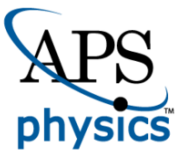

Nonperturbative Gadget for Topological Quantum Codes

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Non-perturbative gadget for topological quantum codes

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Many-body entangled systems, in particular topologically ordered spin systems proposed as resources for quantum information processing tasks, often involve highly non-local interaction terms. While one may approximate such systems through two-body interactions perturbatively, these approaches have a number of drawbacks in practice. Here, we propose a scheme to simulate many-body spin Hamiltonians with two-body Hamiltonians *non-perturbatively*. Unlike previous approaches, our Hamiltonians are not only exactly solvable with exact ground state degeneracy, but also support completely localized quasi-particle excitations, which are ideal for quantum information processing tasks. Our construction is limited to simulating the toric code and quantum double models, but generalizations to other non-local spin Hamiltonians may be possible.

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Many-body entanglement arising in strongly correlated systems is a very promising resource for realizing various ideas in quantum information, such as quantum communication and quantum computation. In particular, topologically ordered spin systems can be employed for reliable storage of quantum information inside the degenerate ground space [1] and for fault-tolerant quantum computation with non-abelian anyonic excitations [2]. These topological approaches may resolve many problems in quantum information science; qubits are encoded in many-body entangled states and are thus naturally protected from decoherence.

Unfortunately, topologically ordered spin systems capable of quantum information processing are very difficult to realize physically. Many proposed topologically ordered spin Hamiltonians, such as the toric code, quantum double model [2], and string-net model [3], involve highly non-local interaction terms; this is a stark contrast to Hamiltonians which occur in nature, which have only geometrically local two-body interactions. Moreover, the resource systems above are known *not* to be supported by any two-body Hamiltonian [4].

Many efforts have been made to construct two-body Hamiltonians which "approximate" non-local resource Hamiltonians. The most commonly used approach is to approximate target Hamiltonians through so-called "perturbative gadgets" [5–10]. The central idea of perturbative gadgets is to design a two-body Hamiltonian whose leading perturbative contribution gives rise to the desired many-body Hamiltonian; unfortunately, most obtained two-body Hamiltonians are not exactly solvable, and their properties are hard to determine except for a few exactly solvable examples [11, 12]. In addition, the perturbative Hamiltonian only approximates the target Hamiltonian, and may give a very weak effective Hamiltonian with a rather small energy gap. Furthermore, quasi-particle excitations (energy eigenstates) arising in perturbative Hamiltonians cannot be created through completely localized manipulations of spins; excitations are always delocalized and the ground state degeneracy might be split for finite system sizes, resulting in fatal errors in practice. While a non-perturbative approach based on the PEPS formalism was developed for simulating the cluster state for measurement-based quantum computation [13], such an approach may not be applicable to degenerate systems with topological order.

Here, we propose a scheme to simulate topologically ordered Hamiltonians through two-body interactions *non-perturbatively*. Our scheme builds on previously established ideas in perturbative gadget studies, such as the use of hopping particles proposed by König [8], and the encoding of single particles into multiple particles used by Brell *et al* [10]. Combining these remarkable insights, we are able to construct the first topologically ordered spin system which satisfies the following; 1) The Hamiltonian has at most two-body, geometrically local interactions. 2) The Hamiltonian has exactly solvable ground states and low-energy excitations, and is provably gapped for all system sizes. 3) The ground space of the Hamiltonian is exactly connected to that of the target Hamiltonian through local unitary transformations, and anyonic excitations are completely localized.

For clarity of presentation, we illustrate the gadget construction for the toric code. A generalization to the quantum double model is also possible [14].

Modified toric code—We begin by defining a modified version of the toric code, also known as the Z_2 lattice gauge model, that we will simulate through a two-body Hamiltonian. Consider a system of qubits defined on edges of a square lattice with periodic boundary conditions. Unlike the conventional toric code, *two* qubits live on each edge in our construction (see Fig. 1(a)), governed by the following Hamiltonian

$$H = -J\sum_s A_s - J\sum_p B_p - J\sum_e C_e$$
$$A_s = \prod_{j\in s} X_j, \quad B_p = \prod_{j\in p} Z_j, \quad C_e = \prod_{j\in e} Z_j,$$

where s, p and e represent "star", "plaquette" and "edge" respectively, as defined in Fig. 1(b)(c)(d). X_j and Z_j

are Pauli X and Z operators on qubit j, and J is some positive constant. The model is exactly solvable since

