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Ultra-Efficient Cooling of Resonators: Beating Sideband Cooling with Quantum Control

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The present state-of-the-art in cooling mechanical resonators is a version of "sideband" cooling. Here we present a method that uses the same configuration as sideband cooling — coupling the resonator to be cooled to a second microwave (or optical) auxiliary resonator — but will cool significantly colder. This is achieved by varying the strength of the coupling between the two resonators over a time on the order of the period of the mechanical resonator. As part of our analysis, we also obtain a method for fast, high-fidelity quantum information-transfer between resonators.

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There is presently a great deal of interest in cooling high-frequency micro- and nano-mechanical oscillators to their ground states. This interest is due to the need to prepare resonators in states with high purity to exploit their quantum behavior in future technologies [1, 2]. The key measure of a cooling scheme is the *cooling factor*, which we will denote by $f_{\rm cool}$. The cooling factor is the ratio of the average number of phonons in the resonator at the ambient temperature, n_T, to the average number of phonons achieved by the cooling method, which we will denote by $\langle n\rangle_{\rm cool}$. The present state-of-the-art for cooling mechanical resonators is *sideband* cooling, which was originally developed in the context of cooling trapped ions [3–5]. This method is a powerful and practical technique, able to achieve large cooling factors, and these have been demonstrated in the laboratory [6–16].

In the context of mechanical resonators, sideband cooling involves coupling the resonator to be cooled (from now on the "target") to a microwave or optical resonator (the "auxiliary") whose frequency is sufficiently high that it sits in its ground state at the ambient temperature. Sideband cooling uses a linear coupling between the resonators, which in practice is usually obtained from a nonlinear "radiation-pressure" interaction by strongly driving the auxiliary [5]. If we denote the annihilation operators for the target and auxiliary by a and b, respectively, then the Hamiltonian of the two linearly-coupled resonators is

$$H = \hbar\omega a^\dagger a + \hbar\Omega b^\dagger b + g\cos(\nu t)x_a x_b, \qquad (1)$$

where $x_a = a+a^\dagger$ and $x_b = b+b^\dagger$ are the position operators of the respective resonators. The coupling is modulated at the difference frequency between the resonators, $\nu = \Omega - \omega$. This down-converts the high frequency of the auxiliary resonator so that the two resonators are effectively on-resonance, and thus exchange energy at the coupling rate g. With this frequency conversion, the auxiliary constitutes a source of essentially zero entropy (and thus zero temperature) for the target resonator [17].

When the rate of the coupling, g, is much smaller than the target resonator frequency ω (so that one is within the rotating-wave approximation (RWA)— see, e.g. [18]), then the linear coupling between the resonators merely amounts to excitation (phonon/photon) exchange between the two. If the damping rate of the auxiliary, κ, is now fast enough, then the excitation exchange, combined with the relatively fast damping of the auxiliary at effectively zero temperature, extracts the phonons out of the target. If the coupling to the resonator is perturbative ($g \ll \omega$), the cooling factor is merely the ratio of the phonon extraction rate to the resonators damping rate γ. The extraction rate cannot always be obtained analytically, but if we denote it by $\Gamma_{\rm cool}$, then $\langle n\rangle_{\rm cool} = n_T/f_{\rm cool}$, where the cooling factor is $f_{\rm cool} = \Gamma_{\rm cool}/\gamma$. Note that the extraction rate is bounded by the rates g and κ. For sideband cooling, the RWA requires $g \ll \omega$ and $\kappa \ll \omega$, limiting the cooling factor.

