Δ(1232) Axial Charge and Form Factors from Lattice QCD

Constantia Alexandrou, Eric B. Gregory, Tomasz Korzec, Giannis Koutsou, John W. Negele, Toru Sato, and Antonios Tsapalis

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The $\Delta(1232)$ axial charge and form factors from lattice QCD

Constantia Alexandrou,[1,2] Eric B. Gregory,[1] Tomasz Korzec,[3] Giannis Koutsou,[4] John W. Negele,[5] Toru Sato,[6] and Antonios Tsapalis[7,8]

[1] *Department of Physics, University of Cyprus, P.O. Box 20357, 1678 Nicosia, Cyprus*
[2] *Computation-based Science and Technology Research Center, The Cyprus Institute, P.O. Box 27456, 1645 Nicosia, Cyprus*
[3] *Institut für Physik, Humboldt Universität zu Berlin, Newtonstrasse 15, 12489 Berlin, Germany*
[4] *Computation-based Science & Technology Research Center, The Cyprus Institute, P.O. Box 27456, 1645 Nicosia, Cyprus*
[5] *Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.*
[6] *Department of Physics, Osaka University, Osaka 560-0043, Japan*
[7] *Hellenic Naval Academy, Hatzikyriakou Ave., Pireaus 18539, Greece*
[8] *Department of Physics, National Technical University of Athens, Zografou Campus 15780, Athens, Greece*

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We present the first calculation on the Δ axial-vector and pseudoscalar form factors using lattice QCD. Two Goldberger-Treiman relations are derived and examined. A combined chiral fit is performed to the nucleon axial charge, N to Δ axial transition coupling constant and Δ axial charge.

Introduction. Numerical solution of Quantum Chromodynamics (QCD), the underlying theory of the strong interactions, has proved a very successful approach in providing a theoretical understanding of baryon structure. During the last few years, simulations of the discretized theory known as lattice QCD have included dynamical quarks with near to physical mass values [1]. A success of these recent simulations is the calculation of the low-lying hadron spectrum [2–4] showing agreement with the experimental values. The lattice set-up can also be applied to compute quantities that are not known experimentally. In this work we report the first calculation of the Δ axial-vector and pseudoscalar form factors.

Understanding the structure of the Δ resonance has great relevance to nuclear phenomenology. The Δ is a rather broad resonance close to the πN threshold. It therefore couples strongly to nucleons and pions making it an important ingredient in chiral expansions [5–8]. The Δ baryon resists experimental probing due to its short lifetime ($\sim 10^{-23}$ s) [9, 10]. Its axial charge and π-Δ coupling constants that are needed as input in chiral Lagrangians are difficult to measure. Baryon chiral expansion calculations that include the Δ explicitly follow one of two strategies as far as the determination of these parameters is concerned. The first is to relegate the axial charge to one of many fit parameters, and fit using lattice [5, 11], experimental [8], or partial-wave calculation data [7]. The second is to use estimates based on phenomenology such as the relation between the nucleon axial charge g_A that is well measured and the Δ axial charge, which can be derived from the large-N_c limit [12] or $SU(4)$ symmetry [13]. The Goldberger-Treiman (GT) relation is then used to get the effective $\pi\Delta\Delta$ coupling.

Quite recently, groups have calculated Δ axial charge through QCD sum rules [14] and χPT [15], and both have noted the lack of an explicit lattice calculation of this quantity. First-principles lattice QCD calculations can probe the structure of the Δ and indeed recent studies have produced calculations of the $\pi N\Delta$ coupling [16, 17] and the electromagnetic form-factors (FFs) of the Δ [18]. Using the lattice QCD techniques developed in Refs. [16–18], we are then well-positioned to calculate the axial charge and the effective pion-Δ coupling, $G_{\pi\Delta\Delta}$, as well as examine the GT relations as a way to relate the Δ axial charge to $G_{\pi\Delta\Delta}$.

