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Long-Term Cycling of Kozai-Lidov Cycles: Extreme Eccentricities and Inclinations Excited by a Distant Eccentric Perturber

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The very long-term evolution of the hierarchical restricted three-body problem is calculated analytically for high inclinations. The Kozai-Lidov Cycles (KLCs) slowly evolve due to the octupole term in the perturber’s potential and exhibit striking features, including extremely high eccentricities and the generation of retrograde orbits with respect to the perturber. These features were found in recent numerical experiments of the non-restricted three body problem and were attributed inaccurately to the comparable and low masses of the two orbiting companions. Our calculation is done by averaging for the first time the double averaged secular equations of motion over the KLCs and finding a new constant of the motion. These very long-term effects are likely to be important in various astrophysical systems thought to involve KLCs, such as hot Jupiters, irregular moons of planets, tight stellar binaries, mergers of compact objects, and many others.

A test particle on a Keplerian orbit, weakly perturbed by the tidal potential of a distant orbiting mass (restricted, hierarchical three body problem), exhibits variations in eccentricity and inclination on secular time scales, t_{secular} . If the potential is approximated by a quadrupole, the variations are periodic and have been obtained analytically [1, 2]. These Kozai-Lidov Cycles (KLCs) are suggested to play an important role in the formation and evolution of many astrophysical systems [e.g. 2–6, 9]. Recently, numerical experiments were reported [9] in which two of the approximations used in Kozai-Lidov theory were relaxed: (i) the octupole potential of the perturber was included, and (ii) the mass of the particle was not assumed to be much smaller than that of the perturber. Very long-term evolution, $t \gg t_{\text{secular}}$, of the cycles was found, exhibiting striking features, including generation of extremely high eccentricities and retrograde orbits (‘orbit flipping’) with respect to the perturber [9]; these phenomena are not possible in Kozai-Lidov theory (see also [8]). In this Letter the very long-term evolution of KLCs due to the small octupole potential is studied. The test particle limit is retained (condition (i) above is relaxed but not (ii)), resulting in similar very long-term evolution. For high inclinations, the long-term evolution is calculated analytically by averaging the secular equations of motion over the KLCs. The analysis identifies a new constant of the motion and yields an analytical criterion for orbit flipping.

Secular Equations Consider a test particle on a Keplerian orbit (semi-major axis a and eccentricity e) subject to perturbation by a distant mass M_{per} on an orbit (a_{per} , e_{per}) around the same central mass M . The coordinate system is defined using the perturber’s orbit,

with the z-axis chosen to be in the direction of the angular momentum vector and the x-axis pointing to the pericenter. It is useful to parametrize the test particle’s orbit by two dimensionless vectors: $\mathbf{j} = \mathbf{J}/\sqrt{GMa}$, where \mathbf{J} is the specific angular momentum vector and G is the universal constant of gravitation; \mathbf{e} , a vector pointing in the direction of the pericenter with magnitude e . The orientation of \mathbf{j} is described by the inclination i and the longitude of ascending node Ω , by $\mathbf{j} = j(\sin i \sin \Omega, -\sin i \cos \Omega, \cos i)$. Usually, the orientation of \mathbf{e} is set by additionally specifying the argument of pericenter ω (angle between \mathbf{e} and $\hat{\mathbf{z}} \times \mathbf{j}$). Here we define the orientation of \mathbf{e} by the co-latitude angle $0 \leq i_e \leq \pi$, and longitude Ω_e defined by $\mathbf{e} = e(\sin i_e \cos \Omega_e, \sin i_e \sin \Omega_e, \cos i_e)$. It turns out that for the cases considered, Ω_e is slowly varying and is useful for describing the very long-term behavior of the system.