Here we demonstrate that one can cool significantly better than traditional sideband cooling by using quantum control to go beyond the RWA, into the ultra-strong coupling regime $g \sim \omega$. We first show that a particular time-dependence of the coupling rate, $g(t)$, can achieve a high-fidelity transfer of quantum states between the target and auxiliary resonators within a *single* resonator period. As pointed out in [19], "state-swapping" is one way to achieve cooling, as this process will load the cold state of the auxiliary into the target [20]. In fact, the phonon/photon exchange of the RWA implements state-swapping in a time of $\pi/(2g)$ [22]. However it was shown in [21] that using this to cool (which means running traditional sideband cooling, but now only for a single swap-time) is little better than the usual approach. In contrast, we show here that numerically optimized control sequences will achieve significantly better cooling factors. This is because they allow one to circumvent the RWA restriction that $g \ll \omega$, and thus swap the energy out of the resonator significantly faster (namely, within a single oscillation period). Further, this method can achieve these lower temperatures over a much wider range of values of the auxiliary damping rate, κ. While our method is quite practical, because it requires a relatively small modification to the existing sideband cooling scheme, and performs at least as well as sideband cooling for any value of g, achieving the lowest temperatures does require ultra-strong coupling ($g \sim \omega$). Previously nano-resonator experiments had only achieved small values of g, but recently a tremendous increase in g was demonstrated in an experiment by Teufel *et al.* [14]. This has brought g within a factor of ten of ω, and further increases appear feasible. The present method is therefore timely, as we anticipate that near-future experiments will be able to realize it. We note that Machnes *et al.* [23] have previously devised a way to go beyond the RWA for trapped-ions, where the auxiliary system is a qubit. However, their method is not feasible for nano-resonators, certainly with near-future technology, because it requires $g \gg \omega$.

To begin our analysis we first consider the problem of engineering a fast, high-fidelity state-swap between two linearly coupled resonators, as this is an important problem in its own right. Fast operations on quantum information are important due to the the ever present effects of decoherence. To obtain such a state-swap, and thus an efficient energy transfer *without* the RWA, we examine the algebra generated by the linear coupling in conjunction with the free Hamiltonians of the resonators. The algebra of these three Hamiltonians suggests that it should be possible to engineer a perfect state-swap by concatenating the evolutions generated by the Hamiltonians in a process of "quantum control" [24]. Up to local operations on each resonator, such a concatenation is equivalent to varying the coupling g with time. This would allow us to obtain efficient energy transfer when $g \sim \omega$, not only achieving faster state-swapping, but also better cooling.

To explore the above conjecture, we simulate the evolution given by the Hamiltonian in Eq.(1), in which g is a function of time. Since Ω is typically much greater than ω (by a factor of at least 100), we may assume that the frequency conversion is exact, and set $\Omega = \omega$ and $\nu = 0$. The corrections to this approximation are of the order of $(\omega/\Omega)^2$. (This is, in fact, an RWA for the frequency Ω, which is distinct from the RWA for the target frequency ω, required by sideband cooling.) We prepare the target resonator in a state that is confined to the space spanned by the 12 lowest Fock states, and completely mixed on that space. The auxiliary is prepared in the ground state, and the resonators evolved for a specified time. This allows us to determine the quality of the swap merely by calculating the purity of the final density matrix for the target resonator. If this state is pure, then the evolution has successfully transferred all the quantum information to the auxiliary resonator. We evolve for a single period of the target resonator, and dividing this time into five equal intervals of duration Δt, we parametrize $g(t)$ by making it piecewise-constant on these intervals. Finally we perform a numerical optimization, using a Quasi-Newton line search method [25], to determine the five piecewise-constant values for $g(t)$. For the simulation we use the basis of Fock states, including the lowest 25 states for each resonator. This achieves an essentially perfect state-swap (a final purity of 0.999977) with the following five values of g/ω: $(1.78, 1.45, 2.44, 1.61, 0.195)$. As a second example, we find that a state-swap with a purity of 0.999991 can be obtained in 0.7 of the resonators period, with the values $(2.76, 0.474, 3.73, 0.78, 2.59)$.

The above results show that, in the absence of decoherence, state-swapping in less than one period is within the "control space" of the linear coupling. But this does not tell us how well we can transfer the cold auxiliary state to the hot resonator in the presence of damping. Damping is equivalent to a continuous measurement process [21, 26], and this inhibits the transfer of energy to the auxiliary due to the quantum Zeno effect. We must therefore simulate the optimized cooling in the presence of damping, but it is impractical to do this with the simulation method used above, as the size of the required superoperators is too large. Fortunately in the case of cooling we are only interested in the average phonon number, given by $\langle n\rangle = \langle a^\dagger a\rangle$, which is a second moment of operators a and $a^\dagger$. Because the dynamics of the resonators is linear (that is, the evolution can be described by a set of linear quantum Langevin equations [27–29]) one can derive a closed set of equations for the variances and covariances of the annihilation operators. Because the means of these operators are zero in thermal states, and remain zero during the evolution, the covariances are equal to the second moments.