In this letter we present the first lattice calculation of the axial-vector and pseudoscalar form factors of the Δ baryon. At non-zero momentum transfer q^2, we find a second pseudoscalar form-factor, yielding a second effective coupling constant, $H_{\pi\Delta\Delta}$. These calculations lead to two GT relations, which are indeed fulfilled by the lattice results.

Axial-vector and Pseudoscalar Matrix Element. We consider the matrix element of a current X between Δ^+ states

$$\langle \Delta(p_f,s_f)|X|\Delta(p_i,s_i)\rangle = \overline{u}_\sigma(p_f,s_f)\left[\mathcal{O}^X\right]^{\sigma\tau} u_\tau(p_i,s_i),$$

where u_σ denotes the Rarita-Schwinger vector-spinor (σ,τ are Lorentz indices), and p_i and p_f are the initial and final momenta of the Δ. For the axial-vector current $A^a_\mu(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi(x)$ the matrix element can be written in terms of four Lorentz-invariant FFs, labeled g_1, g_3, h_1 and h_3:

$$\begin{aligned}\left[\mathcal{O}^{A^3_\mu}\right]^{\sigma\tau} &= -\frac{1}{2}\Bigg[g^{\sigma\tau}\left(g_1(q^2)\gamma_\mu\gamma^5 + g_3(q^2)\frac{q_\mu}{2m_\Delta}\gamma^5\right) \\ &\quad + \frac{q^\sigma q^\tau}{4m_\Delta^2}\left(h_1(q^2)\gamma_\mu\gamma^5 + h_3(q^2)\frac{q_\mu}{2m_\Delta}\gamma^5\right)\Bigg], \quad (1)\end{aligned}$$

where $q = p_f - p_i$ is the momentum transfer. For the pseudoscalar current $P^a(x) = \overline{\psi}(x)\gamma_5\frac{\tau^a}{2}\psi(x)$, the matrix element can be written in terms of two FFs, to be defined below in connection to the GT relations.

Lattice Evaluation. Axial form factors on the lattice are extracted in a standard way from the three-point function

$$\langle G^{\Delta A^3_\mu \Delta}_{\sigma\tau}(t_f,t;\mathbf{p}_f,\mathbf{p}_i;\Gamma_\rho)\rangle = \sum_{\mathbf{x}_2,\,\mathbf{x}_1} e^{-i\mathbf{p}_f\cdot\mathbf{x}_2} e^{+i\mathbf{q}\cdot\mathbf{x}_1}\ \Gamma^{\beta\alpha}_\rho$$
$$\langle\Omega|T\left[\chi^{\sigma\alpha}_\Delta(\mathbf{x}_2,t_2)A^3_\mu(\mathbf{x}_1,t_1)\bar\chi^{\tau\beta}_\Delta(\mathbf{0},0)\right]|\Omega\rangle \quad (2)$$

where $\bar\chi_\Delta$ is an interpolating operator creating a state with the quantum numbers of the Δ^+ [18] and Γ_ρ is a set of projectors, given by $\Gamma_4=\frac{1}{4}(\mathbf{1}+\gamma_4)$ and $\Gamma_k=\mathrm{i}\Gamma_4\gamma_5\gamma_k$. A similar three-point function with $A^3_\mu\to P^3$ is required for the extraction of the pseudoscalar FFs. Technically these are evaluated via the *sequential inversion through the sink* [18] at fixed sink time-slice t_f, while the A^3_μ and P^3 operator insertion is supplied at all intermediate t-slices ($0\le t\le t_f$) and Fourier-transformed for all momenta $\mathbf{q}$ at a small extra CPU cost.

