Topological Quantum Liquids with Quaternion Non-Abelian Statistics

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Topological Quantum Liquids with Quaternion Non-Abelian Statistics

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The ground state manifold (GSM) of a noncollinear magnetic order is usually a "tetrad", meaning the GSM is characterized by the configurations of three perpendicular vectors or nematic-directors. We study three types of tetrad orders in two spatial dimensions, whose ground state manifolds are $\mathrm{SO}(3) = S^3/Z_2$, S^3/Z_4, and S^3/Q_8, respectively. Here Q_8 stands for the finite, non-Abelian quaternion group with eight elements. We demonstrate that after quantum disordering these three types of tetrad orders, the systems enter fully gapped liquid phases described by Z_2, Z_4 and non-Abelian quaternion gauge field theories, respectively. The latter case provides a realization of the *non-Abelian* Toric Code phase proposed by Kitaev based on a finite group G, where here $G = Q_8$, in terms of a rather simple spin-1 $SU(2)$ quantum magnet. This topological phase possesses a 22-fold ground state degeneracy on the torus arising from the 22 excitations with non-Abelian Braiding which form the representations of the Drinfeld double of Q_8.

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The search for quantum liquid states has been one of the main goals of condensed matter theory for decades. There are in general two different routes towards this goal, starting from two opposite limits. The first route is to start with the quantum limit, say the large-N limit of the SU(N) antiferromagnet, and approach the physical system through an $1/N$ expansion. For instance, the $1/N$ expansion within the slave fermion formalism for the SU(N) antiferromagnet leads to the valence bond solid state with no classical counterpart [1]. In our current work, we will take a second route towards the liquid state, which is by quantum disordering the semiclassical state. For instance, it is understood that the valence bond solid state (VBS) naturally emerges if quantum fluctuations destroy the semiclassical Néel order of a spin-1/2 system. This result is based on the observation that the Skyrmion of the Néel order of a spin-1/2 antiferromagnet always carries lattice momentum [2, 3].

In general, the ground state manifold (GSM) of spin states can be written as

$$\mathrm{GSM} = \mathrm{SU}(2)/\ G, \tag{1}$$

where G represents the unbroken subgroup of the SU(2) spin symmetry in the ordered phase. G is at least Z_2 for spin-1/2 systems, because physical order parameters should be invariant under spin rotation by 2π. In the present paper, we will discuss the quantum disordered phases adjacent to semiclassical spin states whose unbroken symmetry G is a discrete subgroup of SU(2), either Abelian or non-Abelian. All these states are "tetrad-like" states *i.e.* the GSM can be represented by three perpendicular vectors or nematic-directors (Fig. 1). We will demonstrate that an exotic non-Abelian topological liquid state can emerge after disordering a tetrad nematic order of a fairly simple spin-1 system. Non-Abelian statistics is a much sought-out phenomenon much discussed in particular in fractional quantum Hall systems [4], and more recently also in certain topological insulators (superconductors) [5]. In the sequel, we will discuss in turn three tetrad states, Type A, B, and C.

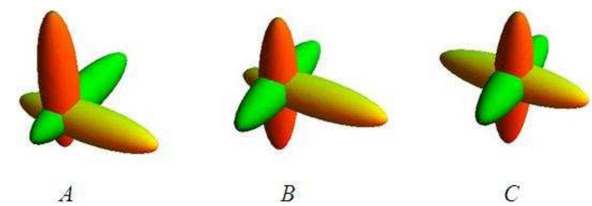

FIG. 1: Three types of tetrad spin order. Type A phase has ground state manifold SO(3), which is equivalent to the configurations of three perpendicular vectors. Type B phase has GSM S^3/Z_4, two of the three perpendicular vectors are headless directors. In type C phase all three vectors are headless directors.