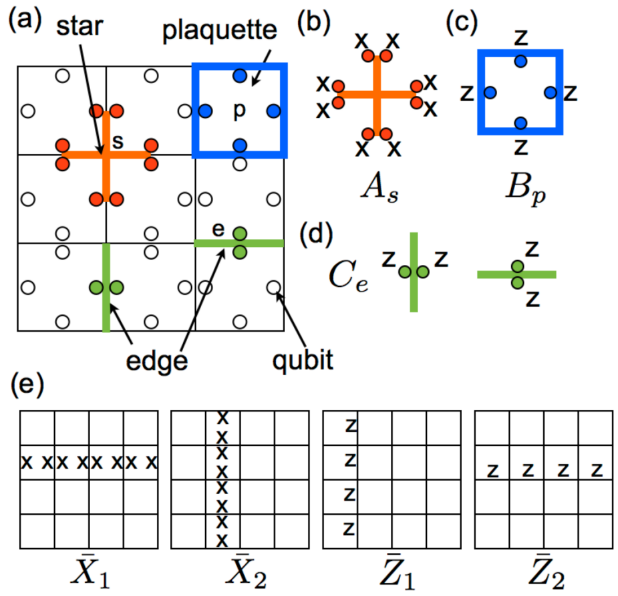

FIG. 1. (a) Construction of the modified toric code. Dots represent qubits. (b) A star term A_s (red online). (c) A plaquette term B_p (blue online). (d) An edge term C_e (green online). (e) Two pairs of logical operators.

interaction terms commute with each other, and it can be considered to be a stabilizer code. The model has four degenerate ground states, as in the toric code. Inside of the ground space, $A_s = B_p = C_e = 1$, meaning $A_s|\psi\rangle = B_p|\psi\rangle = C_e|\psi\rangle = |\psi\rangle$ for all s, p and e when $|\psi\rangle$ is a ground state. Notice that one can create the toric code from this model by applying controlled-NOT gates between pairs of qubits on each edge. Since the toric code and the modified model are connected through local unitary transformations, they are considered to be in the same quantum phase [15, 16]. The ground space of the modified toric code has a four-fold degeneracy, as seen by writing down two pairs of "logical operators" which commute with the Hamiltonian but anti-commute with each other (see Fig. 1(e)). The non-locality of logical operators makes the model of great interest for robust storage of quantum information.

As a first step towards obtaining a two-body Hamiltonian simulating this modified toric code, we group the four qubits in each plaquette into a single composite particle with a 16-dimensional space. While B_p becomes one-body, and C_e is two-body through this grouping, the star term A_s is only reduced to four-body. Below, we provide a scheme to simulate A_s through only two-body terms.

Gadget Hamiltonian– The central idea behind our construction is to add a "gadget particle" at each star (see Fig. 2(a)). The gadget particle has four possible spin values $m_s = 0, 1, 2, 3$. We replace the four-body star term A_s with two-body terms H_{hop} and H_{shield} which involve the gadget particles:

$$H_{gadget} = H_p + H_e + H_{hop} + H_{shield}$$
$$H_p = -J\sum_p B_p, \quad H_e = -J\sum_e C_e. \tag{1}$$

The hopping term is $H_{hop} = \sum_s h_s$ where

$$h_s = -U|m_s = 0\rangle\langle m_s = 0| - t\left(D_s^\dagger + D_s\right)$$
$$D_s^\dagger = \sum_{m_s=0,1,2,3} |m_s + 1\rangle\langle m_s| \otimes A_s(m_s) \pmod 4,$$

where U and t are some positive constants, and m_s represents the spin value of the gadget particle at s. Terms $A_s(m)$ are products of two Pauli X operators as depicted in Fig. 2(b). Since $A_s(m)$ are one-body operators when qubits in a plaquette are viewed as a composite particle, hopping terms are two-body. This hopping term will effectively induce star terms A_s since $A_s = (D_s^\dagger)^4$. The

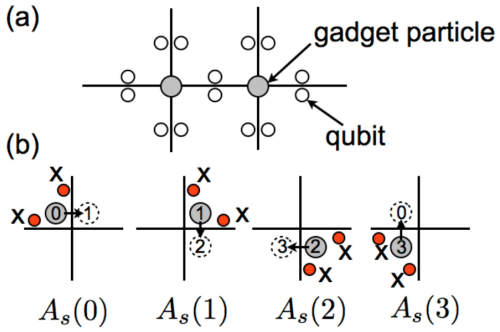

FIG. 2. Construction of the hopping term $D_s^\dagger$. (a) Gadget particles at stars. (b) Terms $A_s(m)$ that are tensor products of two Pauli X operators. Each term acts on two qubits (red online), depending on spin values of gadget particles.