The secular orbital evolution of the test particle is determined by double time-averaging the perturbing potential Φ_{per} over the orbital periods of the test particle and the perturber. The averaged potential expanded to the octupole order (3rd order in a/a_{per}) is given by $\langle \Phi_{\text{per}} \rangle = \Phi_0 \phi = \Phi_0(\phi_{\text{Quad}} + \epsilon_{\text{Oct}} \phi_{\text{Oct}})$, where the dimensionless averaged potential ϕ is expressed as the sum of two components (quadrupole and octupole [12]),

$$\phi_{\text{Quad}} = \frac{3}{4} \left(\frac{1}{2} j_z^2 + e^2 - \frac{5}{2} e_z^2 - \frac{1}{6} \right), \quad (1)$$

$$\phi_{\text{Oct}} = \frac{75}{64} \left[e_x \left(\frac{1}{5} - \frac{8}{5} e^2 + 7e_z^2 - j_z^2 \right) - 2e_z j_x j_z \right], \quad (2)$$

and the normalization parameters are

$$\Phi_0 = \frac{GM_{\text{per}} a^2}{a_{\text{per}}^3 (1 - e_{\text{per}}^2)^{3/2}}, \quad \epsilon_{\text{Oct}} = \frac{a}{a_{\text{per}}} \frac{e_{\text{per}}}{1 - e_{\text{per}}^2}. \quad (3)$$

In the secular approximation, a and ϕ are constant with time while \mathbf{j} and \mathbf{e} evolve according to the following

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equations of motion [7, 10],

$$\frac{d\mathbf{j}}{d\tau} = \mathbf{j} \times \nabla_{\mathbf{j}}\phi + \mathbf{e} \times \nabla_{\mathbf{e}}\phi, \quad \frac{d\mathbf{e}}{d\tau} = \mathbf{j} \times \nabla_{\mathbf{e}}\phi + \mathbf{e} \times \nabla_{\mathbf{j}}\phi, \quad (4)$$

where $\tau = t/t_{\text{sec}}$ and $t_{\text{sec}} = \sqrt{GMa}/\Phi_0$ is the secular timescale. Physical solutions are restricted to those satisfying the physical constraints $j^2 = |\mathbf{j}|^2 = 1 - e^2$, and $\mathbf{e} \cdot \mathbf{j} = 0$.

Kozai-Lidov Cycles (KLCs) When expanded to the quadrupole order (i.e., $\epsilon_{\text{Oct}} = 0$), the averaged perturbing potential is axisymmetric. As a consequence, j_z is conserved and the equations of motion are invariant under rotational transformations around the z axis. In this case, e , i , ω and i_e undergo periodic oscillations (KLCs), which are determined by the two constants of motion j_z and $\phi = \phi_{\text{Quad}}$ [1, 2]. It is convenient to use the constant of motion $C_K = \frac{4}{3}\phi_{\text{Quad}} - \frac{1}{2}j_z^2 + \frac{1}{6}$, which is given by

$$C_K = e^2 - \frac{5}{2}e_z^2 = e^2(1 - \frac{5}{2}\sin^2 i \sin^2 \omega). \quad (5)$$

When $C_K < 0$, ω librates around $\pi/2$ or $-\pi/2$ (librations), while for $C_K > 0$, ω varies monotonically with time taking all values from 0 to 2π (rotations).

For rotations of ω ($C_K > 0$), e reaches minimum at $\omega = 0$ or π , implying $e_{\text{min}}^2 = C_K$.

For librating solutions, e_{min} is obtained at $\omega = \pm\pi/2$. For any KLC, maximum e is obtained at $\omega = \pm\pi/2$, leading to $3e_{\text{max}}^4 + (5j_z^2 - 3 + 2C_K)e_{\text{max}}^2 - 2C_K = 0$.

Equations of motion Since the octupole contribution is small the evolution is described by KLCs on short time scales (of order t_{sec} , in the vicinity of any time t). The properties of these cycles are determined by the (t dependent) values of j_z and C_K . Moreover, given that the total potential ϕ is conserved, we have to a good approximation $\phi_{\text{Quad}} = \text{const}$, implying that

$$C_K + \frac{1}{2}j_z^2 = \text{const}, \quad \Rightarrow \dot{C}_K = -j_z\dot{j}_z. \quad (6)$$

Thus, the problem can be reduced to finding $j_z(t)$.

