_1 ($\langle\bar{\psi}\psi\rangle$) and is a generalization of a Ward identity first considered for noninteracting electrons with quenched disorder [20, 21]. With $\boldsymbol{p}$ the center-of-mass wave vector ($|\boldsymbol{p}| \approx k_{\text{F}}$; k_{F}, ϵ_{F}, and v_{F} denote the Fermi wave number, energy, and velocity), $\boldsymbol{k}$ the hydrodynamic wave vector ($|\boldsymbol{k}| \equiv k \ll k_{\rm F}$), and $m_{\rm e}$ the electron mass it can be written

$$(i\Omega_{n_1-n_2} + \boldsymbol{p}\cdot\boldsymbol{k}/m_{\rm e})\, F_2\,(\boldsymbol{p},\boldsymbol{k}; i\omega_{n_1}, i\omega_{n_2}) = F_1\,(\boldsymbol{p},\boldsymbol{k}; i\omega_{n_1}, i\omega_{n_2})\,, \qquad (1a)$$

F_1 is proportional to the difference between Green functions taken at the fermionic Matsubara frequencies $i\omega_{n_1}$ and $i\omega_{n_2}$. For $i\Omega_{n_1-n_2} = i\omega_{n_1} - i\omega_{n_2} \to 0$, $\boldsymbol{k} \to 0$, F_1 vanishes if $\omega_{n_1}\omega_{n_2} > 0$, but is nonzero if $\omega_{n_1}\omega_{n_2} < 0$. In the latter case, and for noninteracting electrons,

$$F_1\,(\boldsymbol{p},\boldsymbol{k}; i\omega_{n_1}, i\omega_{n_2}) \propto i\,\mathrm{sgn}\,(\Omega_{n_1-n_2})\,\delta(\epsilon_{\boldsymbol{p}} - \epsilon_{\rm F}), \qquad (1b)$$

with $\epsilon_{\boldsymbol{p}}$ the single-particle energy-momentum relation. For $\omega_{n_1}\omega_{n_2} < 0$ there thus is a family of 4-fermion functions that diverge in the hydrodynamic limit of vanishing $\Omega_{n_1-n_2}$ and $\boldsymbol{k}$. Taking moments with respect to $\boldsymbol{p}$ yields an infinite number of soft modes, provided the DOS at the Fermi level, $N_{\rm F} \propto \sum_{\boldsymbol{p}} \delta(\epsilon_{\boldsymbol{p}} - \epsilon_{\rm F})$, is nonzero. In the presence of quenched disorder, in contrast, only the zeroth moment of Eq. (1a) is soft. These soft modes are the Goldstone modes of a spontaneously broken continuous symmetry, namely, rotations in frequency space that transform pairs of fermion fields $\bar\psi$ or ψ (more precisely, pairs of spinors η defined in Eq. (12)) with frequency labels n_1 and n_2, respectively, into linear combinations of the same pair. This symmetry was first discussed by Schäfer and Wegner [21] in the context of disordered electrons, and elaborated on in Ref. 22. For our current purposes we have used a generalization of this transformation to consider phase-space correlation functions, not only s-wave channel modes as in Ref. 22. If n_1 and n_2 are both positive or both negative, then the action is invariant under these rotations. If n_1 and n_2 have opposite signs, then a nonzero DOS spontaneously breaks this invariance. The soft modes in question are thus Goldstone modes; they are not related to a conservation law.

It can be shown that the structure of Eqs. (1) remains unchanged in interacting systems, and that F_2 remains soft [23]. This is consistent with what one expects from FL theory: The symmetry is broken, and Goldstone modes exist, as long as the DOS at the Fermi level is nonzero. The noninteracting DOS, $N_{\rm F}$, gets replaced by the physical DOS, $N(\epsilon_{\rm F})$, and the prefactor of the frequency $\Omega_{n_1-n_2}$ acquires a FL correction. F_2 remains massless, and the frequency continues to scale as a wave number. Conversely, a vanishing DOS implies that the symmetry is restored and the Goldstone modes have zero weight. If this happens, by varying some control parameter, then the system will undergo a symmetry-restoring phase transition from a FL (ordered phase) to a NFL (disordered phase) with the DOS as the OP. In the FL the Goldstone modes are all proportional to the basic Goldstone propagator

$$\mathcal{D}(\boldsymbol{k}, i\Omega) = N(\epsilon_{\rm F})\,\varphi(i\Omega/v_{\rm F}k)/k. \qquad (2a)$$

The explicit form of the function φ depends on the dimensionality. In the limiting case $d \to 1$ one has

$$\varphi(x) \propto |x|/(1 + x^2). \qquad (2b)$$

These considerations show that in the ordered phase there are soft modes whose frequency scales as a wave number, $\Omega \sim k$: A dynamical exponent $z = 1$ is associated with the stable FL FP. At a symmetry-restoring transition, described by a critical FP, $z \neq 1$ in general.