shielding term H_{shield} consists of two-body interactions between gadget particles:

$$H_{shield} = J\sum_s T_\ell(m_s)T_r(m_{s+\hat{x}}) + T_d(m_s)T_u(m_{s+\hat{y}})$$

where $s+\hat{x}$ and $s+\hat{y}$ are unit translations of a "star" s in the horizontal and vertical directions, and

$$T_\ell(m) = 1 - 2\delta_{m,2}, \quad T_r(m) = 2\delta_{m,0} - 1$$
$$T_d(m) = 1 - 2\delta_{m,3}, \quad T_u(m) = 1 - 2\delta_{m,3}$$

with $\delta_{m,m'} = 1$ for $m = m'$ and 0 otherwise. As we will see below, this choice of the shielding term decouples effective interactions between neighboring gadget particles, and makes the model exactly solvable.

Decomposition into subspaces—Now, we solve the gadget Hamiltonian in Eq. (1). It is convenient to decompose the entire Hilbert space into subspaces. Let us denote computational basis states whose gadget values are all $|0\rangle$s: $|\psi(\vec{d})\rangle = |\vec{0}\rangle_{gadget} \otimes |\vec{d}\rangle_{qubit}$ where $|\vec{d}\rangle_{qubit}$ repre-

sents spin values $|d_j\rangle$ for qubits, and $|\vec{0}\rangle_{qubit}$ means that all the qubits are $|0\rangle$s. We define the subspace $\mathcal{M}(\vec{d})$ such that it is spanned by all the states which can be reached from $|\psi(\vec{d})\rangle$ by applying $D_s^\dagger$:

$$\mathcal{M}(\vec{d}) = \big\langle \prod_s (D_s^\dagger)^{\lambda_s} |\psi(\vec{d})\rangle, \quad \text{for all } \lambda_s \big\rangle.$$

We can verify that $\mathcal{M}(\vec{d})$ is an invariant subspace of H_{gadget}. Then, one can solve the gadget Hamiltonian inside each subspace $\mathcal{M}(\vec{d})$ independently.

Ground state subspace— We will first solve for the ground state inside $\mathcal{M}(\vec{0})$, and then will show its lowest energy state to be a ground state. We note that inside $\mathcal{M}(\vec{0})$ $B_p = 1$, and thus plaquette terms need not be considered. Denoting the total number of stars as N, we may view $\mathcal{M}(\vec{0})$ as the Hilbert space of N particles.

$$|\vec{\lambda}\rangle = \bigotimes_s |\lambda_s\rangle = \prod_s (D_s^\dagger)^{\lambda_s} |\psi(\vec{0})\rangle. \tag{2}$$

Noting that $(D_s^\dagger)^4 = A_s, \quad (D_s^\dagger)^8 = I$, these particles can be considered to have eight-dimensional Hilbert spaces, $\lambda_s = 0, \ldots, 7$ [17]. In this "λ-representation", the hopping term H_{hop} can be written as a *one-body* Hamiltonian: $H_{hop} = \sum_s h_s$ where

$$\begin{aligned} h_s = &- U\big(|\lambda_s = 0\rangle\langle\lambda_s = 0| + |\lambda_s = 4\rangle\langle\lambda_s = 4|\big) \\ &- t \sum_{\lambda_s=0}^{7} \big(|\lambda_s + 1\rangle\langle\lambda_s| + h.c\big) \qquad (\text{mod } 8). \end{aligned}$$

However, edge terms C_e are not one-body inside $\mathcal{M}(\vec{0})$.

A key idea behind our gadget arises from the fact that these two-body interactions arising from C_e can be exactly cancelled by adding the shielding term H_{shield}. Inside $\mathcal{M}(\vec{0})$, edge terms have the same action as the following two-body terms involving gadget particles: $C_e = T_\ell(m_s)T_r(m_{s+\hat{x}})$ for a horizontal edge e connecting s and $s+\hat{x}$, and $C_e = T_d(m_s)T_u(m_{s+\hat{y}})$ for a vertical edge e connecting s and $s+\hat{y}$, as one can verify from direct calculations [14]. Then, the edge terms are exactly cancelled: $H_e + H_{shield} = 0$ inside $\mathcal{M}(\vec{0})$. This means the gadget Hamiltonian is one-body in the "λ-representation": $H_{gadget} = \text{const} + \sum_s h_s$.