– *Type A, with $G = Z_2$:* Let us first take $G = Z_2$, and thus the GSM is now $\mathrm{SU}(2)/Z_2 = \mathrm{SO}(3)$. SO(3) is precisely the tetrad manifold, which corresponds to all the configurations of three perpendicular vectors (Fig. 1A). One example of this case is the well understood noncollinear spin density wave (SDW), for which the three perpendicular vectors $\vec{N}_1$, $\vec{N}_2$ and $\vec{N}_3$ that characterize the GSM are defined as $\vec{S}(\vec{r}) = \vec{N}_2 \cos(2\vec{Q}\cdot\vec{r}) + \vec{N}_3 \sin(2\vec{Q}\cdot\vec{r})$; $\vec{N}_1 = \vec{N}_2 \times \vec{N}_3$. Here $\vec{Q}$ is the spiral wave vector of the SDW. It was pointed out in Ref. [6] that if quantum fluctuations destroy the noncollinear spin density wave (SDW), one interesting possibility is that the system enters a Z_2 liquid state. On the torus this Z_2 liquid ground state has a four-fold topological degeneracy [7]. Let us briefly review why this is the case. The most convenient way of parametrizing the manifold SO(3) is by introducing CP^1 spinor fields $z = (z_1, z_2)^t$ as follows:

$$\vec{N}_1 \sim z^\dagger \vec{\sigma} z,\ \vec{N}_2 \sim \mathrm{Re}[z^t i\sigma^y \vec{\sigma} z],\ \vec{N}_3 \sim \mathrm{Im}[z^t i\sigma^y \vec{\sigma} z]. \tag{2}$$

It is straightforward to show that the vectors $\vec{N}_a$ are automatically perpendicular to each other after introducing the spinor z_α. Since all the physical vectors $\vec{N}_a$ are bilinears of z_α, the spinor z_α is effectively coupled to a Z_2 gauge field, which makes z_α equivalent to $-z_\alpha$.

Since the homotopy group $\pi_1[\mathrm{SO}(3)] = Z_2$, the GSM SO(3) supports vortex like topological defects with a Z_2 conservation law. This type of topological defect is often called a vison. Pictorially, the vison can be viewed as a configuration in which (for instance) $\vec{N}_1$ is uniform in space, while $\vec{N}_2$ and $\vec{N}_3$ have a vortex (Fig. 2A). After we destroy the ordered state with quantum fluctuations, the spinor z_α is gapped, but the Z_2 conservation law of the vison still persists. This implies that the disordered phase of the noncollinear SDW is equivalent to the deconfined phase of Z_2 gauge theory, where visons also have a Z_2 conservation law [6, 7]. In this phase the gapped spinor z_α and the vison have mutual semionic statistics, *i.e.* the wave function picks up a minus sign when z_α encircles the vison adiabatically.

As was discussed in Ref. [6], the transition between the ordered phase with GSM SO(3) and the Z_2 deconfined liquid phase is contiuous and belongs to the 3d O(4) universality class. This is because the bosonic spinor field z_α can also be viewed as a four component real vector, whose order-disorder phase transition belongs to the 3d O(4) universality class. The gapped Z_2 gauge field does not introduce singular corrections in the infrared, *i.e.* the 3d O(4) universality class is unaffected by the the presence of the Z_2 gauge field [6]. Because the physical order parameters are bilinears of the spinor z_α, they acquire a relatively large anomalous dimension as compared to the standard order parameters at the Wilson-Fisher fixed point of the $O(4)$ Heisenberg model. Specifically, using a five-loop epsilon expansion in $d = 4 - \epsilon$ dimensions the scaling dimension of these composite bilinears has been found[8, 9] to be $\eta_{\vec{N}_a} \approx 1.37$ in $(2+1)$ dimensions.

– Type B, with $G = Z_4$: Now let us move to the type-B tetrad phase. The GSM can be characterized by one vector and two directors, where again all three vector/directors are perpendicular to each other (Fig. 1B). It is more convenient to describe this manifold using the following slightly different representation of $\vec{N}_a$:

$$\vec{N}_1 \sim \mathrm{tr}[\mathcal{Z}^\dagger \vec{\sigma} \mathcal{Z} \sigma^z], \vec{N}_2 \sim \mathrm{tr}[\mathcal{Z}^\dagger \vec{\sigma} \mathcal{Z} \sigma^x], \vec{N}_3 \sim \mathrm{tr}[\mathcal{Z}^\dagger \vec{\sigma} \mathcal{Z} \sigma^y],$$
$$\mathcal{Z} = \phi_0 1 + i\phi_1 \sigma^x + i\phi_2 \sigma^y + i\phi_3 \sigma^z,$$
$$z = (z_1,\ z_2)^t = (\phi_0 + i\phi_3,\ -\phi_2 + i\phi_1)^t. \qquad (3)$$

