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Large-Scale Magnetic Field Generation by Randomly Forced Shearing Waves

T. Heinemann,[1] J. C. McWilliams,[2] and A. A. Schekochihin[3]

[1] *Institute for Advanced Study, Princeton, New Jersey 08540, USA*
[2] *Department of Atmospheric Sciences, UCLA, Los Angeles, California 90095-1565, USA*
[3] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3NP, United Kingdom*

A rigorous theory for the generation of a large-scale magnetic field by random non-helically forced motions of a conducting fluid combined with a linear shear is presented in the analytically tractable limit of low Rm and weak shear. The dynamo is kinematic and due to fluctuations in the net (volume-averaged) electromotive force. This is a minimal proof-of-concept quasilinear calculation aiming to put the shear dynamo, a new effect recently found in numerical experiments, on a firm theoretical footing. Numerically observed scalings of the wavenumber and growth rate of the fastest growing mode, previously not understood, are derived analytically. The simplicity of the model suggests that shear dynamo action may be a generic property of sheared magnetohydrodynamic turbulence.

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Introduction. Magnetogenesis, or origin of cosmic magnetism, is one of the fundamental problems in theoretical astrophysics. It has long been believed that the magnetic fields observed in most astrophysical bodies owe their existence to the dynamo effect associated with the turbulence of the constituent plasmas. It is not controversial that turbulence of a conducting fluid amplifies magnetic fluctuations at scales comparable to or smaller than the scale of the motions. Small-scale magnetic fluctuations are indeed observed ubiquitously, but in most astrophysical systems, one also finds magnetic fields coherent on scales larger than the scale of the turbulence (e.g., [1]). Generation of such fields, or mean-field dynamo action, is expected to require a combined action of turbulence and some large-scale-coherent feature. One well-known such additional ingredient is net kinetic helicity (or, more generally, reflectional asymmetry) of the motion. Under certain conditions, its presence can cause growth of large-scale ("mean") magnetic field, known as the α-effect [2]. While deriving the α-effect for realistic turbulent systems requires rather drastic closure assumptions, which usually cannot be justified rigorously and have, in fact, been called into question by numerical and analytical considerations [3], it is at least clear that the effect exists in the physically realizable and analytically treatable limit of low Rm [2, 4]. This proof-of-concept analytical result, together with intuitive arguments [5] and a body of numerical evidence [6, 7], have helped build a case for the α-effect as a real physical phenomenon (although whether it can coexist with the small-scale dynamo at large Rm is far from certain [3]).

It has been suggested [8–12] that even in the absence of *mean* helicity, mean-field dynamo action is possible if a large-scale velocity shear is present. The importance of such a possibility can hardly be overestimated, as shear is a ubiquitous feature in astrophysics (usually associated with differential rotation). A recent numerical study [13, 14] showed that the shear dynamo does exist, but its nature has remained poorly understood. The uncertainty is increased by the fact that, while the original derivation of the effect relied on a quantitative outcome of a closure calculation [11], the effect proved difficult to identify by numerical computation of the mean-field-theory coefficients [15] and appeared to go away in rigorously solvable limits: the white-noise-velocity model and low-Rm magnetohydrodynamics [4, 16–18] (but see [19]).

In this Letter, our aim is a minimal proof-of-concept calculation that puts the shear dynamo effect on a firm theoretical footing akin to that enjoyed by the α-effect. We propose a very simple quasilinear mean-field theory that rigorously predicts a large-scale dynamo driven by randomly forced shearing waves in the limit of $\mathrm{Rm} \ll \mathrm{Re} \ll 1$. The effect requires no adjustable parameters. We also recover the scalings of the wavenumber and growth rate of the fastest-growing mode that were observed in a number of numerical studies [13, 14, 20] but have not so far been explained analytically.