The time derivative of j_z arises from the octupole potential alone; using Eqs. (1), and (4), it is given by

$$\dot{j}_z = \frac{75}{64}\epsilon_{\text{Oct}} \left[2j_y j_z e_z - e_y \left(\frac{1}{5} - \frac{8}{5}e^2 + 7e_z^2 - j_z^2 \right) \right]. \quad (7)$$

We focus on KLCs with $j_z^2 \ll 1$ and study the very long-term behavior of the system on time scales $t \sim \epsilon_{\text{Oct}}^{-1}t_{\text{sec}}$. Taking the lowest order terms in j_z , we have

$$\dot{j}_z = -\epsilon_{\text{Oct}} \sin(\Omega_e) f_j(e, i_e) \quad (8)$$

where $f_j = (75/64)e \sin i_e \left[\frac{1}{5} - e^2 \left(\frac{8}{5} - 7 \cos^2 i_e \right) \right]$.

The equation for Ω_e , governed by the quadrupole, satisfies [12]

$$\dot{\Omega}_e = j_z f_\Omega \quad (9)$$

where $f_\Omega = 3(8 - 6/\sin^2 i_e)/8$.

For librating cycles (with ω librating), Ω_e changes by nearly π each cycle (due to the small i_e values, and correspondingly high f_Ω values obtained [12]), implying that \dot{j}_z goes to approximately $-\dot{j}_z$ and the change in j_z after two cycles nearly vanishes (it is higher order in j_z). For rotating cycles, Ω_e varies slowly implying that \dot{j}_z changes slowly. Below we focus on these rotating cycles for which j_z can monotonically change over several secular timescales. We note that for some librating initial conditions, we numerically found significant modulation (including orbit flips, where the orbit changes from prograde to retrograde orientation relative to the perturber) occurring on time scales $t \sim \epsilon_{\text{Oct}}^{-2}t_{\text{sec}}$ much greater than those studied here, $t \sim \epsilon_{\text{Oct}}^{-1}t_{\text{sec}}$.

Using Eq. (6), and assuming that the initial conditions $j_{z,0}, C_{K,0}$ are in the rotating zone ($C_{K,0} > 0$), the condition for rotation is $-j_{z,\text{max}} < j_z < j_{z,\text{max}}$ where

$$j_{z,\text{max}} = \sqrt{2C_{K,0} + j_{z,0}^2}. \quad (10)$$

If j_z crosses this border, \dot{j}_z changes sign during each cycle. For the examples of such cases we checked numerically, after a few cycles j_z moved away from this limit, back into the rotation region.

Averaged equations We next average the equations of motion over the rotating KLCs to obtain approximate equations that describe the very long-term behavior of the system. To the lowest order in $j_z, \epsilon_{\text{Oct}}$, we can average Eqs. (8) and (9) over a KLC by taking the limit $j_z = 0$ in which Ω_e is constant and neglecting the deviation of $\dot{\Omega}_e$ due to the octupole. The latter is important only during short episodes when $|j_z| \lesssim \epsilon_{\text{Oct}}$, in which j_z can change considerably during one KLC and which are not resolved in the lowest order approximation. We obtain

$$\begin{aligned} \dot{\Omega}_e &= j_z \langle f_\Omega \rangle \\ \dot{j}_z &= -\epsilon_{\text{Oct}} \langle f_j \rangle \sin(\Omega_e), \end{aligned} \quad (11)$$

where f_i ($i = \Omega, j$) are averaged over a KLC with $j_z = 0$,

$$\langle f_i \rangle = \frac{1}{\tau_{\text{KLC}}} \oint_{j_z=0} dt f_i = \frac{4}{\tau_{\text{KLC}}} \int_0^{\sqrt{1-e_{\text{min}}^2}} dj \dot{j}^{-1} f_i, \quad (12)$$

where $\tau_{\text{KLC}} = \oint dt$ is the KLC period at $j_z = 0$. Note that $e^2(1 - (5/2)\cos^2 i_e) = C_K$, which together with $e^2 = 1 - j^2$ and $e_{\text{min}}^2 = C_K$ allows a straightforward integration yielding

$$\begin{aligned} \langle f_\Omega \rangle &= \frac{6E(x) - 3K(x)}{4K(x)}, \\ \langle f_j \rangle &= \frac{15\pi}{128\sqrt{10}} \frac{1}{K(x)} (4 - 11C_K) \sqrt{6 + 4C_K}, \\ x &= \frac{3 - 3C_K}{3 + 2C_K}, \end{aligned} \quad (13)$$

where $K(m)$ and $E(m)$ are the complete elliptic functions of the first and second kind respectively. Note that $\langle f_i \rangle$