The kinematics are fixed to a static Δ sink ($\mathbf{p}_f=\mathbf{0},\mathbf{q}=-\mathbf{p}_i$). Denoting for convenience Monte-Carlo averages $G^X_{\sigma\tau}(\Gamma_\rho,\mathbf{q},t)=\langle G^{\Delta X\Delta}_{\sigma\tau}(t_f,t\,;\mathbf{0},-\mathbf{q};\Gamma_\rho)\rangle$ for $X=A^3_\mu$ or P^3, we construct the optimal ratio of three-point to two- point functions

$$R^X_{\sigma\tau}(\Gamma_\rho,\mathbf{q},t) \;=\; \frac{G^X_{\sigma\tau}(\Gamma_\rho,\mathbf{q},t)}{G_\Delta(\Gamma_4,\mathbf{0},t_f)}$$
$$\sqrt{\frac{G_\Delta(\Gamma_4,-\mathbf{q},t_f-t)G_\Delta(\Gamma_4,\mathbf{0},t)G_\Delta(\Gamma_4,\mathbf{0},t_f)}{G_\Delta(\Gamma_4,\mathbf{0},t_f-t)G_\Delta(\Gamma_4,-\mathbf{q},t)G_\Delta(\Gamma_4,-\mathbf{q},t_f)}}$$
$$\xrightarrow[t-t_i\to\infty]{t_f-t\to\infty}\ \Pi^X_{\sigma\tau}(\Gamma_\rho,\mathbf{q})\quad, \qquad (3)$$

where $G_\Delta(\Gamma_4,\mathbf{p},t)$ is the Δ propagator of momentum $\mathbf{p}$. This ratio eliminates unknown field renormalization constants and leading time dependences and tends to a constant at large Euclidean time separations t_f-t and t. A careful optimization in the space of the source-sink Lorentz indices σ,τ,ρ is required and only *two* linear combinations of sequential sources suffice to provide all four axial and two pseudoscalar FFs.

Smearing techniques are implemented resulting in satisfactory suppression of excited state effects allowing the source-sink distance to be fixed at about 1 fm [18]. Lattice computations of the matrix elements of the axial-vector and pseudoscalar currents for all transition momenta vectors $\mathbf{q}$ contributing to a given value of $Q^2=-(p_f-p_i)^2$ are simultaneously analyzed and the over-constrained system determines the form factors through a global χ^2 minimization. We note that $O(500)$ lattice measurements are involved in the extraction of the form factors for Q^2-values up to $\sim$ 2 GeV^2.

The parameters of the lattice ensembles used in this calculation are given in Table I. The quenched Wilson fermions gauge configurations enable the extraction of the FFs with small statistical errors. In addition, we obtained the FFs using dynamical domain-wall valence quarks matched to staggered sea fermions [19]. For the computation of the Δ axial charge we analyzed two additional sets, one with the mixed action with $m_\pi=0.498$ GeV and a second one using $N_f=2+1$ domain wall fermions [20] with $m_\pi=0.297$ GeV in order to have enough data from dynamical simulations for the chiral extrapolation. In all cases the u and d quarks are degenerate whereas the mass of the strange quark in the dynamical simulations is set to its physical mass. For the pion masses of these simulations the Δ is stable.

V	stat.	m_π (GeV)	m_N (GeV)	m_Δ (GeV)
Quenched Wilson fermions $\beta=6.0,\quad a^{-1}=2.14(6)$ GeV				
$32^3\times64$	200	0.563(4)	1.267(11)	1.470(15)
$32^3\times64$	200	0.490(4)	1.190(13)	1.425(16)
$32^3\times64$	200	0.411(4)	1.109(13)	1.382(19)
Mixed action, $a^{-1}=1.58(3)$ GeV				
Asqtad ($am_{\mathrm{u,d/s}}=0.02/0.05$), DWF ($am_{\mathrm{u,d}}=0.0313$)				
$20^3\times64$	264	0.498(3)	1.261(17)	1.589(35)
Asqtad ($am_{\mathrm{u,d/s}}=0.01/0.05$), DWF ($am_{\mathrm{u,d}}=0.0138$)				
$28^3\times64$	550	0.353(2)	1.191(19)	1.533(27)
Domain Wall Fermions (DWF) $m_{\mathrm{u,d}}/m_s=0.004/0.03,\ a^{-1}=2.34(3)$ GeV				
$32^3\times64$	1452	0.297(5)	1.27(9)	1.455(17)

TABLE I: Ensembles and parameters used in this work. We give in the first column the lattice size, in the second the statistics, in the third, fourth and fifth the pion, nucleon and Δ mass in GeV respectively.

