Color Dual Form for Gauge-Theory Amplitudes

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A Color Dual Form for Gauge-Theory Amplitudes

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Recently a duality between color and kinematics has been proposed, exposing a new unexpected structure in gauge theory and gravity scattering amplitudes. Here we propose that the relation goes deeper, allowing us to reorganize amplitudes into a form reminiscent of the standard color decomposition in terms of traces over generators, but with the role of color and kinematics swapped. By imposing additional conditions similar to Kleiss-Kuijf relations between partial amplitudes, the relationship between the earlier form satisfying the duality and the current one is invertible. We comment on extensions to loop level.

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Gauge theories have been long studied as descriptions of fundamental forces of Nature, offering detailed theoretical predictions for experiments and observations. In recent years, on the theoretical side, a remarkable array of simple structures have been uncovered in gauge-theory scattering amplitudes, especially in maximally supersymmetric gauge theories.

The present Letter will focus on one of these recently uncovered structures, the BCJ duality between color and kinematics [1, 2]. The duality is described in terms of graphs with only cubic vertices. In particular, at tree level we can decompose any scattering amplitude of adjoint color representation states as

$$\mathcal{A}_m^{\rm YM} = g^{m-2} \sum_j \frac{c_j n_j}{\prod_{\alpha_j} p^2_{\alpha_j}} \,, \qquad (1)$$

where the sum runs over graphs, labeled by j. At six points, for example, there are 105 cubic graphs. The two basic ones are displayed in fig. 1; the others are given by relabelings of these. The product in the denominator of eq. (1) runs over the Feynman propagators corresponding to each internal line of graph j. The c_j are the color factors obtained by dressing every three vertex with an $\tilde{f}^{abc} = i\sqrt{2} f^{abc}$ structure constant in the usual way, and the n_i are kinematic numerator factors. Representations of the form in eq. (1) can be obtained from Feynman diagrams, or other starting representations. We assign terms to diagrams according to the color factors. If we ecounter a contact term which is missing a propagator of the diagram to which it was assigned, we simply multiply and divide by the appropriate factor of $p^2_{\alpha_i}$. In this way, every term will have all propagators of the assigned diagram. The numerators do not necessarily have to be local, and some can vanish; the key constraint is that a given choice of numerators in eq. (1) yields the correct amplitude.

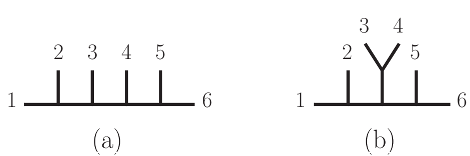

FIG. 1: The two diagram types at six points. Each graph can be taken to represent a color factor, a numerator or a set of Feynman propagators.

The BCJ duality proposes that there exist representations of the amplitude such that for any set of three graphs j_1, j_2, j_3, related by a color Jacobi identity, there is a corresponding numerator relation,

$$c_{j_1} \pm c_{j_2} \pm c_{j_3} = 0 \;\Rightarrow\; n_{j_1} \pm n_{j_2} \pm n_{j_3} = 0 \,, \qquad (2)$$

where the relative signs are dictated by the choice of signs in defining the color factors. In addition, the n_j are required to satisfy the same antisymmetry relations as satisfied by color factors under any relabelings,

$$c_j \to -c_j \;\Rightarrow\; n_j \to -n_j \,. \qquad (3)$$

The n_i do not necessarily need to be manifestly crossing symmetric. Indeed, solutions of the n_j in terms of amplitudes, which violate this condition, may be found in refs. [1, 3]. The conjecture has also been extended to all loop orders [2], offering a rather simple direct construction of gravity loop amplitudes from corresponding gauge-theory ones. One consequence of this duality is that it implies nontrivial relations between the tree-level color-ordered partial amplitudes of gauge theory [1]. These properties have also been studied from the vantage points of string theory [4] and field theory [5, 6].