Shearing Waves. First let us introduce a model velocity field that will be used to obtain a dynamo. Consider an incompressible fluid with an imposed background linear shear, $\boldsymbol{U} = Sx\boldsymbol{e}_y$, and assume that the magnetic field is dynamically weak, so the Lorentz force is negligible. Then the velocity deviation from $\boldsymbol{U}$ satisfies

$$\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + Su_x \boldsymbol{e}_y + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \quad (1)$$

where p is pressure determined from incompressibility $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, ν is viscosity and $\boldsymbol{f}$ is a random body force, assumed to be statistically homogeneous in time and space and to have a characteristic scale ℓ_f.

We now make two simplifying assumptions. First, let $\mathrm{Re} \sim u_{\mathrm{rms}}\ell_f/\nu \ll 1$, so we can neglect the nonlinear term in Eq. (1). Second, let $\partial_z \boldsymbol{u} = 0$ and $\partial_z \boldsymbol{f} = 0$, resulting in a "quasi-2D" velocity with all three vector components but no z-dependence. This velocity will make our calculations particularly transparent. As indicated by numerical experiments [21], it is a favorable but not a uniquely special case as a dynamo. The xy-plane velocity now has a stream function: $\boldsymbol{u}^\perp = \boldsymbol{e}_z \times \boldsymbol{\nabla}\Phi$. Similarly, $\boldsymbol{f}^\perp = \boldsymbol{e}_z \times \boldsymbol{\nabla} F$. We seek solutions of (the linearized) Eq. (1) as superpositions of "shearing waves" [22]:

$$\Phi = \sum_{\boldsymbol{k}_0} \Phi_{\boldsymbol{k}_0}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}}, \quad u_z = \sum_{\boldsymbol{k}_0} u_{z\boldsymbol{k}_0}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}}, \quad (2)$$

where $\boldsymbol{k}_0 = k_{x0}\boldsymbol{e}_x + k_y\boldsymbol{e}_y$, $\boldsymbol{k}(t) = (k_{x0} - Stk_y)\boldsymbol{e}_x + k_y\boldsymbol{e}_y$. The amplitudes of the shearing waves satisfy [23]

$$\partial_t \left[k^2(t)\Phi_{\boldsymbol{k}_0}\right] = -\nu k^4(t)\Phi_{\boldsymbol{k}_0} + k^2(t) F_{\boldsymbol{k}_0}, \quad (3)$$

$$\partial_t u_{z\boldsymbol{k}_0} = -\nu k^2(t) u_{z\boldsymbol{k}_0} + f_{z\boldsymbol{k}_0}. \quad (4)$$

Eq. (3) was obtained by taking $\boldsymbol{e}_z \cdot [\boldsymbol{\nabla} \times \text{Eq. (1)}]$. For simplicity, let us consider the forcing to be white in time (or, equivalently, to have a correlation time much shorter than the viscous relaxation time ℓ_f^2/ν). Then the two-point velocity correlators are

$$\left\langle \Phi_{\boldsymbol{k}_0}(t)\Phi^*_{\boldsymbol{k}'_0}(t')\right\rangle = \delta_{\boldsymbol{k}_0,\boldsymbol{k}'_0} G_\nu(t,t')\frac{k^2(t')}{k^2(t)}\left\langle |\Phi_{\boldsymbol{k}_0}(t')|^2\right\rangle, \quad (5)$$

$$\left\langle u_{z\boldsymbol{k}_0}(t) u^*_{z\boldsymbol{k}'_0}(t')\right\rangle = \delta_{\boldsymbol{k}_0,\boldsymbol{k}'_0} G_\nu(t,t')\left\langle |u_{z\boldsymbol{k}_0}(t')|^2\right\rangle, \quad (6)$$

where $G_\nu(t,t') = \exp\left[-\nu\int_{t'}^t \mathrm{d}t'' k^2(t'')\right]$. Thus, the correlation time of our velocity field is $\tau_c \sim \ell_f^2/\nu$. In a more general case when Re is not small, the velocity correlation time is set by the nonlinear terms, so $\tau_c \sim \ell_f/u_{\rm rms}$ is the typical turnover time of the turbulence. Non-rigorously, this case is included in our analysis. To accommodate it, we introduce the Strouhal number $\mathrm{St} \sim u_{\rm rms}\tau_c/\ell_f$ (following [4]) — then $\mathrm{St} \sim \mathrm{Re}$ for a velocity governed by Eqs. (3) and (4), and $\mathrm{St} \sim 1$ for conventional turbulence.