Lattice results on the Δ axial form factors: In Fig. 1 we show the four axial Δ form-factors, g_1, g_3, h_1 and h_3, as a function of the momentum transfer Q^2. As can be seen, results obtained with the mixed action are in agreement with quenched results for g_1 and g_3. A similar conclusion holds for h_1 and h_3 albeit with much larger statistical errors in the case of the mixed action approach that we therefore omit from the plots for clarity. The value of the matrix element at $Q^2=0$ is connected to the axial charge defined by $\langle\Delta^{++}|A^3_\mu|\Delta^{++}\rangle-\langle\Delta^-|A^3_\mu|\Delta^-\rangle=G_{\Delta\Delta}\mathcal{M}_\mu$ [15]. At $q^2=0$ this is $G_{\Delta\Delta}=-3g_1(0)$.

Pseudoscalar FFs and the Goldberger-Treiman Relations. The Δ axial charge enters in baryon χPT expressions of many important quantities such as the axial charge of the nucleon. Many phenomenological results rely on this value, which is usually treated as a parameter to be determined from fits to experimental or lattice data. It can be related to the $\pi\Delta\Delta$ coupling via the GT relation. In Ref. [21] symmetry arguments in a quartet scheme where N^*_+, N^*_-, Δ_+ and Δ_- form a chiral multiplet, lead to the conclusion that $\pi\Delta_\pm\Delta_\pm$ couplings (with like-charged Δs) are forbidden at tree-level. Quark-model arguments [13] suggest that the $G_{\pi\Delta\Delta}=(4/5)G_{\pi NN}$. Clearly a non-perturbative calculation within lattice QCD of this coupling, as presented

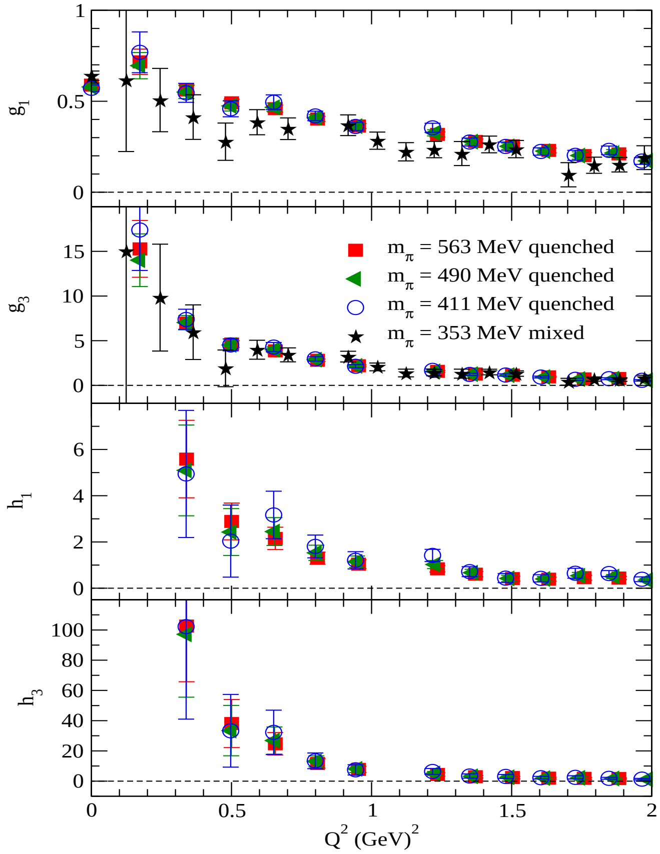

FIG. 1: Results for the axial form-factors, g_1, g_3, h_1 and h_3. Filled squares, filled triangles and open circles denote results in the quenched theory with $m_\pi = 0.563\,, 0.490\,, 0.411$ GeV respectively, and asterisks results using the mixed action with $m_\pi = 0.353$ GeV. For the h_1 and h_3 FFs, the noisier mixed action results are omitted for clarity.