How far does the analogy between color and kinematics extend? We know that in SU(N_c) gauge theory useful trace representations of the color factors exist. In the present Letter we show that the analogy between color and kinematics is sufficiently robust that a representation of the kinematic numerators exists which shares the same algebraic properties as color traces. Moreover we will show that additional interesting constraints can be imposed that uniquely determine the kinematic trace-like representation in terms of kinematic numerators satisfying the duality.

At tree level, the well-known trace-based color decom-

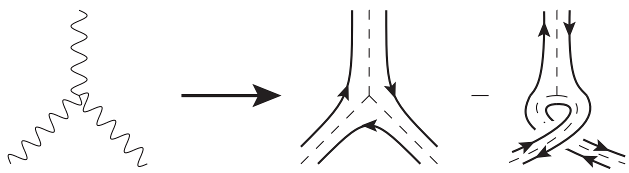

FIG. 2: An antisymmetric vertex in a cubic graph is replaced by a difference of two double-line vertices.

position of gluon amplitudes is [7],

$$\mathcal{A}_m^{\rm tree} = g^{m-2} \sum_{\sigma} {\rm Tr}[T^{a_1} \cdots T^{a_m}] A_m^{\rm tree}(1,2,\ldots,m)\,, \qquad (4)$$

where g is the gauge coupling, $A_m^{\rm tree}$ is a partial amplitude stripped of color factors, and the T^{a_i} are fundamental-representation matrices of an $\mathrm{SU}(N_c)$ Lie group. The sum runs over all non-cyclic permutations of external legs. The labels on momenta, polarizations or spinors, implicit in eq. (4), are also to be permuted in the sum. The color-stripped partial amplitudes can be expressed as a subset of diagrams following the same ordering of legs as in the partial amplitudes, but with no color factors. (For example, see eq. (4.5) of ref. [1].)

We propose that a dual description exists where we can swap the role of color and kinematics in the trace-based color decomposition, in particular by rewriting eq. (4) in a dual form,

$$\mathcal{A}_m^{\rm tree} \;=\; g^{m-2} \sum_{\sigma} \tau_{(12\ldots m)} A_m^{\rm dual}(1,2,\ldots,m)\,, \qquad (5)$$

where the $\tau_{(12\ldots m)}$ are kinematic prefactors satisfying the same cyclic properties as color traces. $A_m^{\rm dual}$ is a dual amplitude defined by replacing all kinematic numerators with color factors. That is, it is generated by the same color-ordered graphs that generate $A_m^{\rm tree}$, except at every vertex we have an $\tilde{f}^{abc}$ instead of a kinematic factor.

If we assume that the duality (2) holds, then gravity amplitudes can be obtained directly from Yang-Mills numerators by replacing the color factors with another copy of the kinematic numerators [1], as proven at tree level [6] and conjectured to hold to all loop orders [2]. Since the kinematic numerators share the same algebraic properties as color factors, it is then straightforward to rearrange gravity amplitudes into a form analogous to the gauge-theory dual form (5),

$$\mathcal{M}_m^{\rm tree} \;=\; i\left(\frac{\kappa}{2}\right)^{m-2} \sum_{\sigma} \tau_{(12\ldots m)} A_m^{\rm tree}(1,2,\ldots,m)\,,$$

Here κ is the gravitational coupling, $A_m^{\rm tree}$ are partial amplitudes of Yang-Mills theory and τ is exactly the same kinematic prefactor as in eq. (5).