The Goldstone modes are *not* related to the density propagator, which is governed by particle-number conservation. The latter, plus the fact that the thermodynamic density susceptibility $\partial n/\partial\mu$ is expected to be uncritical (see below), implies that in the density propagator one has $\Omega \sim k$, or $z = 1$, at both the FL FP and the critical FP. This is consistent with the fact that $\Omega \sim k$ at the stable FP that describes a LL in $d = 1$ [10]. Hence there is more than one dynamical exponent: The critical dynamical exponent z ($\neq 1$ in general), related to the Goldstone modes, and another dynamical exponent $z_{\rm c} = 1$ related to the charge or density dynamics.

The Goldstone modes in the FL phase, and their destruction at the critical FP, provide a physical mechanism for the instability of the FL and the stabilization of a NFL phase. If the fluctuations described by the Goldstone modes become strong enough, and their contribution to the free energy large enough, it is energetically favorable for the system to undergo a transition to a phase where the symmetry is restored, the DOS at the Fermi level vanishes, and the Goldstone modes do not exist. In this sense the Fermi liquid carries within itself the seeds of its own destruction.

In what follows, we construct a scaling theory for a symmetry-restoring FL-to-NFL quantum phase transition where the DOS vanishes. We have also derived an effective field theory that allows for an explicit description of such a transition, the most important aspects of which we will sketch at the end of this Letter.

We start by considering the free energy density f, which quite generally satisfies a scaling relation

$$f(t,T,h) = b^{-(d+z)}\, f(t\,b^{1/\nu}, T\,b^z, h\,b^{y_h})\ . \qquad (3)$$

We have assigned scale dimensions $[L] = -1$ and $[T] = z$ to factors of length and temperature, energy, or inverse time ($\hbar = k_{\rm B} = 1$), which yields $[f] = -d - z$ for the scale dimension of f [24]. b is the RG length rescaling factor. h is the field conjugate to the OP, with scale dimension $[h] = y_h$. t is the dimensional distance from the critical point, and $\nu = 1/[t]$ is the correlation length exponent. For the OP density $N = -(\partial f/\partial h)/T$ Eq. (3) implies

$$N(t,T) = b^{-d+z}\, N(t\,b^{1/\nu}, T\,b^z)\ . \qquad (4)$$

Here we have used the fact that N is the DOS, which scales as an inverse energy times an inverse volume; hence

$y_h = z$. At $T = 0$ and at criticality, respectively, the OP vanishes as a power law,

$$N(t, T = 0) \propto t^{\beta} \quad , \quad N(t = 0, T) \propto T^{(d-z)/z} \ . \qquad (5)$$

with $\beta = \nu(d - z)$. The specific heat coefficient γ is obtained from $C_V = \gamma T = -T\partial^2 f/\partial T^2$. The scaling behavior of γ is the same as that of the DOS:

$$\gamma(t, T) = b^{-d+z}\,\gamma(t\,b^{1/\nu}, T\,b^{z}), \qquad (6)$$

We next consider the OP susceptibility $\chi = \partial N/\partial h$ as a function of t, T, and the wave number k. In general,

$$\chi(t, T; k) = b^{2-\eta}\,\chi(t\,b^{1/\nu}, T\,b^{z}, kb), \qquad \text{(7a)}$$

which defines the exponent η. At $T = 0$ and at criticality, respectively, the homogeneous OP susceptibility diverges:

$$\begin{aligned} \chi(t, T = 0, k = 0) &\propto t^{-\gamma} \quad , \quad \gamma = \nu(2 - \eta), \\ \chi(t = 0, T, k = 0) &\propto T^{-(2-\eta)/z}. \end{aligned} \qquad \text{(7b)}$$

From Eqs. (3), (4), and (7a) we find the exponent relation

$$z = (d - \eta + 2)/2. \qquad (8)$$

This implies that there are only two independent critical exponents, e.g., ν and z (see, however, the remark above regarding multiple exponents z) rather than three as is generally the case at a quantum critical point [25]. For $\partial n/\partial\mu$ we expect no critical behavior since it does not show the perturbative nonanalyticities that are precursors of the critical behavior of other observables [11, 26]. The scaling behavior of the electrical conductivity $\sigma = D_{\rm c}\,\partial n/\partial\mu$ is therefore given by that of the charge diffusion coefficient $D_{\rm c}$, which scales as a length squared divided by a time. Since $D_{\rm c}$ describes the charge or density dynamics the relevant dynamical exponent in this context is $z_{\rm c} = 1$. We thus have [27]

$$\sigma(t, T) = b^{2-z_{\rm c}}\,\sigma(t\,b^{1/\nu}, T\,b^{z}, T\,b^{z_{\rm c}}). \qquad \text{(9a)}$$