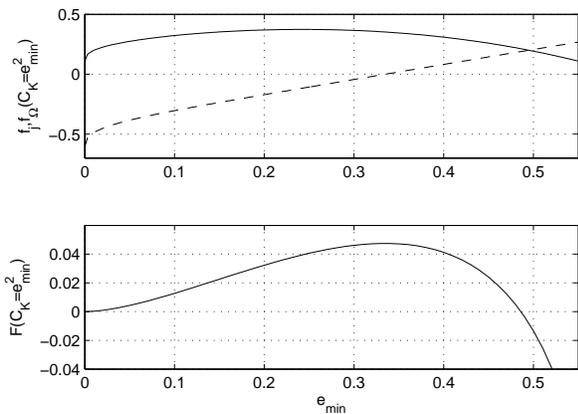


FIG. 1: Upper panel: $\langle f_j \rangle$ (solid) and $\langle f_\Omega \rangle$ (dashed) from Eqs. (13). Lower panel: F from Eq. (15).

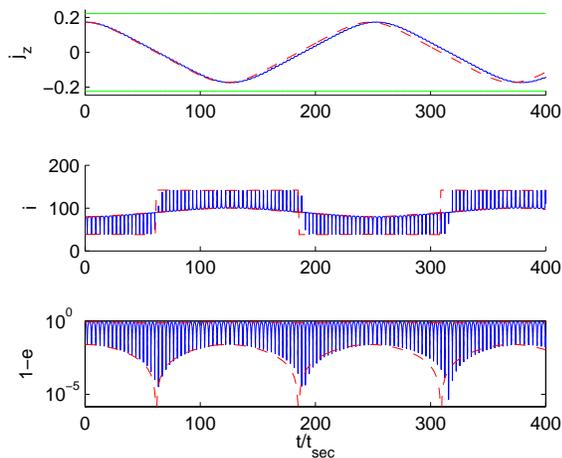


FIG. 2: Results of numerical integrations for $\epsilon_{\text{Oct}} = 0.01$, with initial conditions $\omega_0 = 0$, $\Omega_0 = \pi$, $i_0 = 80^\circ$, $e_0 = 0.1$ where for $\omega = 0$ $e_{\min} = e_0$. The blue solid lines are the result of the integration of the full secular equations, Eqs. (4), while the red dashed lines are the result of the integration of the averaged equations, Eqs. (11), (13) and (6), and using the Kozai-Lidov relations to determine e_{\min} , e_{\max} and i_{\min} , i_{\max} . The two green horizontal lines in the top panel represent $\pm j_{\text{max}}$, given by Eq. (10)

are functions of C_K alone since the KLC over which the averaging is made has $j_z = 0$. The upper panel of Fig. 1 shows plots of $\langle f_\Omega \rangle$ and $\langle f_j \rangle$.

Eqs. (11), (13) and (6) form a closed set of equations for the slowly varying j_z , C_K and Ω_e . An example of a numerical integration of these equations, compared with the results of a direct integration of the full secular equations, Eq. (4), is shown in Fig. 2 for $\epsilon_{\text{Oct}} = 0.01$. As can be seen, the approximate equations describe the long-term evolution to a good approximation.

These equations break down if $|j_z|$ crosses the threshold $j_{z,\text{max}}$, given by Eq. (10), in which case Ω_e receives kicks, \dot{j}_z changes sign, j_z moves to the rotation region after a few secular time scales and the averaged equa-

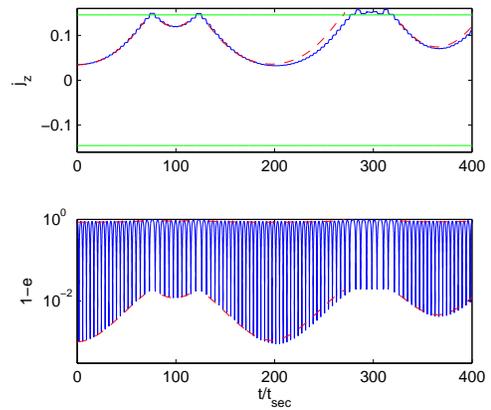


FIG. 3: Results of numerical integrations for $\epsilon_{\text{Oct}} = 0.01$, with initial conditions $\omega_0 = 0$, $\Omega_0 = 0$, $i_0 = 88^\circ$, $e_0 = 0.1$. The line styles are as in Fig. 2.

tions become valid again. An example of such behavior is seen in Fig. 3. In this example, the effective equations, Eqs. (11), (13) and (6), were integrated numerically only in the intervals where $C_K > 0$ (dashed lines).