The τ's are generated by expressing each numerator in terms of a set of objects which satisfy the cyclic symmetry

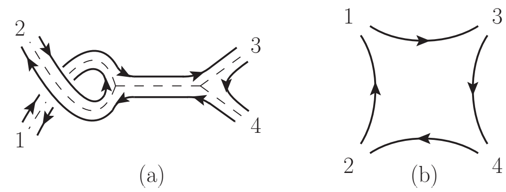

FIG. 3: Sewing of two vertices in a double-line graph (a). The ordering of the external legs follows the arrow around the graph. This graph corresponds with the kinematic quantity $\tau_{(1342)}$. The same double-line graph is displayed in (b) in a form emphasizing that it is the same quantity whether we sew the two three-point τ's in the 12 channel or 13 channel.

of color traces. For example, at the three-point level we demand that,

$$n_{123} = \tau_{(123)} - \tau_{(132)}\,,$$

where n_{123} is just the three vertex, as illustrated in fig. 2. In general, we will use parentheses around the subscript labels on the τ's to indicate which color trace the quantity is analogous to. In particular, $\tau_{(123)}$ is analogous to ${\rm Tr}[T^{a_1}T^{a_2}T^{a_3}]$.

For any number of legs, we can associate diagrams to the τ's in a manner completely parallel to the standard 't Hooft double-line formalism for color. Just as the color traces "trivialize" the color Jacobi identities, the τ's will do the same for the kinematic numerator factors, emphasizing the parallelism between color and kinematics. For example, in fig. 3(a) we display the double-line diagram for $\tau_{(1342)}$ obtained by sewing together two double-line three-point graphs. For the duality (2) to hold, the kinematic expression associated with each double-line graph should depend on only the topological structure of the graph, rather than on the detailed structure of vertices and internal lines in the underlying cubic graph. That is, a more appropriate way to draw the graph in fig. 3(a) is shown in fig. 3(b). In much the same way as single traces used in the tree-level color decomposition depend on only the cyclic ordering of legs, we demand that the τ's also depend on only the ordering. The property that one should obtain the same object when sewing in either channel is of course reminiscent of a key feature of string theory. However, at present we take the diagrams only as guides, since we do not have rules for directly combining lower-point τ's into higher-point ones.

To be more explicit, consider some examples. At four points there are three graphs contributing to eq. (1). These numerators are expressed in terms of τ via

$$n_{12(34)} = \tau_{(1[2,[3,4]])}\,, \qquad (6)$$

where the parenthesis on the indices of n indicates the associated propagators, *i.e.* in this case we have one, $i/(k_3+k_4)^2$. For τ the brackets signify an antisymmetric combination, *i.e.* $\tau_{(12[3,4])} \equiv \tau_{(1234)} - \tau_{(1243)}$. The two other channels at four points are just relabelings of this channel. We note that by expressing the n_j in terms of τ, the Jacobi-like equation $n_{12(34)} - n_{23(41)} - n_{42(31)} = 0$ holds automatically.

Can we find an explicit form of τ with the desired cyclicity? It is not difficult to check at four points that

$$\tau_{(1234)} \;=\; \frac{1}{6}\big(n_{12(34)} + n_{23(41)}\big)$$

indeed satisfies cyclicity and after using the duality relations (2) returns the numerators when combined as in eq. (6). The other τ's are given by relabelings. Interestingly, this solution satisfies some additional properties, namely invariance under a reversal of arguments and also an identity reminiscent of the U(1) decoupling identity for amplitudes,

$$\tau_{(1234)} + \tau_{(1342)} + \tau_{(1423)} = 0\,.$$

More generally, our ability to express the τ's directly in terms of the graph numerators n_j is precisely dependent on having τ satisfying identities of the same form as Kleiss-Kuijf identities,

$$\tau_{(1\{\alpha\}m\{\beta\})} \;=\; (-1)^{|\beta|}\sum_{\{\sigma\}}\tau_{(1\{\sigma\}m)}, \tag{7}$$

where the sum is over the "ordered permutations" $\{\sigma\}\in \mathrm{OP}(\{\alpha\},\{\beta^T\})$, that is, all permutations of $\{\alpha\}\bigcup\{\beta^T\}$ that maintain the order of the individual elements belonging to each set within the joint set. The notation $\{\beta^T\}$ represents the set $\{\beta\}$ with the ordering reversed, and $|\beta|$ is the number of elements in $\{\beta\}$. For gauge-theory partial amplitudes, these relations were conjectured in ref. [8] and proven in ref. [9]. They are a consequence of the antisymmetric nature of the vertices describing the n_j, as noted in ref. [1]. Indeed, any cyclic object, such as τ, that can be expressed as a linear combination of the n_j with prefactors that respect the symmetry and relabeling properties of the n_j automatically will satisfy the Kleiss-Kuijf relations.