An important quantity to watch is the net (volume-, but not time-, averaged) helicity

$$\mathcal{H}(t) = \langle \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle_{xy} = -2\sum_{\boldsymbol{k}_0} k^2(t) u_{z\boldsymbol{k}_0}(t)\Phi^*_{\boldsymbol{k}_0}(t). \quad (7)$$

We can ensure that its statistical (or, equivalently, time) average vanishes, $\langle\mathcal{H}(t)\rangle = 0$, by stipulating $\left\langle f_{z\boldsymbol{k}_0}(t) F^*_{\boldsymbol{k}_0}(t')\right\rangle = 0$. This removes the possibility of the standard α-effect [2, 4].

Mean-Field Theory. The evolution equation for the magnetic field $\boldsymbol{B}$ in the presence of linear shear is

$$\partial_t\boldsymbol{B} + Sx\partial_y\boldsymbol{B} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} + SB_x\boldsymbol{e}_y + \eta\nabla^2\boldsymbol{B}, \quad (8)$$

where η is the magnetic diffusivity. Since the velocity field is independent of z, we can separate the dependence of $\boldsymbol{B}$ on z by expanding $\boldsymbol{B} = \sum_{k_z}\boldsymbol{B}(k_z)\exp(\mathrm{i}k_z z)$. Only the projection $\boldsymbol{B}^\perp$ onto the xy-plane needs to be calculated because $B_z = (\mathrm{i}/k_z)\boldsymbol{\nabla}\cdot\boldsymbol{B}^\perp$. For each k_z, $\boldsymbol{B}^\perp$ will satisfy a closed equation with k_z appearing as a parameter and no mode coupling in k_z.

We now seek the solutions of Eq. (8) again in the form of a superposition of shearing waves, $\boldsymbol{B}^\perp = \sum_{\boldsymbol{k}_0}\boldsymbol{B}^\perp_{\boldsymbol{k}_0}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{r}}$, where the perpendicular wave numbers $\boldsymbol{k}_0$ and $\boldsymbol{k}(t)$ are defined in the same way as in the velocity decomposition [Eq. (2)]. $\boldsymbol{B}^\perp_{\boldsymbol{k}_0}$ satisfies

$$\begin{aligned}\partial_t\boldsymbol{B}^\perp_{\boldsymbol{k}_0} &= SB_{x\boldsymbol{k}_0}\boldsymbol{e}_y - \eta\left[k^2(t) + k_z^2\right]\boldsymbol{B}^\perp_{\boldsymbol{k}_0} \\ &+ \sum_{\boldsymbol{k}'_0}\Phi_{\boldsymbol{k}'_0}[\boldsymbol{e}_z\times\boldsymbol{k}'(t)]\cdot[\boldsymbol{k}(t)\hat{\mathbf{I}} - \hat{\mathbf{I}}\boldsymbol{k}'(t)]\cdot\boldsymbol{B}^\perp_{\boldsymbol{k}_0-\boldsymbol{k}'_0} \\ &- \mathrm{i}k_z\sum_{\boldsymbol{k}'_0} u_{z\boldsymbol{k}'_0}\boldsymbol{B}^\perp_{\boldsymbol{k}_0-\boldsymbol{k}'_0}, \end{aligned} \quad (9)$$

