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# The effects of interaction on quantum spin Hall insulators

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We study the  $S_z$ -conserving quantum spin Hall insulator in the presence of Hubbard  $U$  from a field theory point view. The main findings are the following. (1) For arbitrarily small  $U$  the edges possess power-law correlated antiferromagnetic XY local moments. Gapless charge excitations arise from the Goldstone-Wilczek mechanism. (2) The vortex instanton of the XY moments violates charge conservation of the edge, hence are usually forbidden. (3) For samples with finite width, electron tunneling between opposite edges allow vortex instantons to proliferate when  $K$ , the XY stiffness constant, satisfies  $4\pi K + (4\pi K)^{-1} < 4$ . So long as the preceding inequality is violated, the edge modes remain gapless despite the sample width is finite. This is fundamentally different from the free electron limit. (4) The phase transition from the topological insulator to the large  $U$  antiferromagnetic insulator is triggered by the condensation of magnetic excitons. (5) In the large  $U$  antiferromagnetic insulating phase the magnetic vortices carry charges proportional to the square magnitude of the antiferromagnetic order parameter.

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The subject of topological insulator (TI) has attracted considerable attention recently[1]. A signature of this type of band insulator is the presence of itinerant boundary states in the bulk band gap[2–6]. Moreover, unlike those in usual band insulators, these itinerant in-gap states are robust against any modification of the free-electron Hamiltonian so long as they (i) respect time reversal symmetry, and (ii) do not close the bulk bandgap. Because of the robustness, these boundary states can evade Anderson’s localization in the presence of (time-reversal invariant) disorder[7]. At the present time non-interacting TIs are fairly well understood. What remains open is the effect of electron-electron interaction on TIs[8].

Recently two independent Monte-Carlo simulations[9, 10] were performed on the simplest kind of two dimensional interacting TI. The Hamiltonian studied in these works is  $H_0 + H_u$  where  $H_0$  is the  $S_z$ -conserving free electron model introduced by Kane and Mele[2]:

$$H_0 = \sum_{\sigma=\pm 1} \left\{ - \sum_{\langle ij \rangle} c_{i\sigma}^+ c_{j\sigma} + i t' \sum_{\langle\langle ij \rangle\rangle} \sigma \nu_{ij} c_{i\sigma}^+ c_{j\sigma} \right\}. \quad (1)$$

Here  $i, j$  label the sites of a honeycomb lattice, the first term describes the nearest neighbor hopping, and the second term is a spin dependent second neighbor hopping. Here  $\nu_{ij} = (\hat{d}_1 \times \hat{d}_2)_z / |(\hat{d}_1 \times \hat{d}_2)_z|$  where  $\hat{d}_1$  and  $\hat{d}_2$  are unit vectors along the two bonds the electron traverses when hopping from  $j$  to  $i$ .  $H_u$  is given by  $H_u = U \sum_i n_{i\uparrow} n_{i\downarrow}$ . According to Ref.[9], for  $t' \gtrsim 0.03$  (see Fig. (1)) there are only two phases as a function of  $U$ . The large  $U$  phase is an easy-plane (XY) antiferromagnetic (AF) Mott insulator; at small  $U$  it is an (interacting) TI with gapless spin and charge edge excitations.

The present work is motivated by the following considerations. Consider a system with edges (Fig. (2)(a)). At

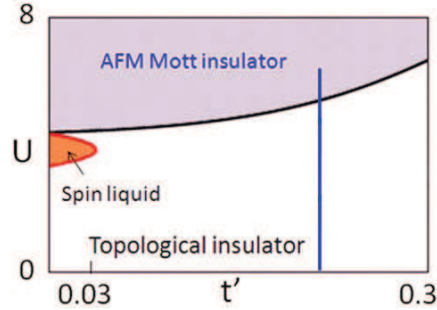


FIG. 1: (color on-line) A schematic reproduction of the phase diagram of the  $H_0 + H_u$  reported in Ref.[9]. The blue cut is considered in the text.