At five points we have

$$\begin{aligned} n_{12(3(45))} \;&=\; \tau_{(1[2,[3,[4,5]]])}\,,\\ \tau_{(12345)} \;&=\; \frac{1}{20}\sum_{\sigma} n_{12(3(45))}\,, \end{aligned}$$

where the sum runs over cyclic permutations. The numerator $n_{12(3(45))}$ is the graph with Feynman propagators $i/(k_3+k_4+k_5)^2$ and $i/(k_4+k_5)^2$. One can straightforwardly verify that this τ satisfies the relations (7).

At six points we can express the numerators of the two diagrams shown in fig. 1(a) and (b) in terms of the τ via

$$\begin{aligned} n_{12(3(4(56)))} \;&=\; \tau_{(1[2,[3,[4,[5,6]]]])}\,,\\ n_{(12)(34)(56)} \;&=\; n_{12(3(4(56)))} - n_{12(4(3(56)))}\,. \end{aligned}$$

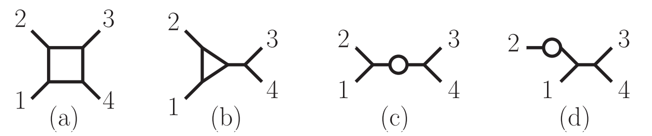

FIG. 4: Cubic diagrams appearing in one-loop four-point amplitudes.

The decomposition of τ in terms of the numerators is more complicated, in part because non-trivial rearrangements are possible using the duality (2). One such solution is

$$\begin{aligned} \tau_{(1\ldots6)} \;=\; \tfrac{1}{1890}\sum_{\sigma}\Big(&32n_{12(3(4(56)))} - 3n_{12(4(3(56)))}\\ &- \tfrac{31}{2}n_{12(3(6(45)))} - \tfrac{31}{2}n_{12(6(3(45)))} + 2n_{36(1(2(45)))}\\ &+ 2n_{36(2(1(45)))} + 2n_{36(4(1(25)))} - n_{26(1(4(35)))}\\ &- n_{26(4(1(35)))} - n_{35(1(2(46)))} - n_{35(2(1(46)))}\\ &+ n_{24(1(3(56)))} + n_{24(3(1(56)))} - n_{26(1(3(45)))}\\ &- n_{26(3(1(45)))}\Big)\,, \end{aligned} \tag{8}$$

where here the sum runs over the cyclic permutations of labels. The reader may also verify that $\tau_{(1\ldots6)}$ satisfies the Kleiss-Kuijf relations (7), given the algebraic properties of the kinematic numerators.

We have explicitly verified through nine points that an expression for τ in terms of kinematic numerators (as in eq. (8)) exists and that it automatically satisfies the Kleiss-Kuijf-like relations (7). The explicit expression for τ at $m=7$ contains more than 600 numerators and grows rapidly as the number of legs increases. Generally, it is better to think of the n_j numerators as functions of the τ. A general solution is

$$n_{12(3(\ldots m))\cdots)} = \tau_{(1[2,[3,[\ldots m]]\cdots])}\,.$$

The remaining numerators can be obtained by solving the duality relations (2), because they automatically hold for numerators expressed in terms of the τ's.