where $\hat{\mathbf{I}}$ is a unit dyadic. We take the large-scale mean field to be the xy-average of the total magnetic field, i.e. $\bar{\boldsymbol{B}} = \boldsymbol{B}^\perp_0$. The dynamical equation for the mean field is given by the $\boldsymbol{k}_0 = 0$ component of Eq. (9):

$$\begin{aligned}\partial_t\bar{\boldsymbol{B}} = S\bar{B}_x\boldsymbol{e}_y - \eta k_z^2\bar{\boldsymbol{B}} - \mathrm{i}k_z\sum_{\boldsymbol{k}_0} u_{z\boldsymbol{k}_0}\boldsymbol{B}^\perp_{-\boldsymbol{k}_0} \\ - \sum_{\boldsymbol{k}_0}\Phi_{\boldsymbol{k}_0}\left[\boldsymbol{e}_z\times\boldsymbol{k}(t)\right]\boldsymbol{k}(t)\cdot\boldsymbol{B}^\perp_{-\boldsymbol{k}_0}\end{aligned} \quad (10)$$

(note that $\bar{B}_z = 0$ because $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{B}} = \mathrm{i}k_z\bar{B}_z = 0$).

We now calculate $\boldsymbol{B}^\perp_{-\boldsymbol{k}_0}$ in Eq. (10) in terms of $\bar{\boldsymbol{B}}$ via Eq. (9). This is particularly easy in the limit $\mathrm{Rm} \ll \min(1, \mathrm{St}, \mathrm{Sh}^{-1})$, where $\mathrm{Rm} \sim u_{\rm rms}\ell_f/\eta$ and $\mathrm{Sh} \sim S\ell_f/u_{\rm rms}$. We also assume $k_z\ell_f \ll 1$, which will be verified *a posteriori* for the fastest growing dynamo mode. With these approximations, the dominant terms in Eq. (9) are $\eta k^2(t)\boldsymbol{B}^\perp_{\boldsymbol{k}_0}$ and the $\boldsymbol{k}'_0 = \boldsymbol{k}_0$ components of the wavenumber sums, giving

$$\boldsymbol{B}^\perp_{-\boldsymbol{k}_0} = -\frac{\mathrm{i}k_z u^*_{z\boldsymbol{k}_0}\bar{\boldsymbol{B}} + \Phi^*_{\boldsymbol{k}_0}\left[\boldsymbol{e}_z\times\boldsymbol{k}(t)\right]\boldsymbol{k}(t)\cdot\bar{\boldsymbol{B}}}{\eta k^2(t)}. \quad (11)$$

Substituting this into Eq. (10), we get

$$\partial_t\bar{\boldsymbol{B}} = S\bar{B}_x\boldsymbol{e}_y - [\eta + \beta(t)]k_z^2\bar{\boldsymbol{B}} + \mathrm{i}k_z\boldsymbol{e}_z\times\hat{\boldsymbol{\alpha}}(t)\cdot\bar{\boldsymbol{B}}, \quad (12)$$

where $\beta(t) = \sum_{\boldsymbol{k}_0}|u_{z\boldsymbol{k}_0}(t)|^2/\eta k^2(t) \ll \eta$ (negligible "turbulent diffusivity" in the limit of low Rm) and

$$\hat{\boldsymbol{\alpha}}(t) = 2\sum_{\boldsymbol{k}_0}\frac{u_{z\boldsymbol{k}_0}(t)\Phi^*_{\boldsymbol{k}_0}(t)}{\eta k^2(t)}\boldsymbol{k}(t)\boldsymbol{k}(t). \quad (13)$$

Eq. (12) has the form of a standard mean-field equation [2] with mean electromotive force $\boldsymbol{\mathcal{E}} = \hat{\boldsymbol{\alpha}}\cdot\bar{\boldsymbol{B}} - \mathrm{i}k_z\beta\,\boldsymbol{e}_z\times\bar{\boldsymbol{B}}$, but it is of stochastic nature: $\hat{\boldsymbol{\alpha}}(t)$ and $\beta(t)$ fluctuate with the correlation time τ_c of the velocity field. Note that $\langle\hat{\boldsymbol{\alpha}}(t)\rangle = 0$ because we have constructed our velocity field in such a way that $\langle u_{z\boldsymbol{k}_0}(t)\Phi^*_{\boldsymbol{k}_0}(t)\rangle = 0$ (cf. [8, 15, 19, 24]).