To describe the damping, we use the Markovian version of the Brownian-motion master equation [29, 30]. If we define the vector $\mathbf{x} \equiv (a, a^\dagger, b, b^\dagger)^{\mathrm{t}}$, then the matrix of covariances is $C \equiv \langle \mathbf{x}\mathbf{x}^{\mathrm{t}}\rangle - \langle \mathbf{x}\rangle\langle \mathbf{x}^{\mathrm{t}}\rangle$. The equation of motion for C is

$$\dot{C} = AC + CA^{\mathrm{t}} + G, \qquad (2)$$

where

$$A = \begin{pmatrix} -i\omega - \gamma/2 & \gamma/2 & -ig & -ig \\ \gamma/2 & i\omega - \gamma/2 & ig & ig \\ -ig & -ig & -i\omega - \kappa/2 & 0 \\ ig & ig & 0 & i\omega - \kappa/2 \end{pmatrix},$$

$$G = \begin{pmatrix} -\gamma\left(n_T + \frac{1}{2}\right) & \gamma\left(n_T + \frac{1}{2}\right) & 0 & 0 \\ \gamma\left(n_T + \frac{1}{2}\right) & -\gamma\left(n_T + \frac{1}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & \kappa(n_{\mathrm{aux}} + 1) \\ 0 & 0 & \kappa n_{\mathrm{aux}} & 0 \end{pmatrix},$$

and n_{aux} is the initial average number of photons in the auxiliary resonator.

We now wish to determine the function $g(t)$ that gives the minimum value of $\langle a^\dagger a\rangle$ after a fixed time interval. To