$\mathcal{Z}$ is a SU(2) matrix, sometimes called the SU(2) slave rotor field[23]. $\mathcal{Z}$ has an action by $\mathrm{SU}(2)_{\mathrm{left}}$ (left multiplication) and by $\mathrm{SU}(2)_{\mathrm{right}}$ (right multiplication). While $\mathrm{SU}(2)_{\mathrm{left}}$ transformations correspond to the physical SU(2) spin rotation symmetry, $\mathrm{SU}(2)_{\mathrm{right}}$ transformations contain the gauge symmetry as its subgroup. symmetry, symmetry as its subgroup.

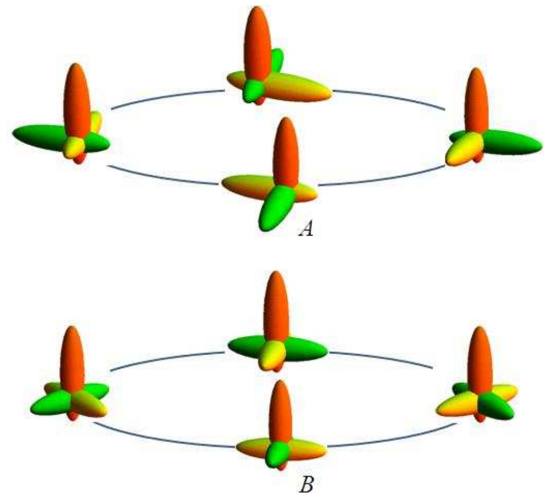

FIG. 2: The configuration of vison defect in tetrad order A and half-vison defect in tetrad order B.

Now let us take $\vec{N}_1$ a vector, while $\vec{N}_2$ and $\vec{N}_3$ are both headless directors. In order to make $\vec{N}_2$ and $\vec{N}_3$ headless, we can couple $\mathcal{Z}$ to a gauge field taking values in a group with group elements:

$$Z_4 = \{1,\ i\sigma^z,\ -1,\ -i\sigma^z\}. \qquad (4)$$

The gauge field always acts on $\mathcal{Z}$ by right multiplication. Under the gauge transformation

$$\mathcal{Z} \to \mathcal{Z}(\pm i\sigma^z), \qquad (5)$$

both $\vec{N}_2$ and $\vec{N}_3$ reverse direction, while $\vec{N}_1$ remains invariant. Therefore the type B tetrad phase can be understood as the condensate of the SU(2) rotor field $\mathcal{Z}$ (or spinor z_α) when it is coupled to the Z_4 gauge field with gauge group Z_4 from Eq. 4. Unlike the type A case, $\vec{N}_2$ and $\vec{N}_3$ are no longer themselves physical order parameters due to the presence of the Z_4 gauge field; rather, the physical order parameter $Q_i^{ab} = N_i^a N_i^b - \frac{1}{3}(\vec{N}_i)^2$, $i = 2, 3$ is of quadrupolar type.

In addition to the vison defect discussed in the type A phase, the type B phase also has a "half-vison" defect, *i.e.* the configuration in which $\vec{N}_1$ is uniform in space, while $\vec{N}_2$ and $\vec{N}_3$ have a half vortex (Fig. 2B). This defect has a logarithmically divergent instead of a confining energy because $\vec{N}_2$ and $\vec{N}_3$ are nematic directors. We can also describe this half vortex as a Z_4 gauge flux $\pm i\sigma^z$ in the condensate of $\mathcal{Z}$. After encircling this flux, $\mathcal{Z}$ undergoes a gauge transformation as in Eq. 5, and $\vec{N}_2$ and $\vec{N}_3$ reverse their directions.

