Nonzero $\theta_{13}$ for Neutrino Mixing in the Context of $A_4$ Symmetry
Ernest Ma and Daniel Wegman
Phys. Rev. Lett. 107, 061803 — Published 5 August 2011
DOI: 10.1103/PhysRevLett.107.061803
Nonzero $\theta_{13}$ for neutrino mixing in the context of $A_4$ symmetry

Ernest Ma and Daniel Wegman

Department of Physics and Astronomy, University of California, Riverside, California 92521, USA

Abstract

In the original 2004 paper which first derived tribimaximal mixing in the context of $A_4$, i.e. the non-Abelian finite symmetry group of the tetrahedron, as its simplest application, it was also pointed out how $\theta_{13} \neq 0$ may be accommodated. On the strength of the new T2K result that $0.03(0.04) \leq \sin^2 2\theta_{13} \leq 0.28(0.34)$ for $\delta_{CP} = 0$ and normal (inverted) neutrino mass hierarchy, we perform a more detailed analysis of how this original idea may be realized in the context of $A_4$. 
Neutrino oscillations require nonzero neutrino masses as well as nonzero neutrino mixing angles. The current combined world data imply \[1\]

\[
7.05 \times 10^{-5} \text{ eV}^2 \leq \Delta m_{21}^2 \leq 8.34 \times 10^{-5} \text{ eV}^2, \tag{1}
\]

\[
2.07 \times 10^{-3} \text{ eV}^2 \leq \Delta m_{32}^2 \leq 2.75 \times 10^{-3} \text{ eV}^2, \tag{2}
\]

\[
0.36 \leq \sin^2 \theta_{23} \leq 0.67, \quad 0.25 \leq \sin^2 \theta_{12} \leq 0.37, \tag{3}
\]

\[
\sin^2 \theta_{13} \leq 0.035 \text{ (90\% CL)}. \tag{4}
\]

However, the T2K Collaboration recently announced that a new measurement \([2]\) has yielded a nonzero $\theta_{13}$ at 90\% confidence level, i.e.

\[
0.03(0.04) \leq \sin^2 2\theta_{13} \leq 0.28(0.34) \tag{5}
\]

for $\delta_{CP} = 0$ and normal (inverted) neutrino mass hierarchy.

For several years now, the mixing matrix $U_{l\nu}$ linking the charged leptons ($e, \mu, \tau$) to the neutrino mass eigenstates ($\nu_1, \nu_2, \nu_3$) has often been assumed to be of tribimaximal form \([3]\), i.e.

\[
U_{TB} = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix}, \tag{6}
\]

which predicts $\theta_{13} = 0$. This is particularly appealing because it was derived in 2004 \([4]\) from the simple application of the symmetry group $A_4$, first used for understanding maximal $\nu_\mu - \nu_\tau$ mixing in 2001 \([5]\). However, even in that original 2004 paper \([4]\), the possibility of $\theta_{13} \neq 0$ was already anticipated. Although the new T2K result \([2]\) is only 2.5$\sigma$ away from zero, it is the most solid experimental indication to date of this possibility. Here we offer a more detailed analysis of how $\theta_{13} \neq 0$ may be realized in the context of $A_4$.

As is well-known, $A_4$ is the group of the even permutation of 4 objects. It is also the symmetry of the perfect three-dimensional tetrahedron \([6]\). It has 12 elements and 4 irreducible
The multiplication rule
\[ \mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3} + \mathbf{3}'. \tag{7} \]

The first step in understanding neutrino mixing is to show that \( A_4 \) allows the charged-lepton mass matrix to be diagonalized by the Cabibbo-Wolfenstein matrix \([7, 8]\)
\[ U_{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{8} \]
where \( \omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2 \), with three independent eigenvalues, i.e. \( m_e, m_\mu, m_\tau \).