If $z < 1$ (see below) this yields for the electrical resistivity $\rho = 1/\sigma$ at criticality

$$\rho(t = 0, T) \propto T. \qquad \text{(9b)}$$

These scaling predictions all pertain to the critical FP. Also of interest are the OP and the OP susceptibility in the ordered phase, $|t| = \infty$, where $\eta = d$, which implies $z = 1$. From Eq. (4) we have

$$N(|t| = \infty, T) \propto \text{const.} + T^{d-1}. \qquad (10)$$

This is one example of the perturbative nonanalyticities mentioned above. The same power law holds at $T = 0$ as a function of the distance ω from the Fermi surface: $N(T = 0, \omega) \propto$ const. $+\,\omega^{d-1}$. It is analogous to the Coulomb anomaly in disordered systems, where $N(T = 0, \omega) \propto$ const. $+\,\omega^{(d-2)/2}$ [28]. The latter is a precursor of the quantum phase transition in disordered systems (the Anderson-Mott transition [16]), where the DOS vanishes and serves as an OP [29]. The current theory suggests that an analogous statement holds in clean ones. For the OP susceptibility we find from Eq. (7a)

$$\chi(|t| = \infty, T, k) = k^{d-2}\,f_{\chi}(T/k) \propto T/k^{3-d}. \qquad (11)$$

In the second relation we have used the result of an explicit calculation [30], which yields $f_{\chi}(x \to 0) \propto x$. This divergence of the OP susceptibility, or the 2-point local-DOS correlation function, which is observable by tunneling experiments, is a consequence of the Goldstone modes. It is analogous to the $1/k^{4-d}$ divergence of the longitudinal susceptibility in the ordered phase of a Heisenberg ferromagnet [31]. For a 2-d FL it predicts a $1/k$ divergence with a prefactor that is linear in T.

The preceding scaling considerations are expected to be valid between the lower critical dimension $d_{\rm c}^- = 1$ and some upper critical dimension $d_{\rm c}^+$. Equation (11) suggests $d_{\rm c}^+ = 3$, but this requires further corroboration. For $d > d_{\rm c}^+$ one expects the critical behavior to be mean-field like and governed by a Gaussian FP. An approach that focuses on the Gaussian fixed point and its stability will also allow for an ϵ-expansion about $d_{\rm c}^+$ which will complement the current expansion about $d_{\rm c}^-$.

We now sketch the derivation of an effective field theory that allows for an explicit description of a quantum phase transition of the type we have discussed above. A complete account will be given elsewhere [30]. This effective theory is in the spirit of the matrix field theories that were pioneered by Wegner [21, 32], and generalized by others [18, 22], for disordered systems. We consider a fermionic action and define electron bispinors

$$\eta_n(\boldsymbol{x}) = \left(\bar\psi_{n\uparrow}(\boldsymbol{x}), \bar\psi_{n\downarrow}(\boldsymbol{x}), \psi_{n\downarrow}(\boldsymbol{x}), -\psi_{n\uparrow}(\boldsymbol{x})\right)/\sqrt{2} \qquad (12)$$

where $\bar\psi$ and ψ are fermionic fields with Matsubara frequency index N and spin projection $\uparrow\downarrow$, as well as adjoints $\eta_n^+(\boldsymbol{x}) = C\eta_n(\boldsymbol{x})$ with $C = i\sigma_1 \otimes \sigma_2$, where $\sigma_{1,2}$ are Pauli matrices. We confine the tensor product $\eta_n^+(\boldsymbol{x}) \otimes \eta_m(\boldsymbol{y})$ to a spin-quaternion-valued bosonic matrix field $Q_{nm}(\boldsymbol{x}, \boldsymbol{y})$ by means of a Lagrange multiplier $\Lambda_{nm}(\boldsymbol{x}, \boldsymbol{y})$. The Ward identity then takes the form of Eqs. (1) with

$$\begin{aligned} F_2(\boldsymbol{p}, \boldsymbol{k}; i\omega_{n_1}, i\omega_{n_2}) \;=\; &\langle \mathrm{tr}\, Q_{n_2 n_1}(\boldsymbol{p} + \boldsymbol{k}/2, \boldsymbol{p} - \boldsymbol{k}/2) \\ &\times \mathrm{tr}\, Q_{n_1 n_2}(\boldsymbol{p} - \boldsymbol{k}/2, \boldsymbol{p} + \boldsymbol{k}/2)\rangle \end{aligned} \qquad (13)$$