$U = 0$  the bulk has a band gap and the only low energy excitations are the “helical” edge modes described by the following Hamiltonian

$$H_{E0} = \mp i v \int dx \Psi^\dagger \sigma_z \partial_x \Psi, \quad (2)$$

where  $\mp$  applies to the top/bottom edges. In Eq. (2)  $v$  is the edge velocity (which will be set to 1 in the rest of the paper), and  $\Psi$  is a two component fermion field whose first/second component corresponds to spin up/down. For small  $U$  the bulk is non-magnetic[9, 10] but the edge will develop local moments. To see that we Hubbard-Stratonovich decouple the Hubbard  $U$  term in the imaginary-time Euclidean action as

$$e^{-U \int d\tau \sum_j n_{j\uparrow} n_{j\downarrow}} \sim \int D[S^\pm] e^{-U \int d\tau \sum_j [S_j^+ S_j^- - S_j^+ c_{j\downarrow}^\dagger c_{j\uparrow} - S_j^- c_{j\uparrow}^\dagger c_{j\downarrow}]} \quad (3)$$

We note that upon time reversal  $S_j^\pm \rightarrow -S_j^\mp$  and  $c_{j\sigma}^+ c_{j-\sigma} \rightarrow -c_{j-\sigma}^+ c_{j\sigma}$  hence the decoupled action is time-reversal invariant. (Note that the imaginary time  $\tau$  transforms as  $it$  hence is invariant under time reversal.) In mean-field theory  $S^\pm$  takes on space-time independent values (hence spontaneously break the time reversal symmetry) which introduce two mass terms into  $H_0$ :

$$H_{E0} - Um [\cos \theta_0 \int dx \Psi^\dagger \sigma_x \Psi + \sin \theta_0 \int dx \Psi^\dagger \sigma_y \Psi]. \quad (4)$$

Because the non-interacting theory has a logarithmically diverging susceptibility with respect to these mass terms mean-field theory predicts the edges will have long range AF XY order for arbitrarily small positive  $U$ . Of course, such long range order is destroyed by spin wave fluctuations and the time reversal symmetry is restored. The resulting edges have power-law decaying AF XY corre-

lation. However since the local moments introduces a single particle gap, one might wonder where are the gap-less charge excitations.

First, to make sure our statement concerning AF moment formation at infinitesimal  $U$  is correct for the edges of a *two dimensional* TI, we perform a mean-field calculation on a cylinder for  $t' = 0.2$  and  $0 \leq U \leq 5$  (this corresponds to the cut associated with the blue line interval in Fig. (1)). The results are shown in Fig. (3); from which it is clear that while the order parameter deep in the bulk (the blue curve) vanishes for  $U \lesssim 2.4$ , the edge order parameter (the red curve) survives to the lowest  $U$  value. There are two types of fluctuations above the mean-field vacuum: the massive modulus fluctuation in  $m$  and the massless Goldstone mode (i.e., spin wave) fluctuations. In the presence of these fluctuations we need to replace Eq. (4) by the following action  $S_E = \int dx d\tau \mathcal{L}_E$ :

$$\mathcal{L}_E = \bar{\Psi} \partial_\tau \Psi \mp i \Psi^\dagger \sigma_z \partial_x \Psi - U \bar{m} [\cos \theta(x, \tau) \Psi^\dagger \sigma_x \Psi + \sin \theta(x, \tau) \Psi^\dagger \sigma_y \Psi] + U [\delta m(x, \tau)^2 - \delta m(x, \tau) \Psi^\dagger \sigma_x \Psi]. \quad (5)$$

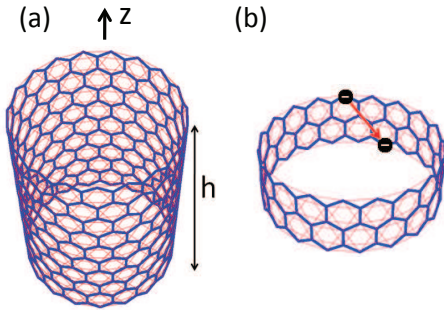


FIG. 2: (color on-line)(a) The spin-Hall insulator defined on a cylinder with height  $h$ . The blue/red bonds denote the nearest/second neighbor hopping. (b) When the cylinder is short, the electron can directly tunnel from one edge to the other.