This has been achieved in two ways. One is the original proposal of 2001 [5]. The other was discovered later in 2006 [9]. In the former, the lepton assignments are \( L_i = (\nu_i, l_i) \sim \mathbf{3}, \quad l_1 \sim \mathbf{1}, \quad l_2 \sim \mathbf{1}', \quad l_3 \sim \mathbf{1}'', \) with 3 Higgs doublets \( \Phi_i = (\phi_i^0, \phi_i^-) \sim \mathbf{3}. \) In the latter, they are \( L_i = (\nu_i, l_i) \sim \mathbf{3}, \quad l_i \sim \mathbf{3}, \) with 4 Higgs doublets \( \Phi_i = (\phi_i^0, \phi_i^-) \sim \mathbf{3}. \) Assuming \( v_1 = v_2 = v_3 \) for the vacuum expectation values of \( \Phi_i \), which correspond to a \( Z_3 \) residual symmetry (lepton triality) [10, 11, 12, 13], the seemingly impossible result of a diagonal charged-lepton matrix is always obtained from \( U_{CW} \) of Eq.(8), independent of the values of \( m_e, m_\mu, m_\tau \). This is a highly nontrivial result, which motivates how the otherwise arbitrary \( 3 \times 3 \) neutrino mass should be organized. It argues strongly for an underlying non-Abelian symmetry with a three-dimensional irreducible representation, the smallest of which is \( A_4 \).

We now consider the neutrino mass matrix in the original \( A_4 \) basis. Let there be 6 heavy Higgs triplets [14]:
\[ \xi_1 \sim \mathbf{1}, \quad \xi_2 \sim \mathbf{1}', \quad \xi_3 \sim \mathbf{1}'', \quad \xi_i \sim \mathbf{3} \quad (i = 4, 5, 6), \tag{9} \]
where \( \xi_i = (\xi_i^+, \xi_i^+, \xi_i^0) \). Then
\[ \mathcal{M}_\nu = \begin{pmatrix} a + b + c & a & e \\ f & a + \omega b + \omega^2 c & d \\ e & d & a + \omega^2 b + \omega c \end{pmatrix}, \tag{10} \]
where $a$ comes from $\langle \xi_1^0 \rangle$, $b$ from $\langle \xi_2^0 \rangle$, $c$ from $\langle \xi_3^0 \rangle$, $d$ from $\langle \xi_4^0 \rangle$, $e$ from $\langle \xi_5^0 \rangle$, $f$ from $\langle \xi_6^0 \rangle$. As it stands, there is of course no prediction at all. For a pattern to emerge, the way $A_4$ breaks into its subgroups must be considered. For $b = c$ and $e = f = 0$, which breaks $A_4$ to $Z_2$, the neutrino mass matrix, written in the basis where the charged-lepton mass matrix is diagonal, is given by

$$M^{(e, \mu, \tau)}_{\nu} = U^\dagger C W M_{\nu} U^* C W = \begin{pmatrix} a + (2d/3) & b - (d/3) & b - (d/3) \\ b - (d/3) & b + (2d/3) & a - (d/3) \\ b - (d/3) & a - (d/3) & b + (2d/3) \end{pmatrix},$$

which is indeed diagonalized by $U_{TB}$ of Eq.(6), with eigenvalues $m_1 = a - b + d$, $m_2 = a + 2b$, and $m_3 = -a + b + d$. It has been shown [15] how this pattern is obtained from $A_4$ alone with the help of lepton number.

Deviations from tribimaximal mixing may be obtained for $b \neq c$. This will allow $\nu_1$ to mix with $\nu_3$ and $\theta_{13}$ becomes nonzero. However, this same mixing will move $\theta_{12}$ to a larger value [4] so that $\tan^2 \theta_{12} > 0.5$ which is not favored by current data. To allow $\tan^2 \theta_{12} < 0.5$, it was proposed [4] that $e = -f \neq 0$ in Eq.(10). This is maintained by an assumed residual symmetry of the $\xi \Phi \Phi$ soft terms of the Higgs potential under which $\xi_5 \leftrightarrow -\xi_6$ and $\Phi_2 \leftrightarrow \Phi_3$. As a result, the neutrino mass matrix under $U_{TB}$ is no longer diagonal, but is given by [4]

$$M^{(1,2,3)}_{\nu} = \begin{pmatrix} m_1 & 0 & m_4 \\ 0 & m_2 & m_5 \\ m_4 & m_5 & m_3 \end{pmatrix},$$

where $m_1 = a - (b + c)/2 + d$, $m_2 = a + b + c$, $m_3 = -a + (b + c)/2 + d$, $m_4 = \sqrt{3}/2(c - b)$ and $m_5 = -i\sqrt{2}e$. If $m_4 = 0$, then $\nu_2$ mixes with $\nu_3$ and it can be shown that the allowed range of $\theta_{23}$ from Eq.(3) implies $\sin^2 2\theta_{13} \leq 0.04$ which lies on the outer edge of the allowed region of Eq.(5). In the following, we consider both $m_4$ and $m_5$ to be nonzero and study various numerical solutions to the T2K data.