in this work, provides valuable input to phenomenology.

Partial Conservation of the Axial Current (PCAC) when applied to the hadronic world leads to important phenomenological predictions such as the GT relation, originally derived for the nucleon state. Similarly, a non-diagonal GT relation, applicable to the axial $N-\Delta$ transition is formulated and relates the axial $N-\Delta$ coupling c_A to the $\pi N\Delta$ effective coupling. PCAC at the hadron level reads: $\partial^\mu A^a_\mu = f_\pi m_\pi^2 \pi^a$. In the SU(2) symmetric limit of QCD with m_q denoting the up/down mass, the pseudo-scalar density is related to the divergence of the axial-vector current through the axial Ward-Takahashi identity (AWI) $\partial^\mu A^a_\mu = 2m_q P^a$. Taking matrix elements of the LHS of the AWI identity between Δ states we can define two Lorentz-invariant $\pi\Delta\Delta$ form factors, $G_{\pi\Delta\Delta}(q^2)$ and $H_{\pi\Delta\Delta}(q^2)$ factoring out the pion pole as

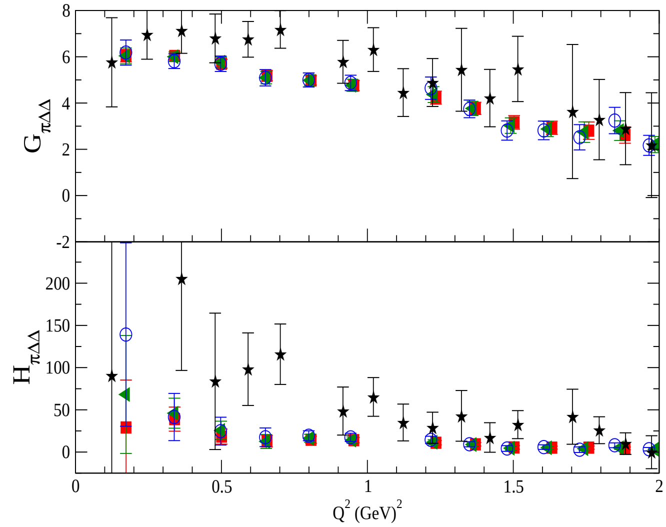

FIG. 2: The pseudoscalar Δ form factors $G_{\pi\Delta\Delta}$ and $H_{\pi\Delta\Delta}$. The notation is the same as that of Fig. 1.

dictated by PCAC

$$\langle \Delta(p_f, s_f)|P^3|\Delta(p_i, s_i)\rangle = -\frac{1}{2m_q}\frac{f_\pi m_\pi^2}{(m_\pi^2 - q^2)} \times$$
$$\overline{u}_\sigma \left[g^{\sigma\tau} G_{\pi\Delta\Delta}(q^2) + \frac{q^\sigma q^\tau}{4m_\Delta^2} H_{\pi\Delta\Delta}(q^2)\right]\gamma^5 u_\tau \ , \qquad (4)$$