In the following we first ignore the  $\delta m$  fluctuation and focus on the effect of the spin waves. In Ref.[11] Goldstone and Wilczek showed that integrating out the gapped fermions in Eq. (5) in the presence of a background electromagnetic gauge field and the total effective action looks like

$$S_{eff} = \int dx d\tau \left\{ \frac{K}{2} (\partial_\mu \theta)^2 - \frac{ie}{2\pi} (A_0 \partial_x \theta - A_x \partial_\tau \theta) \right\}. \quad (6)$$

Here  $e$  is the electron charge, and the last two terms are

the famous chiral anomaly. Using the method of Abanov and Wiegmann[12] we have derived the stiffness term  $K = (m^2/2\pi) \int_0^\Lambda \frac{p dp}{(p^2 + m^2)^2} = (1 - \frac{m^2}{\Lambda^2 + m^2})/4\pi$ , (here  $\Lambda$  is a momentum cutoff). Hence as  $\Lambda/m \rightarrow \infty$   $K \rightarrow 1/4\pi$ .

So far in deriving  $K$  we have ignored the  $\delta m$  fluctuation. The easiest way to account for these fluctuations is to replace the fermion bi-linears in Eq. (5) by their bosonization formulae[13]:

$$\begin{aligned} \mathcal{L}_E &= \frac{1}{2} (\partial_\mu \phi)^2 - U \bar{m} C_\Lambda \cos[\theta(x, \tau) - \sqrt{4\pi} \phi(x, \tau)] \\ &+ U [\delta m(x, \tau)^2 - \delta m(x, \tau) C_\Lambda \cos(\sqrt{4\pi} \phi)], \end{aligned} \quad (7)$$

where  $C_\Lambda$  is a cutoff dependent constant. Now it is straightforward to integrate out  $\delta m$  and  $\phi$ . The results are (i) the generation of the term  $\sim \int dx d\tau \cos 2\theta$  which is irrelevant for  $(\pi K)^{-1} < 2$  (ii) an order  $U$  correction to  $K$ :  $K \rightarrow K + O(U)$ . Hence for large  $\Lambda$  and small  $U$  the effect of  $\delta m$  fluctuations is to make Eq. (6) the action of a repulsive Luttinger liquid[14].

In the absence of  $A_\mu$ , Eq. (6) implies the XY correlation function

$$\langle e^{i\theta(0,0)} e^{-i\theta(x,\tau)} \rangle \sim (x^2 + \tau^2)^{-1/4\pi K}, \quad (8)$$

hence, as claimed earlier, the long-range order in mean-field theory is destroyed. The last two terms of Eq. (6) imply the space and time gradients in  $\theta$  produce excess charge and current densities at the edges:

$$\rho_E = \frac{e}{2\pi} \partial_x \theta, \quad J_E = -\frac{e}{2\pi} \partial_t \theta. \quad (9)$$