If we now average Eq. (12) over forcing realizations and look for exponential growth of $\langle\bar{\boldsymbol{B}}(t)\rangle$, we will find that, under the approximations we have made, no such growth occurs to the lowest order in the standard cumulant expansion [25] used to calculate $\langle\hat{\boldsymbol{\alpha}}(t)\cdot\bar{\boldsymbol{B}}(t)\rangle$ [26].

While it is possible that the mean field grows at a higher order in the expansion, the key question that needs to be addressed at lowest order is, in fact, not necessarily whether the statistical average of the large-scale field $\bar{\boldsymbol{B}}(t)$ exhibits exponential growth, but whether the mean large-scale magnetic energy $\langle|\bar{\boldsymbol{B}}(t)|^2\rangle/2$ does.

Large-Scale Energy. In order to address this last question, we introduce the mean-field covariance vector

$$\boldsymbol{C} = (\bar{B}_x^*\bar{B}_x, \bar{B}_y^*\bar{B}_y, \Re\bar{B}_x^*\bar{B}_y, \Im\bar{B}_x^*\bar{B}_y), \tag{14}$$

where $\Re$ and $\Im$ denote real and imaginary parts. The evolution equation for $\boldsymbol{C}$ follows directly from Eq. (12):

$$(\partial_t + 2\eta k_z^2)\boldsymbol{C} = (\hat{\boldsymbol{S}} + k_z\hat{\boldsymbol{A}})\cdot\boldsymbol{C}, \tag{15}$$

where we have introduced the matrices

$$\hat{\boldsymbol{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2S & 0 \\ S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{A}} = \begin{bmatrix} 0 & 0 & 0 & 2\alpha_{yy} \\ 0 & 0 & 0 & 2\alpha_{xx} \\ 0 & 0 & 0 & -2\alpha_{xy} \\ \alpha_{xx} & \alpha_{yy} & 2\alpha_{xy} & 0 \end{bmatrix}. \tag{16}$$

In the following, we will use arabic numerals to refer to the components of the vectors and matrices in Eq. (15).

We now average Eq. (15) with respect to forcing realizations using the cumulant expansion [25] to calculate $\langle\hat{\boldsymbol{A}}(t)\cdot\boldsymbol{C}(t)\rangle$. Since the Kubo number $\mathrm{Ku}\sim k_z\hat{\boldsymbol{A}}\tau_c\sim k_z\ell_f\mathrm{Rm}\,\mathrm{St}\ll 1$, the expansion can be truncated at the lowest order in Ku. The result is that $\langle\boldsymbol{C}(t)\rangle$ satisfies, for $t\gg\tau_c$,

$$(\partial_t + 2\eta k_z^2)\langle\boldsymbol{C}\rangle = (\hat{\boldsymbol{S}} + k_z^2\hat{\boldsymbol{D}})\cdot\langle\boldsymbol{C}\rangle, \tag{17}$$

where the term originating from $\hat{\boldsymbol{\alpha}}$ now has the form of a (negative) tensor diffusivity

$$\hat{\boldsymbol{D}} = \int_0^\infty \mathrm{d}t'\langle\hat{\boldsymbol{A}}(t)\cdot\hat{\boldsymbol{A}}(t-t')\rangle. \tag{18}$$

We have also assumed $S\tau_c\ll 1$, which allowed us to neglect the matrix exponentials of $\hat{\boldsymbol{S}}t'$ in Eq. (18).