where $\omega_{n_1}\omega_{n_2} < 0$. This identifies $q_{nm}(\boldsymbol{p}_1, \boldsymbol{p}_2) \equiv \Theta(-nm)\,Q_{nm}(\boldsymbol{p}_1, \boldsymbol{p}_2)$ as the Goldstone modes. The corresponding elements λ of the Lagrange multiplier field Λ are also soft modes. The electron-electron interaction couples q to $P_{nm}(\boldsymbol{p}_1, \boldsymbol{p}_2) \equiv \Theta(nm)\,Q_{nm}(\boldsymbol{p}_1, \boldsymbol{p}_2)$, and integrating out P, and the corresponding part of Λ, generates terms to all orders in q and λ. An analogous procedure in the presence of quenched disorder provides a perturbative derivation, order by order in powers of q, of the generalized nonlinear sigma-model for the Anderson-Mott transition problem [16, 18]. We have derived the action to order q^4, which suffices for a 1-loop calculation. The effects of λ can be absorbed into diagram rules.

This effective theory can be analyzed by RG methods in $d = 1 + \epsilon$ by means of a systematic loop expansion in analogy to the disordered case in $d = 2 + \epsilon$ [16]. The 1-point function is proportional to the DOS:

$$P^{(1)} = \langle \mathrm{tr}\, Q_{nn}(\boldsymbol{x}, \boldsymbol{x}) \rangle \big|_{i\omega_n \to i0} \propto N(\epsilon_{\mathrm{F}}) \equiv N_{\mathrm{F}}\, Z^{1/2}. \quad (14)$$

Physically, $Z^{1/2} = (1 + \delta Z)^{1/2}$ is the physical DOS normalized by the bare or free-electron DOS; technically, it is the field-renormalization constant. It is related to, but not the same as, the residue of the pole in the Green function. We have performed a 1-loop calculation and have found that δZ is negative, logarithmically divergent in $d = 1$, and proportional to $1/\epsilon$ in $d = 1 + \epsilon$. As a function of K_{s} it is of $O(K_{\mathrm{s}}^2)$ for small K_{s}. In a naive extrapolation the DOS thus vanishes at a critical value $K_{\mathrm{s}}^{\mathrm{c}} = O(\epsilon^{1/2})$. The 2-point function

$$P^{(2)} = \langle q_{n_1 n_2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\, q_{n_3 n_4}(\boldsymbol{k}_3, \boldsymbol{k}_4) \rangle \quad (15a)$$

has a contribution proportional to $\delta_{n_1 n_3}\, \delta_{n_2 n_4}$ that constitutes the Goldstone propagator $\mathcal{D}$, Eqs. (2). In $d = 1$

$$\mathcal{D}(\boldsymbol{k}, i\Omega) = Z\, H\, |\Omega| / (\boldsymbol{k}^2/G^2 + H^2\Omega^2). \quad (15b)$$

The bare values of G (which is the loop expansion parameter) and H are $1/v_{\mathrm{F}} N_{\mathrm{F}}$ and N_{F}, respectively. Our explicit calculation has found that G and H are not singularly renormalized at 1-loop order. We have performed a structural analysis of the loop expansion which confirms this and shows that at 2-loop order there is a singular renormalization of G due to insertion diagrams; an inspection of skeleton diagrams will require a full 2-loop calculation. This strongly suggests a critical FP at 2-loop order with a FP value of the renormalized coupling constant $g = b^{\epsilon} G$ given by $g^* = O(\epsilon^{1/2})$. Choosing the independent exponents to be ν and z this leads to

$$\nu = 1/2\epsilon + O(1) \quad , \quad z = 1 + O(\epsilon). \quad (16)$$

The $O(\epsilon)$ term in z requires a 2-loop calculation; an educated guess is as follows. In the bare theory, $G^2 H \propto m_{\mathrm{e}}/n$, with n the electron density. This quantity one does not expect to be renormalized, so $H \sim G^{-2} \sim b^{-2\epsilon}$, or $z = 1 - \epsilon$ [33]. Hence $z < z_{\mathrm{c}}$, which implies Eq. (9b).

In summary, we have described a FL-to-NFL transition that is characterized by a vanishing DOS at the Fermi level. Promising systems to observe such a transition are optical lattices where the interaction strength between the fermions can by tuned. Pseudogap phases observed in optical lattices [34] and high-T_{c} superconductors [35] may be manifestations of the physics discussed above. This notion requires further investigation. For instance, it would be very interesting to look experimentally for the precursor effect in a FL described by Eqs. (10, 11).

This work was supported by the National Science Foundation under Grant Nos. DMR-09-29966, and DMR-09-01907.

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