Because of this, all energy eigenstates inside $\mathcal{M}(\vec{0})$ can be written in the tensor product form $|\vec{\alpha}\rangle = \bigotimes_s |\alpha_s\rangle$ where $|\alpha_s\rangle = \sum_{\lambda_s} \alpha_s(\lambda_s)|\lambda_s\rangle$. The lowest energy state is $|\psi_{GS}(\vec{0})\rangle = \bigotimes_s |\alpha_0\rangle$, where $\alpha_0(\lambda) = \alpha_0(\lambda + 4)$ for all λ. Therefore, returning from the λ-representation, we can write the ground state as

$$\begin{aligned} |\psi_{GS}(\vec{0})\rangle &= \prod_s \sum_{\lambda=0}^{7} \alpha_0(\lambda)(D_s^\dagger)^\lambda |\psi(\vec{0})\rangle \\ &= \prod_s (I + A_s) \sum_{\lambda=0}^{3} \alpha_0(\lambda)(D_s^\dagger)^\lambda |\psi(\vec{0})\rangle. \end{aligned}$$

We see that there is a finite energy gap inside $\mathcal{M}(\vec{0})$, since H_{gadget} acts as a one-body Hamiltonian.

Unitary Connection—This lowest energy state $|\psi_{GS}(\vec{0})\rangle$ is connected to the ground state of the modified toric code through the following local unitary transformation:

$$U = \prod_s U_s, \quad U_s \equiv \sum_{m_s=0}^{3} |m_s\rangle\langle m_s| \prod_{m<m_s} A_s(m). \tag{3}$$

In particular, we have $U|\psi_{GS}(\vec{0})\rangle = |\tilde{\alpha_0}\rangle_{gadget}^{\otimes N} \otimes |\psi_{Toric}(\vec{0})\rangle_{qubit}$ where $|\tilde{\alpha_0}\rangle = \sum_{m=0}^{3} \alpha_0(m)|m\rangle$, and $|\psi_{Toric}(\vec{0})\rangle_{qubit} = \prod_s (I + A_s)|\vec{0}\rangle$ is a ground state of the modified toric code. We may verify that the gadget Hamiltonian has three other ground states $|\psi_{GS}(\vec{d_i})\rangle$, $i = 1, 2, 3$, inside $\mathcal{M}(\vec{d_i})$, connected in the same way to the ground states $|\psi_{Toric}(\vec{d_i})\rangle$ of the modified toric code.

It is then simple to find the logical operators for the gadget Hamiltonian; they are those of the modified toric code conjugated by U: $U^\dagger \bar{X}_1 U$, $U^\dagger \bar{X}_2 U$, $U^\dagger \bar{Z}_1 U$ and $U^\dagger \bar{Z}_2 U$. The ground space is topologically ordered since it meets the criteria for the stability against local perturbations proposed in [18].

Anyonic excitations, which are also energy eigenstates, can be created by applying "segments" of logical operators combined with local operations on gadget particles in a similar way to the conventional toric code. As a result, excitations can be created only through *completely localized* manipulations of spins in small regions. This is in striking contrast to perturbative Hamiltonians where anyonic excitations are *delocalized*, and cannot be created through completely localized manipulations of spins.

Energy gap—Finally, we show that $|\psi_{GS}(\vec{d_i})\rangle$ are the ground states of the gadget Hamiltonian. To do so, we prove that the lowest energy states within other non-ground-state subspaces $\mathcal{M}(\vec{d})$ have a finite higher energy than the lowest energy state within $\mathcal{M}(\vec{0})$.

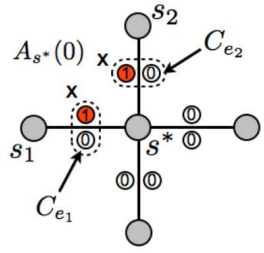

FIG. 3. A non-ground-state subspace $\mathcal{M}(\vec{d})$. $A_{s^*}(0)$ anticommutes with two edge terms C_{e_1} and C_{e_2}.