Matrix elements of the AWI identity, $\langle\Delta|\partial_\mu A^\mu|\Delta\rangle = 2m_q\langle\Delta|P|\Delta\rangle$ now lead to a matrix equation, satisfied at finite q^2,

$$m_\Delta \left[g^{\sigma\rho}(g_1 - \tau g_3) + \frac{q^\sigma q^\rho}{4m_\Delta^2}(h_1 - \tau h_3)\right] =$$
$$\frac{f_\pi m_\pi^2}{(m_\pi^2 - q^2)} \left[g^{\sigma\rho} G_{\pi\Delta\Delta} + \frac{q^\sigma q^\rho}{4m_\Delta^2} H_{\pi\Delta\Delta}\right] \ , \qquad (5)$$

where $\tau = -q^2/(2m_\Delta)^2$. We display the pseudoscalar FFs in Fig. 2. The results using dynamical quark simulations have increased statistical errors and are consistent with the quenched results. The quark mass m_q, extracted from AWI, and f_π, calculated from the pion-to-vacuum amplitude, are taken from Ref. [16]. Our lattice results are consistent with pion-pole dominance that is manifested by a monopole dependence for $g_3(Q^2)$ *and* $h_1(Q^2)$, while results on $h_3(Q^2)$ are consistent with a dipole-dependence. we obtain a pair of GT-type relations, valid at finite q^2,

$$f_\pi G_{\pi\Delta\Delta}(q^2) = m_\Delta g_1(q^2)\,, f_\pi H_{\pi\Delta\Delta}(q^2) = m_\Delta h_1(q^2) \quad .(6)$$

The validity of the GT relations is examined by evaluating the ratios $f_\pi G_{\pi\Delta\Delta}(Q^2)/m_\Delta g_1(Q^2)$ and $f_\pi H_{\pi\Delta\Delta}(Q^2)/m_\Delta h_1(Q^2)$ as shown in Fig. 3. For the former ratio, for which statistical errors are smaller, lattice data show that indeed this ratio is compatible with unity

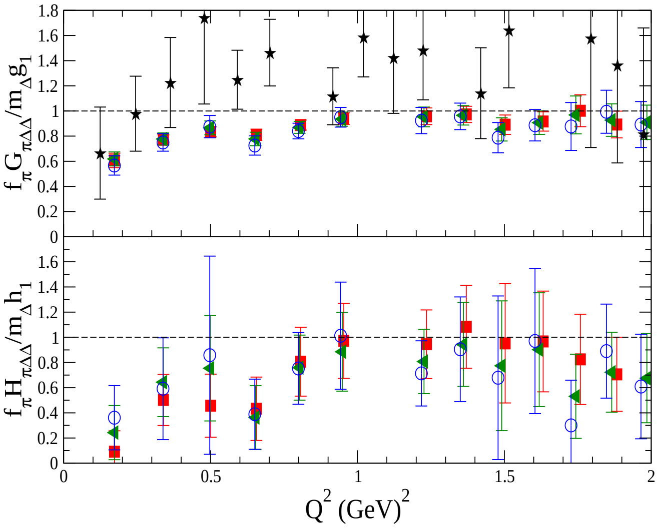

FIG. 3: The Goldberger-Treiman ratios from Eq. (6). The notation is the same as in Fig.1. In the lower graph we omitted for clarity results using the mixed action that carry larger statistical errors.