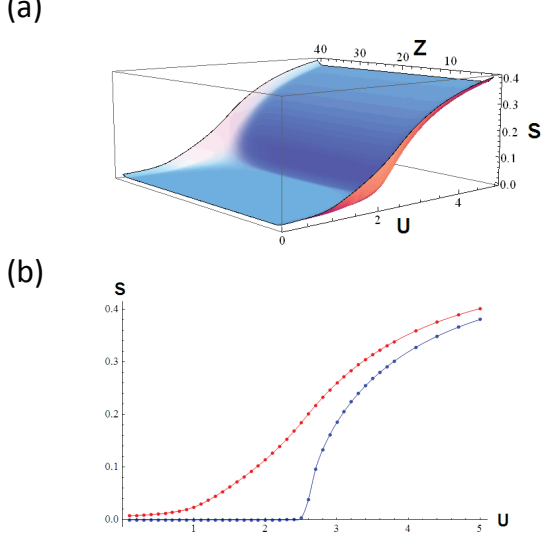


FIG. 3: (color on-line) (a) The mean-field antiferromagnetic XY order parameter as a function of  $z$  (see Fig. (2)(a)) and  $U$ . The calculation is done for a cylinder with 35 unit cell in the periodic direction and 40 unit cells in the  $z$  direction. The value  $t'$  is 0.2. (b) The order parameter as a function of  $U$  deep in the bulk  $z = 20$  (blue), and at the edge  $z = 1$  or 40 (red). It is clear that while the order parameter in the bulk vanishes for  $U \lesssim 3$  (the small rounding is due to finite size effect), the edge order parameter survives to the lowest  $U$  value.

Because of Eq. (9) gapless spin wave excitations induce charge and current density fluctuations with the following power-law correlation functions

$$\Pi_{\rho/J}(x, t) \propto \langle \partial_{x/t} \theta(0, 0) \partial_{x/t} \theta(x, t) \rangle \sim \pm \frac{t^2 - x^2}{(x^2 + t^2)^2}.$$

Thus through the Goldstone-Wilczek mechanism gapless charge excitations emerge!

The next question concerns the space-time vortices (instantons) of the  $\theta$  field. Because of Eq. (9) a vorticity- $m$  instanton at the space-time location  $(x_0, t_0)$  will cause  $\oint_{\partial D} dx_\mu \partial_\mu \theta = -(2\pi/e) \oint_{\partial D} dx_\mu \epsilon^{\mu\nu} J_{E,\nu} = -(2\pi/e) \int_D d^2x \partial_\mu J_{E,\mu} = 2\pi m$ . Here  $D$  is an arbitrary disk containing  $(x_0, t_0)$  and  $J_{E,\mu} = (\rho_E, J_E)$  is the edge 2-current. This implies

$$\partial_t \rho_E + \partial_x J_E = -me \delta(x - x_0) \delta(t - t_0), \quad (10)$$

hence vortex instantons violate the *edge* charge conservation, hence under usual circumstances should be forbidden. Nonetheless such instantons can occur through the tunneling of electrons from one edge to the other[15](Fig. (2)(b)). Annihilate a right(left) moving

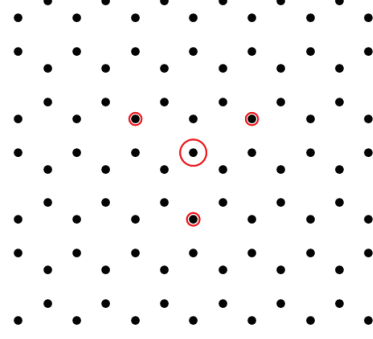


FIG. 4: (color on-line) An example of the XY order parameter profile associated with a localized magnetic exciton. The size of the red circles is proportional to the magnitude of the order parameter.

electron at the edge removes a charge  $e$  and spin  $1/2$  ( $-1/2$ ). Hence

$$\psi_{R/L}(x) \sim \text{Exp}\{\mp i\theta(x)/2 + i2\pi \int_x^\infty dy \Pi(y)\}, \quad (11)$$

where  $[\theta(x), \Pi(y)] = i\delta(x-y)$ . The above result resemble the usual 1D bosonization formula. This is not surprising since if we identify  $\theta$  with  $2\sqrt{\pi}\phi$  Eq. (9) becomes the bosonization expression for the charge and current densities. It can be shown straightforwardly that for  $4\pi K + (4\pi K)^{-1} < 4$  the inter-edge electron tunneling is a relevant perturbation. Under such condition, inter-edge electron tunneling will gap out the edge modes. However, if  $4\pi K + (4\pi K)^{-1} > 4$  the inter-edge electron tunneling is irrelevant. In that case the quantum spin Hall effect will survive even when  $h$ , the sample width, is finite! Such stabilization of the TI state for finite width sample is a pure interaction effect.