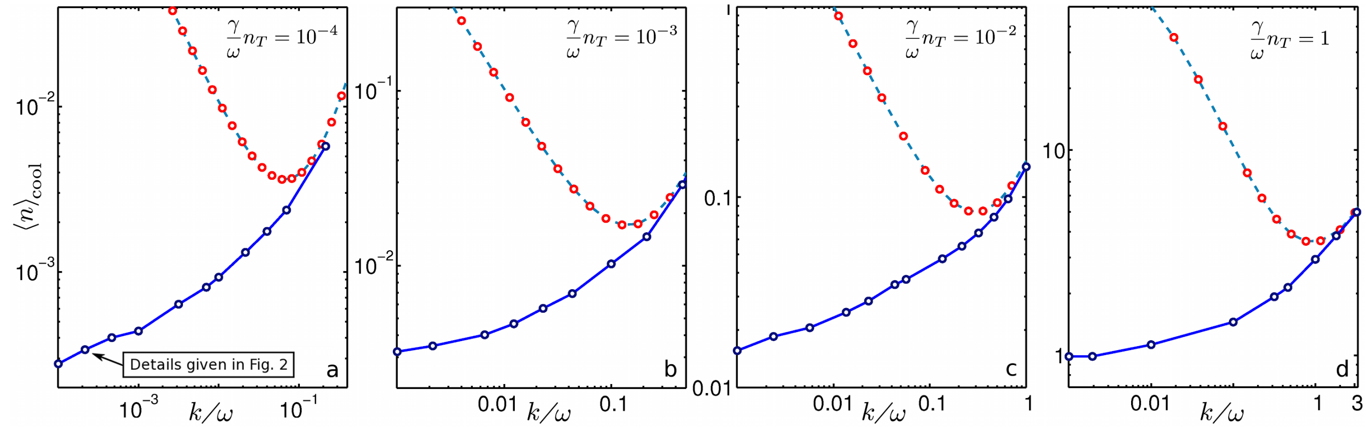

FIG. 1. (Color online) The average phonon number, $\langle n\rangle_{\rm cool}$, achieved by the cooling method presented here, compared to sideband cooling, as a function of the damping rate of the auxiliary resonator, κ, and for four values of $\gamma n_T/\omega$. The values obtained by sideband cooling are given by the circles/dashed line (red online). The circles/solid line give the values achieved by the present cooling method, in which g is a time-varying pulse. The time required for the cooling pulse, τ, varies with κ. As examples, for plot (a), from left to right, to obtain the values given by the circles we used $\tau\omega/(2\pi) = 0.5, 0.55, 0.73, 1, 1, 1.5, 2, 2, 3, 5.3, 6$. For plot (c), the pulse times are $\tau\omega/(2\pi) = 0.55, 0.6, 0.6, 0.6, 0.8, 0.7, 0.8, 1, 1.6, 1.6, 1.8, 2, 2$.

do so we take the same approach as above, choosing g to be piecewise-constant. We wish to determine the optimal cooling over a broad range of the relevant parameters, and compare this to sideband cooling. Measuring time in units of $1/\omega$, the important parameters are the damping rates of the target and auxiliary (respectively γ/ω and κ/ω), and the average number of thermal phonons in the target at temperature T, n_T. By examining the equations of motion, we see that so long as $n_T \gg 1$, to a good approximation the evolution, and thus the cooling, depends only on the product of γ and n_T, rather than each separately. Since $n_T \gg 1$ is the relevant regime for present experiments, we therefore need to determine the optimal cooling as a function of κ/ω and $(\gamma/\omega)n_T$.

The thermal occupation of the auxiliary at the ambient temperature, $n_{\rm aux}$, is very small with present technology. For example, a 10 GHz stripline resonator at 50 mK has $n_{\rm aux} = 6.7 \times 10^{-5}$, and at 100 mK has $n_{\rm aux} = 8.3 \times 10^{-3}$. We expect $n_{\rm aux}$ to be significant only if the cooled value of $\langle a^\dagger a\rangle$ is close to $n_{\rm aux}$, and we verify this below.

The final parameter is the time over which we perform the cooling. The optimal cooling will be obtained when the control pulse swaps the energy into the auxiliary in the shortest time. As the damping rates increase, we expect the Zeno effect to lengthen this minimum swap-time. For each value of κ we obtained the (approximately) optimal time by hand. As expected, we find that this time increases with κ (since γ/ω remains small).

We now perform the optimization over $g(t)$, with $n_{\rm aux} = 0$, and plot the results in Figure 1, along with the values of $\langle n\rangle_{\rm cool}$ that are achieved using sideband cooling (these are obtained by optimizing over the coupling strength and the detuning [5]). We see from Fig. 1 that our "optimal control" cooling scheme is superior to sideband cooling when κ is *less than* the value for which sideband cooling achieves its best performance. The second key result is that the improvement provided by optimal control increases as $\gamma n_T/\omega$ decreases. For $(\gamma/\omega)n_T = 10^{-4}$, 10^{-3}, and 10^{-2}, the smallest values we obtained for $\langle a^\dagger a\rangle$ are better than sideband cooling by factors of approximately 13, 5, and 5, respectively (see Figs. 1(a)-(d)). We note that a simple estimate of the lowest achievable temperature is as follows: ideally the time for the control to swap the energy is $\pi/g \sim \pi/\omega$, and the bath injects approximately γn_T phonons during this time. Thus one expects that $\langle n\rangle_{\rm cool} \sim \pi(\gamma/\omega)n_T$. This closely matches the results in Fig. 1 when $\gamma n_T/\omega \ll 1$.

Most of the cooling results in Fig. 1 are obtained using no more than 24 time-segments (that is, 24 piecewise-constant values for $g(t)$) per period. In many cases 10 segments is sufficient for optimal cooling. While the piecewise-constant functions we have used for $g(t)$ show that the control timescales are feasible, these functions are rather artificial. The actual experimental waveforms will not have infinitely sharp transitions between segments. To show that such sharp transitions are unnecessary, for a single value of κ we perform the optimization for a 12-segment pulse, now with linear transitions that have the same duration as the constant segments. The piecewise-constant pulse achieves $\langle n\rangle_{\rm cool} = 3.39 \times 10^{-4}$, and the piecewise-linear pulse performs very similarly, giving $\langle n\rangle_{\rm cool} = 3.46 \times 10^{-4}$. Both pulses are displayed in Fig. 2. Note that the coupling is turned off at the end of the piecewise-linear pulse, which is necessary to leave the resonator in its ground state. This removes the need to turn off the coupling adiabatically, as would be required by sideband cooling. To determine the waveform for a specific experiment, one would ideally parametrize