When the rotor field $\mathcal{Z}$ that couples to the Z_4 gauge field is gapped out, the system is described by a pure Z_4-gauge theory – a 'Z_4-liquid' phase. If the system in this phase is defined on the torus, then there can be four different fluxes through each cycle of the torus: 0, $\pi/2$, π, $3\pi/2$. Each of these different flux combinations corresponds to an independent topological sector. There

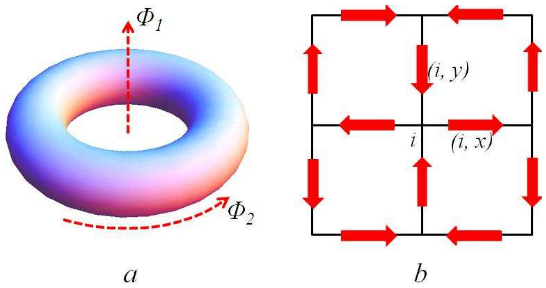

FIG. 3: a, the topological degeneracy can be counted as the number of inequivalent commuting gauge fluxes Φ_1 and Φ_2 through both cycles of the torus. b, The ring exchange terms Eq. 10 are ring product of gauge field on four links of each square, and in the ring exchange terms the links are connected in the sequence of the arrows around each plaquette.

is thus a 16 fold topological degeneracy on the torus. Recently, one of the authors of the present paper proposed that the type B phase is an intermediate phase of the Hubbard model on the honeycomb lattice [10], sandwiched between a fully gapped spin liquid phase and a pure Néel order with ground state manifold S^2. In Ref. [10], the vector $\vec{N}_1$ is the Néel order, whereas the directors $\vec{N}_2$ and $\vec{N}_3$ are spin nematic orders. The transition between type B tetrad order and the Z_4 liquid phase also belongs to the 3d O(4) universality class, for the same reason as the type A case.

– *Type C, with $G = Q_8$:* Now let us move on to the type C phase, whose ground state is characterized by three perpendicular nematic directors. This phase can be obtained from a system of spin-1 $SU(2)$ quantum spins $\hat{S}^a_i$ possessing both two-spin and four-spin interactions [11, 12]. For a spin-1 system it is often convenient to introduce SU(3) Schwinger bosons $\vec{b}_i$ as [13]. When $\langle \vec{b}^* \rangle$ is parallel with $\langle \vec{b} \rangle$, the spin symmetry is broken, while there is no spin polarization on any site, thus the system only has nematic order. Since $\langle \vec{b}^* \rangle \parallel \langle \vec{b} \rangle$, the Schwinger boson $\vec{b}$ can be rewritten as $\langle \vec{b}_i \rangle = e^{i\theta} \vec{N}_i$. $\vec{N}_i$ is in fact a nematic director, because the transformation $\vec{N} \to -\vec{N}$ can be cancelled by the transformation $\theta \to \theta + \pi$, which is part of the U(1) gauge symmetry associated with the Schwinger boson $\vec{b}$. The vectors $\vec{N}_i$ are precisely the nematic directors in Fig. 1C.

Experimentally it was observed that the triangular lattice spin-1 material $NiGa_2S_4$ has no global spin order with time-reversal symmetry breaking at low temperature, but it still has gapless excitations with linear dispersion [14]. It has been proposed[11, 12] that the candidate ground state of this system is characterized by a 'tetrad' of three perpendicular nematic directors N^a_i on three different sublattices denoted by $i = 1, 2, 3$. This proposed ground state thus has exactly the same GSM as that of Fig. 1C. The physical order parameter of this state is the quadrupolar spin order parameter[24]:

$$Q^{ab}_i \sim N^a_i N^b_i - \frac{1}{3}(\vec{N}_i)^2 \sim \langle \hat{S}^a_i \hat{S}^b_i - \frac{2}{3} \rangle. \tag{6}$$

Here $\hat{S}^a_i$ is the spin-1 operator on sublattices $i = 1, 2, 3$. This equation defines N^a_i, and it is precisely the nematic director N^a_i in the previous paragraph introduced through Schwingber boson [13]. This GSM is equivalent to that of biaxial nematic order [25] of a liquid crystal [15, 16]. "triatic" nematic spin order was also found in numerical work on SU(2) spin-1/2 models with both two-spin and four-spin interactions on the triangular lattice [17].