In the interacting TI phase the system exhibits quantized spin-Hall conductance. This can be understood as follows. In the presence of an electric field between the two edges, a voltage difference  $V$  develops. This induces a difference in  $\partial_x \theta$  between the two edges ( $E_1$  and  $E_2$ )  $(\partial_x \theta)_{E_1} - (\partial_x \theta)_{E_2} = \frac{eV}{2\pi K}$ . Because the spin current is  $K \partial_x \theta$ , this gives  $J_{S_z}^{\text{tot}} = \frac{e}{2\pi} V$  hence the spin Hall conductance is  $\frac{e}{2\pi}$  which is the same as the free electron value. As to the two terminal conductance, it was pointed out in the context of quantum wires that in the presence of translation invariance interaction effect does not change the two terminal conductance of a Luttinger liquid[16].

Finally we consider the bulk transition between the AF Mott insulator and the TI. For this discussion let's use the periodic boundary condition, and consider the blue cut in Fig. (1). First we approach the Mott insulator

from the TI side. In the presence of  $U$  its lowest-energy exciton is magnetic (we don't use the word "triplet" because  $SU(2)$  is broken by the spin-orbit hopping down to  $U(1)$ ). An example of the XY order parameter profile associated with a magnetic exciton is shown in Fig. (4). Upon increasing  $U$  the transition into the AF insulator is triggered by the condensation of magnetic excitons. At the transition the modulus and the phase coherence of the AF XY order parameter develop simultaneously.

It is also instructive to approach the transition from the AF Mott insulator side. In this case one naturally expects the XY order to be destroyed by the condensation of vortices[17]. Because the AF XY order parameter, the triplet superconducting order parameter and the quantum spin Hall order parameter (which introduces the spin-dependent hopping term in Eq. (1)) form a Wess-Zumino-Witten five-tuplet for the free-graphene bandstructure[18], one expects the following charge density-skyrmion density relation[12]

$$\rho = \frac{1}{2\pi} \epsilon_{abc} n^a \partial_x n^b \partial_y n^c. \quad (12)$$

Here  $n^{1,2,3}$  are the components of the unit vector, with  $n^1$  associated with the quantum spin Hall order parameter (the strength of the imaginary second neighbor hopping in Eq. (1), which is fixed) and  $n^{2,3}$  associated with the AF XY order parameters. The vortex charge is therefore proportional to  $n_1(1 - n_1^2)$ . Since the XY order parameter vanishes at the Mott  $\rightarrow$  TI transition, the condensed vortices are charge neutral. Were it not true the vortex condensed phase can not be an insulator. In addition, since the modulus of the XY order parameter vanishes at the transition the vortices do not see a background magnetic flux which frustrates the vortex condensation. This implies the universality class of the transition is three-dimensional XY like as claimed in Ref.[9].

In summary, despite the apparent differences, the power-law correlated antiferromagnetic XY edges do exhibit properties expected for the quantum spin Hall insulators. The essential physics is the Goldstone-Wilczek mechanism; through which the space-time gradients of the phase angle of the XY order parameter are proportional to the charge and current densities. The space-time vortices of the XY order parameter violate edge charge conservation hence are prohibited in thermodynamic samples. This is the mechanism through which the gapless charge and spin excitation are protected at the edges. At the moment we do not have a good picture for the "spin liquid dome" in Fig. (1). The main findings of this paper are summarized in the abstract.

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