The atmospheric neutrino mixing is assumed to be maximal, i.e. $\sin^2 2\theta_{23} = 1$, which is also the assumption of T2K in obtaining their new result. The solar neutrino mixing is
taken to be \( \sin^2 2\theta_{12} = 0.87 \pm 0.3 \) \cite{1}. We also use \( \Delta m^2_{32} = 2.40 \times 10^{-3} \) eV\(^2\) which is the value used by T2K, and \( \Delta m^2_{21} = 7.65 \times 10^{-5} \) eV\(^2\). For the central value of \( \theta_{12} = 34.43^{\circ} \), we have \( \tan^2 \theta_{12} = 0.47 \) which is rather close to the tribimaximal prediction of 0.5. Using this and assuming the central value of \( \sin \theta_{13} = 0.168 \) (\( \sin^2 \theta_{13} = 0.11 \)), the zero entry of the neutrino mass matrix of Eq.(12) implies the condition

\[
0.007655m'_1 - 0.020990m'_2 + 0.013342m'_3 = 0, \tag{13}
\]

where \( m'_{1,2,3} \) are the mass eigenvalues of Eq.(12). Hence they are related to the measured \( \Delta m^2_{32} \) and \( \Delta m^2_{21} \) by

\[
m'_2 &= \pm \sqrt{m'_1^2 + \Delta m^2_{21}}, \tag{14}
m'_3 &= \pm \sqrt{m'_1^2 + \Delta m^2_{21}/2 \pm \Delta m^2_{32}}. \tag{15}
\]

There is only one solution to Eq.(13), i.e.

\[
m'_1 = 0.0246 \text{ eV}, \quad m'_2 = -0.0261 \text{ eV}, \quad m'_3 = -0.0552 \text{ eV}, \tag{16}
\]

which exhibits normal mass hierarchy. From this solution, we then obtain \( m'_{1,2,3,4,5} \) and the original \( A_4 \) parameters \( a, b, c, d, e \). The \( \nu_e \) mass observed in nuclear beta decay is given by \( \sum_i |U_{ei}|^2 |m'_i| = 0.026 \) eV. The effective mass \( m_{ee} \) for neutrinoless double beta decay is

\[
m_{ee} = |a + (2/3)d| = |(2/3)m_1 + (1/3)m_2|. \tag{17}
\]

We plot in Figs. 1 to 3 the solutions for \( |m'_{1,2,3}| \) and \( m_{ee} \) as a function of \( \sin^2 2\theta_{13} \) in the range 0.03 to 0.135 [corresponding to the upper bound given in Eq.(4)] for \( \sin^2 \theta_{23} = 1 \) and the values \( \sin^2 2\theta_{12} = 0.84, 0.87, 0.90 \). Thus \( m_{ee} \) is predicted to be at most 0.04 eV. As for the \( \nu_e \) mass in nuclear beta decay, it can be read off as approximately given by \( (2|m'_1| + |m'_2|)/3 \). We also plot in Fig. 4 the values of \( m_{1,2,3,4,5} \) for \( \sin^2 2\theta_{12} = 0.87 \). This shows that \( m_4 \) and \( m_5 \), i.e. the parameters of \( A_4 \) which deviate from tribimaximal mixing, are indeed small. In
terms of $A_4$ symmetry, the following breaking patterns are in effect: in the charged-lepton sector, $A_4$ breaks to $Z_3$ (which may be verified experimentally from Higgs-boson decay [13]); in the neutrino sector, $A_4$ breaks first to $Z_2$ (the tribimaximal limit), and then $Z_2$ is also broken with the pattern $b \neq c$ and $f = -e$, which may be maintained by a suitably chosen set of soft terms.

In conclusion, on the strength of the recent observation [2] of a nonzero $\theta_{13}$ for neutrino mixing, the original $A_4$ proposal of 2004 [4] is updated. We find that solutions are indeed possible with the most recent data but only in a normal hierarchy of neutrino masses, i.e. $|m_1| < |m_2'| < |m_3'|$. We confirm that the parameters of $A_4$ which deviate from tribimaximal mixing, i.e. $m_4$ and $m_5$, are indeed small. We also make predictions on the effective $m_{ee}$ in neutrinoless double beta decay.

This work is supported in part by the U. S. Department of Energy under Grant No. DE-FG03-94ER40837.

References


Figure 1: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.84$. 

$\sin^2(2\theta_{12}) = .84$
Figure 2: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.87$. 
Figure 3: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.90$. 
Figure 4: The $A_4$ parameters $m_{1,2,3,4,5}$ of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.87$. 