Analytic solution The averaged equations of motion, Eqs. (11), (13) and (6), admit a constant of the motion of the form

$$C = F(C_K) - \epsilon_{\text{Oct}} \cos \Omega_e. \quad (14)$$

Indeed, using (11) and (6) we find, $\dot{C} = \epsilon_{\text{Oct}} j_z \sin \Omega_e (F' \langle f_j \rangle + \langle f_\Omega \rangle)$, and by defining

$$\begin{aligned} F(C_K) &= - \int_0^{C_K} \frac{\langle f_\Omega \rangle(c)}{\langle f_j \rangle(c)} dc \\ &= 32 \frac{\sqrt{3}}{\pi} \int_{\frac{3-3C_K}{3+2C_K}}^1 \frac{K(x) - 2E(x)}{(41x - 21)\sqrt{2x + 3}} dx \end{aligned} \quad (15)$$

we obtain $\dot{C} = 0$. The numerical value of $F(e_{\min}^2)$ as a function of e_{\min} is shown in the bottom panel of Fig. 1 and tabulated in [12].

Note that F diverges at $e_{\min,\text{inf}} = (4/11)^{0.5}$ where $\langle f_j \rangle = 0$ and has a maximum at $e_{\min,m} \approx 0.335$. For $e_{\min} > e_{\min,\text{inf}}$, the integration limits in Eq. (15) must be chosen differently. Here we focus on $e_{\min} < e_{\min,\text{inf}}$. This constant of motion holds most of the information about the system.

Flip criterion We next use the constant of motion, Eq. (14), to derive a criterion for the initial conditions which allow j_z to change sign, hence allowing i to increase above 90° , so that the orbit becomes retrograde relative to the perturber.

During a flip, $j_z = 0$ and Eq. (6) implies that $C_K = C_{K,0} + 0.5 j_{z,0}^2$. Given the constant of motion Eq. (14), and that the term $\epsilon_{\text{Oct}} \cos \Omega_e$ can change by at most $2\epsilon_{\text{Oct}}$, a required condition for a flip is that $\epsilon_{\text{Oct}} > \epsilon_{\text{Oct},c}$ where

$$\epsilon_{\text{Oct},c} = \frac{1}{2} \max(|\Delta F(x)|) \quad (16)$$

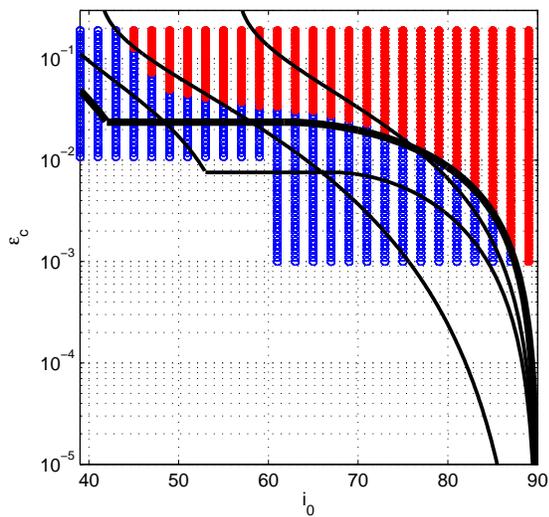


FIG. 4: The thick black line is the analytical threshold for a flip, Eq. (16), at $e_0 \rightarrow 0$ (this reduces to (17) for $i_0 > 61.7^\circ$). The results from numerical integrations of the full secular equations are shown as red circles (cases that had a flip) and blue circles (cases that did not flip). All simulations had $\omega_0 = 0, e_0 = 0.001$ and Ω scanned between 0 to 2π , covering the rotating KLCs initial conditions. Note that $e_{\min} = e_0$ when $\omega_0 = 0$. The thin black lines are the approximate analytical thresholds resulting from Eq. (16) for $e_0 = 0.2, 0.335 (= e_{\min, m}), 0.5$ (left to right at high ϵ_c).