An interesting consequence of the above is that we can define an alternative color decomposition in place of eq. (4), where instead of color traces we use objects that satisfy Kleiss-Kuijf relations as well. These objects are given by taking the above equations for the τ in terms of the numerators n_j and replacing them with color factors c_j. This gives a valid set of objects to use in place of color traces in eq. (4). Using the explicit formulas given above, it is straightforward to confirm though six points that the amplitude (4) is unchanged under this substitution.

Can this construction be extended to loop level? As an initial peek at this question, we turn to the simple case of a one-loop four-point $\mathcal{N}=4$ super-Yang-Mills amplitude, first obtained in ref. [10]. We follow the same diagrammatic double-line formalism as at tree level.

The four types of cubic diagrams contributing to the four-point one-loop amplitude are shown in fig. 4. The numerators associated with these diagrams in the BCJ representation are

$$n_{(\mathrm{a})} = stA^{\mathrm{tree}}\,, \qquad n_{(\mathrm{b})} = n_{(\mathrm{c})} = n_{(\mathrm{d})} = 0\,,$$

where s and t are standard four-point Mandelstam invariants. Although this seems like a trivial state of affairs, expanding each numerator in terms of the double-line graphs reveals unexpected structure. In this expansion, there are multiple lines flowing around the double-line graphs, and we indicate the external legs attached to each line with a $(12\ldots)$ in the subscript of τ. Since each one-loop graph carries two independent lines, each τ will have two sets of parentheses in the subscripts. The graphs that appear in the expansion of the present example are $\tau_{(1)(234)}$, $\tau_{(12)(34)}$ and $\tau_{(1234)()}$, along with relabelings of these. The decomposition of the four numerators in terms of the τ's is straightforward and closely parallels a $\mathrm{U}(N_c)$ color decomposition.

One immediate solution to the decomposition can be obtained by setting $\tau_{(1234)()}$ proportional to the box numerator, and the other two τ functions to zero. A similar construction also works for the five-point amplitude [11] of $\mathcal{N}=4$ super-Yang-Mills theory. In this case $\tau_{(12345)()}$ are set proportional to the pentagon numerator given in ref. [12].[1]

A more interesting solution to the four-point decomposition is

$$\begin{aligned}\tau_{(1234)()} &= \frac{1}{62} stA^{\mathrm{tree}}\,, \qquad \tau_{(12)(34)} = \frac{3}{31} stA^{\mathrm{tree}}\,,\\ \tau_{(1)(234)} &= -\frac{3}{62} stA^{\mathrm{tree}}\,.\end{aligned}$$

With this solution the τ functions satisfy the same identities as the color-ordered partial amplitudes, namely,

$$\tau_{(\{\alpha\})(\{\beta\})} = (-1)^{|\beta|} \sum_{\{\sigma\}} \tau_{(\{\sigma\})()}\,,$$

where the sum is over the "cyclically ordered permutations" $\mathrm{COP}(\{\alpha\},\{\beta^T\})$, that is, all permutations of $\{\alpha\}\bigcup\{\beta^T\}$ that maintain the cyclic orderings of $\{\alpha\}$ and $\{\beta^T\}$ separately, and with one leg fixed (see ref. [11]).

To go beyond these simple examples, one needs a self-consistent assignment of loop momentum labels, with an understanding of the proper way to associate kinematic information with the double-line graphs. We leave the study of loop level to future work.

There are also a number of other interesting open questions. For example, it would be very useful to find a direct recursive procedure for building the τ's. Such a construction would automatically produce numerators that satisfy the color-kinematics duality. We also note that the double-line graphs describing the kinematic trace representation are reminiscent of open string diagrams. This brings up the interesting question of whether the properties we have described here can be unraveled in string theory. More generally, the trace-like representation described here emphasizes a group-theoretic origin for the duality. Because the same kinematic numerators appear in gravity theories, the same underlying group theory should carry over to gravity. It remains an important challenge to understand the origin of this duality and to fully map out its implications.

[1] We thank J. J. M. Carrasco and H. Johansson for pointing out that the color-kinematics duality holds for this form.

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