We note that $\hat{\boldsymbol{D}}$ is block diagonal: its elements are zero where those of $\hat{\boldsymbol{A}}$ are not, and vice versa. It follows that $\langle C_4\rangle = \langle\Im\bar{B}_x^*\bar{B}_y\rangle$ evolves independently of the other components of $\langle\boldsymbol{C}\rangle$:

$$\partial_t\langle C_4\rangle = -k_z^2(2\eta - D_{44})\langle C_4\rangle. \tag{19}$$

Since $D_{44}/\eta\sim\mathrm{Rm}^3\mathrm{St}\ll 1$ [27], we conclude that C_4 always decays, which means that $\bar{B}_x$ and $\bar{B}_y$ asymptotically have the same complex phase.

With $C_4 = 0$, we are left with a rank-three eigenvalue problem. If we let $\langle\boldsymbol{C}\rangle\propto\exp(2\gamma t)$, the resulting dispersion relation will be a cubic equation in $\gamma+\eta k_z^2$. This equation can be solved perturbatively in the limit $k_z^2\hat{\boldsymbol{D}}\sim k_z^2\hat{\boldsymbol{A}}^2\tau_c\ll S$ or, equivalently, $(k_z\ell_f)^2\,\mathrm{Rm}^2\,\mathrm{St}\,\mathrm{Sh}^{-1}\ll 1$. In the end, this means that the only element of the tensor $\hat{\boldsymbol{D}}$ that survives to give a non-negligible contribution is

$$D_{12} = 2\int_0^\infty \mathrm{d}t'\big\langle\alpha_{yy}(t)\alpha_{yy}(t-t')\big\rangle, \tag{20}$$

where $t\gg\tau_c$. Then γ satisfies

$$(\gamma+\eta k_z^2)^3 - \frac{k_z^2 S^2 D_{12}}{4} = 0 \tag{21}$$

and so, assuming that $D_{12} > 0$, the real root of this equation gives the dynamo growth rate [28]

$$\gamma = -\eta k_z^2 + \left(\frac{k_z^2 S^2 D_{12}}{4}\right)^{1/3}. \tag{22}$$

The vertical wave number and the growth rate of the fastest growing mode are

$$k_z^{\mathrm{pk}} = \frac{|S|^{1/2}}{\sqrt{2}}\left(\frac{D_{12}}{27\eta^3}\right)^{1/4} \sim \ell_f^{-1}\mathrm{Sh}^{1/2}\,\mathrm{St}^{1/4}\,\mathrm{Rm}^{5/4} \tag{23}$$

(confirming $k_z^{\mathrm{pk}}\ell_f\ll 1$) and

$$\gamma_{\max} = \frac{|S|}{3}\left(\frac{D_{12}}{3\eta}\right)^{1/2} \sim \frac{u_{\mathrm{rms}}}{\ell_f}\,\mathrm{Sh}\,\mathrm{St}^{1/2}\,\mathrm{Rm}^{3/2}. \tag{24}$$

The structure of this mode is such that

$$\frac{\langle|\bar{B}_x|^2\rangle}{\langle|\bar{B}_y|^2\rangle} = \frac{D_{12}}{6\eta} \sim \mathrm{St}\,\mathrm{Rm}^3, \tag{25}$$

which is independent of shear.