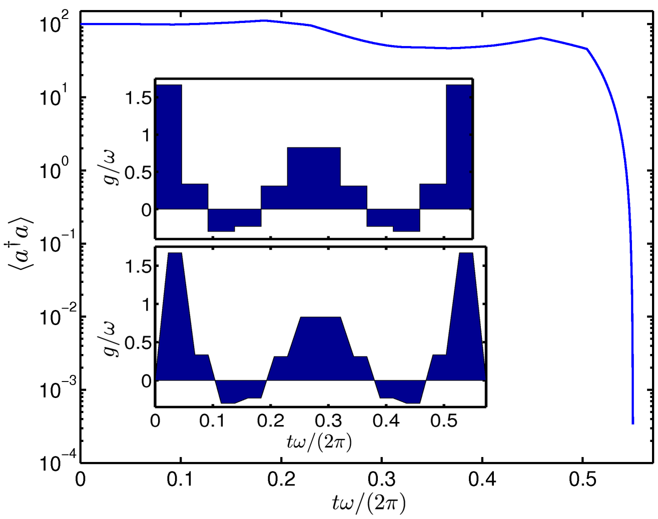

FIG. 2. (Color online) The optimal 12-segment piecewise-constant control pulse for the coupling rate $g(t)$ (inset (a)), for one of the points in Fig. 1a, along with the time-dependence of the average number of phonons. The parameters are $n_T = 100$, $\gamma = 1\times10^{-6}\omega$, $\kappa = 2.15\times10^{-4}\omega$. Inset (b): An optimal pulse with bounded transition-rates between segments, for the same parameters, that gives essentially the same cooling. The duration of the pulse in (a) is 0.55 resonator periods, and that in (b) is 0.573.

$g(t)$ so that it is well-matched to the waveforms produced by the electronics. Since fast electronics generates transitions shorter than 1 ns, the required control pulses should be feasible for resonators with frequencies up to 100 MHz. We have also examined the sensitivity of the cooling to noise on the control pulse: with independent Gaussian errors of 0.1% on each segment of the pulse in Fig. 2 (inset (a)), $\langle n\rangle_{\rm cool}$ increases by about 30%.

As noted above, the linear interaction in Eq.(1) is usually obtained by using the nonlinear interaction $x_a b^\dagger b$, and strongly driving the auxiliary [3–5]. This induces an effective frequency shift in the auxiliary of $\Delta\omega = g^2/(2\omega)$, with the result that changing g also changes the auxiliary frequency. Optimization shows that this frequency shift must be cancelled to realize the above cooling. This could be achieved by modulating the frequency of the driving field, or by applying a second field to modulate the frequency of the target.

We now determine the effect of thermal photons in the auxiliary resonator ($n_{\rm aux} > 0$). We perform the optimization again for the left-most four points plotted in Fig 1(a), for which $\gamma = 10^{-6}\omega$ and $n_T = 100$. With $n_{\rm aux} = 0$ we have $\langle n\rangle_{\rm cool} = 10^{-4}\times(2.8, 3.4, 4, 4.7)$. For $n_{\rm aux} = 10^{-4}$, these become $\langle n\rangle_{\rm cool} = 10^{-4}\times(3.8, 4.4, 5, 5.7)$. The increase in the average phonon number is approximately the addition of $n_{\rm aux}$. This confirms our intuition that thermal photons are only significant when $\langle n\rangle_{\rm cool} \sim n_{\rm aux}$.

To summarize, we have shown that by modulating a linear coupling, an essentially perfect state-swap can be performed between two resonators within a single oscillation period. This can be used to prepare a mechanical resonator in the ground state, with fidelity higher than possible with traditional sideband cooling.

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Note added: after submitting this work we learned of concurrent work by Cerrillo *et al.* (arXiv:1104.5448) that also treats pulsed cooling of resonators.

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