where x is in the range $C_{K,0} < x < C_{K,0} + 0.5j_{z,0}^2$ and $\Delta F(x) = F(x) - F(C_{K,0})$. For cases where initially $e_0 \ll 1$ implying that $C_K \ll 1$ and $j_{z,0} = \cos i_0$, and for $j_{z,0}^2 < 2e_{\min, m}^2$ ($i_0 > 61.7^\circ$), Eq. (16) reduces to

$$\epsilon_{\text{Oct},c} = \frac{1}{2}F\left(\frac{1}{2}\cos^2 i_0\right). \quad (17)$$

This analytic theoretical threshold for orbit flips is shown in thick black line in Fig. 4. For comparison, the results of numerical integrations of Eqs. (4) for $10/\epsilon_{\text{Oct}}$ secular times for ϵ_{Oct} scanned over 0.001 to 0.2, i_0 scanned over 40° to 90° , with $\omega_0 = 0, e_0 = 0.001$ and Ω_0 scanned over 0 to 2π are shown in filled red (flipped) and open blue (no flip) circles. The analytical curve describes the flip condition to better than 10% for $i_0 \gtrsim 80^\circ$, better than 20% for $i_0 > 70^\circ$, and to a factor less than 2 for $i_0 > 50^\circ$. The deviation at low inclinations (large $|j_z|$) is not surprising, given that our formalism assumes $j_z^2 \ll 1$. It is encouraging that the overall behavior is captured quite well for j_z up to 0.5. The thresholds resulting from Eq. (16) for $e_0 = 0.2, 0.335, 0.5$ (left to right at high $\epsilon_{\text{Oct},c}$) are shown as thin black lines. Note the

discontinuity in the cases $e_0 < e_{\min, m}$ which arises from the presence of a maximum in F . Note also the presence of flips for very small octupoles at $e_0 = e_{\min, m}$ where $\langle f_\Omega \rangle = 0$ and Ω_e can remain constant for a long while allowing j_z to change considerably.

Discussion The analysis presented in this letter shows that the very long-term evolution, numerically studied in the context of hot Jupiter migration in [9], occurs already in the test particle approximation in which the mass of the perturber is much bigger than that of the planet, and will thus be important for stellar mass perturbers. The octupole potential causes a slow, cyclic modulation of the Kozai-Lidov cycles. Note that similar evolution occurs in other perturbing potentials where there are small deviations from axisymmetry [11].

Consequently, the distribution of orbital parameters of KLC-migrated hot Jupiters due to a stellar perturber may be significantly affected by the dynamics of the octupole perturbations described in this Letter.

Consider for example, the parameters used to numerically study the statistical properties of hot Jupiter migration in [5], $a = 5\text{AU}$, $a_{\text{per}} = 500\text{AU}$, $M = M_{\text{per}} = M_\odot \sim 1000M_J$, where the contribution of the octupole term was neglected. In this example, the planet's orbital period is ~ 10 yrs, the perturber's orbital period is $\sim 10^4$ yrs, and the secular time scale is $t_{\text{secular}} \sim 10^6$ yrs. We performed a few test runs with these parameters and with $\epsilon_{\text{Oct}} = 0.01$ (corresponding to $e_{\text{per}} \approx 0.6$), including general-relativistic (GR) precession and found that extremely high eccentricities $1 - e_{\text{max}} \sim 10^{-4}$ can be reached within $\sim 10^8$ yrs for the high inclinations $i > 80^\circ$ considered (e_{max} may be limited by other physical effects such as tidal precession). For these parameters, a flip was suppressed due to the GR precession, but for equally likely parameters with slightly closer perturbers, $a_{\text{per}} \lesssim 300\text{AU}$, the effect of GR precession is overcome by the octupole and flips are also attainable.

A numerical investigation of the problem studied in this letter is published simultaneously by Yoram Lithwick and Smadar Naoz.

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