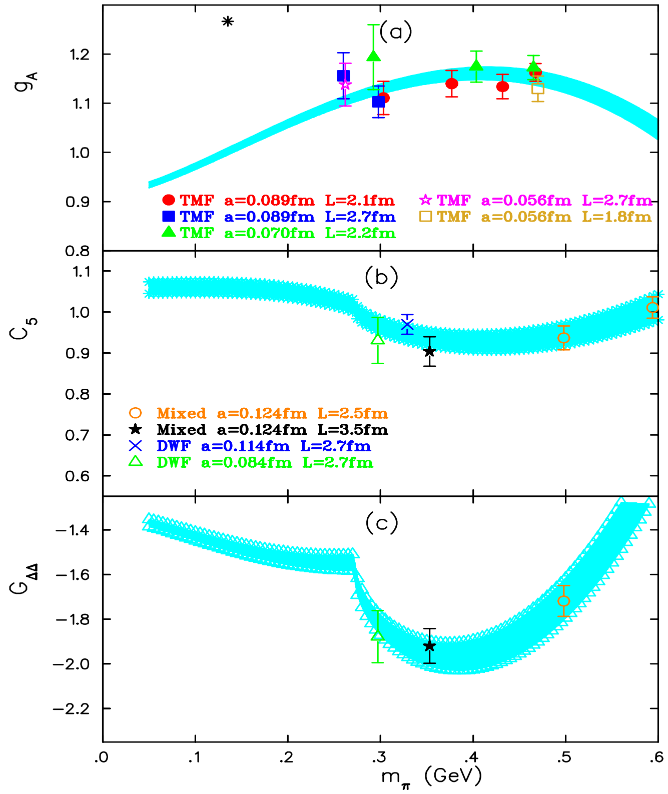

FIG. 4: Combined chiral fit: (a) Nucleon axial charge, g_A, fitted to lattice data obtained with $N_f = 2$ twisted mass fermions (TMF) [22]. The physical value is shown by the asterisk; (b) Real part of axial N to Δ transition coupling $C_5(0)$ [17]; (c) Real part of Δ axial charge $G_{\Delta\Delta} = -3g_1(0)$.

for $Q^2 \gtrsim 0.8\text{GeV}^2$ This behavior is similar to the one obtained for the pseudoscalar nucleon and $N-\Delta$ couplings $G_{\pi NN}$ and $G_{\pi N\Delta}$ [16] for the same ensembles. We expect that the behavior at low Q^2 will be affected by pion cloud effects as the mass of the pion decreases towards the physical point.

Δ axial charge and combined chiral fits. Having, for the first time, a set of lattice results for the axial nucleon charge [22], the axial $N-\Delta$ transition coupling, C_5 [17] and the Δ axial charge, allows us to perform a combined fit to all three quantities using heavy baryon χPT in the small scale expansion [15, 23, 24]. For g_A we use results obtained with twisted mass fermions since different discretization schemes yield results that are agreement [25]. The combined fit has seven free parameters, namely the values of the three axial coupling constants of the nucleon, the $N-\Delta$ and the Δ, three parameters related to the m_π^2-terms in the chiral expansions of g_A, C_5 and $G_{\Delta\Delta}$ and a constant entering the chiral expression of C_5 [24]. As can be seen in Fig. 4, lattice data for all three observables are approximately constant within the mass range considered. The best fits are shown by the bands that take into account the statistical errors of the lattice results. As have been observed in all recent lattice studies, the physical value of g_A is underestimated and this combined fit does not provide a possible resolution to this puzzle. Having lattice results at pion masses below 300 MeV will be essential to check the validity of these chiral expansions.

Conclusions. We have presented the first calculation of the axial-vector and pseudoscalar form factors of the Δ using lattice QCD. From the most general decomposition of the axial-vector and pseudoscalar vertex we derived two Goldberger-Treiman relations whose validity is satisfied at the same level of accuracy as that found for the nucleon case [16]. At zero momentum transfer the Δ matrix element yields the phenomenologically important Δ axial charge, which in this work is obtained for pion masses in the range of about 300 MeV to 500 MeV. As in the case of the nucleon axial charge, it shows a weak dependence on the pion mass in this mass range. Using lattice results for the axial nucleon charge, the axial N to Δ transition coupling and the Δ axial charge we performed, for the first time, a combined fit to all three quantities that provides a reasonable description to the lattice results. However, these state-of-the-art lattice results and chiral perturbation calculations, yield a value for the nucleon axial charge lower than its experimental value. Such a discrepancy between lattice and experimental results are seen in several key hadronic observables [25] calling for high accuracy lattice calculations with pion mass below 300 MeV in order to gain some insight on the chiral behavior of these fundamental quantities.

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