We first consider a subspace $\mathcal{M}(\vec{d})$ defined by $|\psi(\vec{d})\rangle = A_{s^*}(0)|\psi(\vec{0})\rangle$ where $\vec{d}$ has non-zero components for two qubits acted on by $A_{s^*}(0)$, as shown in Fig. 3. We notice that $A_{s^*}(0)$ commutes with all terms except two edge terms C_{e_1} and C_{e_2}. Therefore, solving H_{gadget} inside

$\mathcal{M}(\vec{d})$ is equivalent to solving

$$A_{s^*}(0)H_{gadget}A_{s^*}(0)^\dagger = H_{gadget} + V$$

inside $\mathcal{M}(\vec{0})$, where $V = 2J(C_{e_1} + C_{e_2})$.

Below, we show that the lowest energy states for $H_{gadget} + V$ inside $\mathcal{M}(\vec{0})$ have finite higher energy than those of H_{gadget} for appropriate choices of parameters U, t and J. For simplicity of discussion, we neglect a constant correction resulting from plaquette term H_p by writing $H_{gadget} = H_{hop} = \sum_s h_s$ inside $\mathcal{M}(\vec{0})$. Then, one may write $H'_{gadget} = \sum_{s\neq\{s^*,s_1,s_2\}} h_s + H^*$ with

$$H^* = \sum_{s=\{s^*,s_1,s_2\}} h_s + 2J(C_{e_1} + C_{e_2})$$

where s_1 and s^* are connected by e_1, and e_2 connects s_2, s^* (Fig. 3).

Returning to the λ-representation, we note that all particles except s^*, s_1, s_2 are non-interacting and are governed under the same Hamiltonian h_s as before. Let us denote the lowest energy eigenvalue of h_s as E_0. Noting that E_0 is upper bounded by $-U$, it suffices to show that $H^* > -3U > 3E_0$ for the existence of an energy gap.

Let $H^* = H_1 + H_2$ where $H_1 = -t\sum_{s=\{s^*,s_1,s_2\}}(D_s^\dagger + D_s)$ and $H_2 = -U\sum_{s=\{s^*,s_1,s_2\}}|m_s = 0\rangle\langle m_s = 0| + 2J(C_{e_1} + C_{e_2})$. Since one cannot minimize H_1 and H_2 simultaneously, we obtain a lower bound for H^* by finding minimal energy eigenvalues for H_1 and H_2 individually. One can verify that $H_1 \geq -6t$ by directly finding energy eigenvalues of H_1. Similarly, one can verify that $H_2 \geq \min(-3U + 4J, -2U - 4J)$. Here, let us choose U and J such that $J = U/8$, and $H_2 \geq -\frac{5}{2}U$. H'_{gadget} has a provably higher ground state energy than H_{gadget} when $H^* > -6t - \frac{5U}{2} > -3U > 3E_0$, so we simply set $U > 12t$. This proof may be easily generalized to arbitrary $\mathcal{M}(\vec{d})$ when $U > 16t$.

A drawback of this proof is that a small value of $t = U/16$ gives a weak constant gap for h_s and thus the gap inside $\mathcal{M}(\vec{0})$ is $\sim 10^{-4}U$. Tighter analysis presented in [14] shows that when $J = 0.09U$, $t = 0.375U$, the system has a quite reasonable energy gap of $> 0.075U$ both inside and outside $\mathcal{M}(\vec{0})$.

Particle dimension— In this construction, a gadget particle is four-dimensional, and a composite particle is eight-dimensional after removing the internal degree of freedom for B_p. This can be improved through defining a similar construction on a triangular lattice, leading to six-dimensional gadget particles and four-dimensional composite particles. In addition, a more elaborate construction reduces the dimension of the gadget particles to three while keeping the dimension of composite particles at four [14].

Discussion— Our gadget construction can be generalized to the quantum double model, which may be universal for topological quantum computation, in a rather straightforward way [14]. We expect that similar generalizations are possible for other interesting, but highly non-local topologically ordered Hamiltonians. In addition, our non-perturbative gadget may find use in adiabatic quantum computation and Hamiltonian complexity problems.

In our construction, we have heavily taken advantage of the fact that the terms being simulated commute. Whether non-commuting terms can be simulated in this way remains an open question. Perhaps insights from related problems in theoretical computer science will prove fruitful, opening new connections.

Conclusion— In this paper, we argue the necessity of simulating topological quantum codes *non-perturbatively*, and propose a model which is two-body, exactly solvable, and supports completely localized quasi-particle excitations. While our construction involves large particle dimensions and "gadgety" interaction terms which put it beyond modern experimental capabilities, there has recently been remarkable theoretical and experimental progress in engineering custom interactions between particles [19–21]. We hope our construction will provide a stepping stone towards physical realizability of topological quantum codes.

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