It is still most convenient to describe this phase with the SU(2) rotor variable $\mathcal{Z}$, but now $\mathcal{Z}$ is coupled to a discrete non-Abelian gauge field taking values in the non-Abelian Quaternion group Q_8,

$$Q_8 = \{\pm 1,\ \pm i\sigma^x,\ \pm i\sigma^y,\ \pm i\sigma^z\}. \tag{7}$$

Again, the gauge field acts on the rotor field $\mathcal{Z}$ by right multiplication. As a consequence of the action of this gauge group, $\vec{N}_i$ in Eq. 3 become headless nematic directors.

Now we will describe this gauge theory based on the non-Abelian quaterion group Q_8 in more detail. Following the general construction in Ref. [7], we define an 8-dimensional Hilbert space $\mathcal{H}$ on each link (i, μ) of the lattice, whose basis elements we denote by $|g_{i,\mu}\rangle$. Here i denotes a lattice site, $\mu = \hat{x}, \hat{y}$ a unit vector in a (positive) lattice direction, and $g_{i,\mu}$ denotes any of the eight elements of the group Q_8. Now we define, for any group element $h \in Q_8$, and on every link $i, \pm\mu$ of the lattice, operators $T^h_{i,\pm\mu}$ and $Q^h_{i,\pm\mu}$ with the following action on the basis vector $|g_{i,\mu}\rangle$ residing on that link:

$$\begin{aligned}
& T^h_{i,+\mu}|g_{i,\mu}\rangle = \delta_{h,g_{i,\mu}}|g_{i,\mu}\rangle, \quad T^h_{i+\mu,-\mu}|g_{i,\mu}\rangle = \delta_{h^{-1},g_{i,\mu}}|g_{i,\mu}\rangle, \\
& Q^h_{i,+\mu}|g_{i,\mu}\rangle = |hg_{i,\mu}\rangle, \quad Q^h_{i+\mu,-\mu}|g_{i,\mu}\rangle = |g_{i,\mu}h^{-1}\rangle, \\
& (T^h_{i,\mu})^\dagger = T^h_{i,\mu} = T^{h^{-1}}_{i+\mu,-\mu}, \\
& (Q^h_{i,\mu})^{-1} = (Q^h_{i,\mu})^\dagger = Q^{(h^{-1})}_{i,\mu}.
\end{aligned} \tag{8}$$

$T^h_{i,\mu}$ and $Q^f_{i,\mu}$ turn out to satisfy the following algebra:

$$\begin{aligned}
& Q^f_{i,\mu} T^h_{i,\mu} Q^{(f^{-1})}_{i,\mu} = T^{fh}_{i,\mu}, \\
& Q^f_{i+\mu,-\mu} T^h_{i,\mu} Q^{(f^{-1})}_{i+\mu,-\mu} = T^{hf^{-1}}_{i,\mu}.
\end{aligned} \tag{9}$$

The dynamics of the discrete gauge field is given by the following 'ring exchange' term [26]:

$$\begin{aligned}
H^q_{\text{ring}} = & \sum_h -K\ T^{h_{i+\hat{x},\hat{y}}}_{i+\hat{x},\hat{y}} T^{h_{i+\hat{x}+\hat{y},-\hat{x}}}_{i+\hat{x}+\hat{y},-\hat{x}} T^{h_{i+\hat{y},-\hat{y}}}_{i+\hat{y},-\hat{y}} T^{h_{i,\hat{x}}}_{i,\hat{x}} \\
& \times\ \text{tr}[G_{h_{i+\hat{x},\hat{y}}} G_{h_{i+\hat{x}+\hat{y},-\hat{x}}} G_{h_{i+\hat{y},-\hat{y}}} G_{h_{i,\hat{x}}}] + H.c.
\end{aligned} \tag{10}$$

G_h is the two dimensional representation Eq. 7 of the group element $h \in Q_8$, and $\sum_h$ denotes summation over all group elements $h_{i,\mu} \in Q_8$ on each link.

The direction of μ in the ring exchange term on each plaquette follows the arrows in Fig. 3. Here we always assume $K > 0$, which favors the gauge flux through each plaquette to be 1.