Discussion. The scalings derived above, viz., $k_z^{\mathrm{pk}}\propto S^{1/2}$, $\gamma_{\max}\propto S$, and the independence of $\langle|\bar{B}_x|^2\rangle/\langle|\bar{B}_y|^2\rangle$ of S, are precisely the ones reported in the numerical experiments [13, 14, 20]. One should keep in mind that most of these simulations were not done in the asymptotic regime $\mathrm{Rm}\ll\min\left(1,\mathrm{St},\mathrm{Sh}^{-1}\right)$ or at particularly small $S\tau_c$. The fact that the scalings we have derived nevertheless appear to hold even for parameter values at the boundary of the analytically tractable regime might be interpreted as a testimony to the robustness of the underlying physical effect [29]. Indeed, we note that Eq. (22) rigorously holds for any "quasi-2D" velocity field superimposed on a uniform shear flow with the only provisos that it has a well defined characteristic length scale, a correlation time much shorter than the inverse rate of shear, and the property that $\langle u_{z\boldsymbol{k}_0}\Phi_{\boldsymbol{k}_0}^*\rangle = 0$. Our theory shows that such a velocity field is always capable of dynamo action provided sufficiently large scales in the z-direction are accessible to the mean field (i.e., provided the system is large enough). A field of randomly forced shearing waves at low Re, given by Eqs. (3) and (4), is a physically realizable example of such a velocity field. For this field, using Eqs. (5), (6), (13), (20), and $S\tau_c\ll 1$, we get

$$D_{12} = 4\sum_{\boldsymbol{k}_0} k_y^4\frac{\big\langle|\Phi_{\boldsymbol{k}_0}(t)|^2\big\rangle\big\langle|u_{z\boldsymbol{k}_0}(t)|^2\big\rangle}{\nu\eta^2 k^6(t)}, \tag{26}$$

which is positive, as assumed in Eq. (22).

The key ingredient in the dynamo loop are fluctuations in the $\hat{\boldsymbol{\alpha}}$-tensor (13), which, in conjunction with stretching of the mean field by the background shear flow, provide a positive feedback [30]. This is evocative of the dynamo models known as the "stochastic α-effect", which are based on introducing a fluctuating scalar α_{yy} [8, 19, 24] — this has usually been done based on *ad hoc* non-rigorous models of how this α comes about. The theory we have presented here is the first calculation of this kind done from first principles.

Conclusion. We have presented a minimal analytically tractable model of the shear dynamo. The simplicity of the model suggests that the effect is robust, while its rigorous validity in the realizable limit of low Rm, weak shear and for a velocity field consisting of randomly forced shearing waves at low Re suggests that it is physical and does not depend on *ad hoc* closure assumptions. Much remains to be understood before it can be assessed whether the shear dynamo offers a panacea for (non-helical) generation of large-scale magnetic fields in astrophysical systems. A further effort in this direction appears worthwhile in view of the great success enjoyed by shear-induced dynamos in astrophysically motivated numerical experiments and a basic similarity of the field structure that they generate [13, 14, 20, 31]. A companion paper on the quasilinear elemental shear dynamo, exploring broader parameter regimes, is [32]. A major outstanding task is to understand how the shear dynamo mechanism of generating large-scale fields coexists with the fluctuation dynamo of small-scale fields, which will inevitably be present at sufficiently large Rm [14] and, therefore, in any real astrophysical situation (cf. [3]).

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[27] In fact, it is not hard to prove, using Eq. (13), then Eqs. (5) and (6), that $D_{44} = 0$ exactly.

[28] The non-resistive part of the dynamo growth rate (22) can also be obtained via heuristic arguments given in [8], see their Eq. (13).

[29] Note that while the distinction between the growth of the large-scale magnetic *vector* field and of the large-scale magnetic *energy* was not appreciated by [13, 14], their numerical results on the dependence of the field's growth rate and vertical scale on S all referred to the root-mean-square large-scale field integrated over the numerical box and so a comparison with our predictions for the large-scale energy is appropriate. As the vector large-scale field at any given z randomly changed sign in their simulations on timescales of order a few growth times, it is plausible that its long-term time average was indeed zero, as suggested by our theory, although this was not checked at the time. A more extensive investigation of the feasibility of mean-field growth in broader parameter regimes than attempted here can be found in [32].

[30] Comparing Eqs. (7) and (13), we see that fluctuations in $\hat{\boldsymbol{\alpha}}$ are related to fluctuations in kinetic helicity, although, for a general multiscale velocity field, the correspondence

is by no means exact (cf. [33]).
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