The SU(2) rotor field $\mathcal{Z}_i$ is defined on the vertices i of the square lattice. Right- and left- multiplication of $\mathcal{Z}$ by SU(2) transformations $\mathrm{SU(2)}_{\mathrm{right}}$ and $\mathrm{SU(2)}_{\mathrm{left}}$ is generated by the operators J^a_R and J^a_L, satisfying the commutation relations (see also Ref. 18)

$$[J^a_{R,L}, J^b_{R,L}] \;=\; i\epsilon_{abc} J^c_{R,L}, \quad [J^a_R, J^b_L] = 0. \tag{11}$$

In particular, J^a_L and J^a_R act as follows:

$$e^{i\vec{\theta}\cdot\vec{J}_R} \mathcal{Z} e^{-i\vec{\theta}\cdot\vec{J}_R} = \mathcal{Z} e^{-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}}; \quad e^{i\vec{\theta}\cdot\vec{J}_L} \mathcal{Z} e^{-i\vec{\theta}\cdot\vec{J}_L} = e^{i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}} \mathcal{Z} \tag{12}$$

The quaternion gauge group is a subgroup of the $\mathrm{SU(2)}_{\mathrm{right}}$ transformation. deduced in

The full Hamiltonian with both, rotor and gauge fields reads

$$\begin{aligned} H &= H_{rot} - \sum_{i,\mu,h_{i,\mu}} t \; \mathrm{tr}[\mathcal{Z}_i T^{h_{i,\mu}}_{i,\mu} G_{h_{i,\mu}} \mathcal{Z}^\dagger_{i+\mu}] + H^q_{\mathrm{ring}} \\ H_{rot} &= \sum_i \sum_a \frac{U_R}{2} J^{a2}_{R,i} + \frac{U_L}{2} J^{a2}_{L,i} \end{aligned} \tag{13}$$

This Hamiltonian is subject to the following quaternion gauge group constraint:

$$e^{i\pi J^a_{R,i}} = Q^{i\sigma^a}_{i,+\hat{x}} Q^{i\sigma^a}_{i,-\hat{x}} Q^{i\sigma^a}_{i,+\hat{y}} Q^{i\sigma^a}_{i,-\hat{y}}. \tag{14}$$

$a = x, y, z$. The unitary operator $T^{h_{i,\mu}}_{i,\mu} G_{h_{i,\mu}}$ appearing in Eq. 13 is the analogue of the conventional term $e^{i\vec{A}_{i,\mu}\cdot\vec{\sigma}}$ where $\vec{A}_{i,\mu}$ is the gauge potential. The quaternion group gauge constraint Eq. 14 generates the following gauge transformations on both $\mathcal{Z}$ and $T^{h_{i,\mu}}_{i,\mu}$:

$$T^{h_{i,\mu}}_{i,\mu} \to T^{f_i h_{i,\mu} f^{-1}_{i+\mu}}_{i,\mu}, \quad \mathcal{Z}_i \to \mathcal{Z}_i G_{f_i}, \tag{15}$$

where $f_i \in Q_8$. The Hamiltonian Eq. 13 is invariant under this gauge transformation. We have formulated this model on the square lattice, but generalizations to other lattices are straightforward. Again, the quantum phase transition between the ordered phase and quaternion liquid phase belongs to the 3d O(4) universality class because the Q_8 gauge field is always gapped.

When $U_L, U_R \gg t$, the SU(2) rotor field $\mathcal{Z}_i$ is gapped out, and the system is described by the pure quaternion group gauge theory Eq. 10, plus the gauge constraints. In the spin Hamiltonian, the rotor field $\mathcal{Z}_i$ can be gapped by turning on the following term on the spin Hamiltonian considered in Ref. [11, 12]:

$$H' = \sum_{\ll i,j \gg} J' \hat{Q}_i \cdot \hat{Q}_j, \quad J' > 0. \tag{16}$$

where $\hat{Q}_i$ is the five-component quadrupole order parameter introduced in Ref. [11–13]. Eq. 16 is an antiferroquadrupole interaction between the 2nd neighbor sites on the triangular lattice. This term energetically disfavors the system to form a three sublattice tetrad nematic order, and we propose that it will drive the system into the phase described by the pure quaternion nonabelian gauge theory. This gapped phase is a realization of the *non-Abelian toric code* phase built on a finite group G, proposed by Kitaev [7]. In the present case $G = Q_8$. Due to the non-Abelian nature of the group Q_8, this gauge theory is known to possess a rich set of gapped excitations exhibiting non-Abelian statistics, which are characterized by the representations of the so-called Drinfeld double [19–21] of the group Q_8. These excitations are the following:

(i) *magnetic* excitations are located at the centers of the plaquettes of the lattice (see Fig. 3), and are characterized by the product of group elements around a plaquette. Since the product can be taken over different closed loops enclosing the same "magnetic flux", a magnetic excitation is not characterized by a group element g itself, but by its conjugacy class $\mathcal{C}_g = \{h^{-1}gh : \; h \in G\}$.

(ii) *electric* charges are located at the vertices of the lattice (see fig. 3). An electric charge represents a violation of the vertex constraint of Eq. 14 and corresponds to an irreducible representation α of the group G. Transporting an electric charge α around a magnetic flux $\mathcal{C}_g$ along a closed path yields the representation matrix $D^{(\alpha)}(g)$ of the group element g.

(iii) the most general excitation contains both, magnetic and electric charges (often called a "dyon"), and is represented by a pair $(\mathcal{C}_g, a)$ as follows: when there is no magnetic charge, $\mathcal{C}_g = \mathcal{C}_{g=1}$, then $a = \alpha$ is an electric charge, i.e. a representation of the group G. However when the magnetic charge associated with a "dyon" is not vanishing, i.e. when $\mathcal{C}_g \neq \mathcal{C}_{g=1}$, its electric charge a is an irreducible representation $a = \hat{\alpha}$ of the *Normalizer* $N(g) = \{h \in G : hg = gh\}$ of g (consisting of all those group elements commuting with g), which is in general not the entire group G, but only a subgroup thereof. – Let us count the total number of excitations for the Drinfeld double of the quarternion group Q_8. We use the following facts: there are 5 conjugacy classes $\{+1\}, \{-1\}, \{\pm i\sigma^a\}$ where $a = x, y, z$; the number of irreducible representations of any finite group equals the number of conjugacy classes; the centralizer of any of the three conjugacy classes $\{\pm i\sigma^a\}$ is the Abelian cyclic group Z_4 of four elements generated by $i\sigma^a$. Thus, there are 5 excitations of the form $(\mathcal{C}_1, \alpha)$, 5 of the form $(\mathcal{C}_{-1}, \alpha)$, and 4 of the form $(\mathcal{C}_{i\sigma^a}, \hat{\alpha})$, for each $a = x, y, z$, where $\hat{\alpha}$ labels the representations of Z_4. This amounts to a total of $5+5+3\times 4 = 22$ excitations. Since for a general 2D topological field theory the ground state degeneracy on the torus equals the number of topological excitations ("particles"), this degeneracy is 22 in the present case. The fusion rules for these 22 particles can be obtained from the modular S-matrix (through the Verlinde Formula) which, in turn, can be obtained in the standard manner[20] from the Drinfeld double construction.

In general, for topological phases which are Drinfeld doubles of a finite group, the number of topological sectors on the torus corresponds precisely to the number of commuting pairs of gauge inequivalent magnetic fluxes through the two cycles of the torus (Fig. 3a). (The two fluxes need to commute in order to keep the system in its ground state.)

It is important to note that the statistics of the excitations is only well-defined in the disordered liquid phase (described by pure Q_8 gauge theory). The ordered phase with a $\mathcal{Z}$-condensate has gapless Goldstone modes, which make adiabatic braiding operations impossible. Another important difference between ordered and disordered phase is that these non-Abelian defects have logarithmic divergent energy in the ordered phase, while in the disordered phase they all have finite energy.

Summary: In this work we studied a fully gapped topological spin liquid state with non-Abelian excitations. Despite its complicated effective model description, we propose that such state can be realized by disordering a rather simple spin order of a spin-1 quantum SU(2) magnet.

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[24] Different such quadrupolar spin order can exist on the three sublattices of the triangular lattice
[25] The non-Abelian vortices of biaxial nematic order were also discussed in the context of liquid crystals [22].
[26] Note that by definition $h_{i,\mu} = h_{i+\mu,-\mu}$.