Equilibration of Luttinger Liquid and Conductance of Quantum Wires

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Equilibration of Luttinger liquid and conductance of quantum wires

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Luttinger liquid theory describes one-dimensional electron systems in terms of non-interacting bosonic excitations. In this approximation thermal excitations are decoupled from the current flowing through a quantum wire, and the conductance is quantized. We show that relaxation processes not captured by the Luttinger liquid theory lead to equilibration of the excitations with the current and give rise to a temperature-dependent correction to the conductance. In long wires, the magnitude of the correction is expressed in terms of the velocities of bosonic excitations. In shorter wires it is controlled by the relaxation rate.

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The conductance of quantum wires measured at low temperatures is quantized in units of $2e^2/h$ [1]. This universality is commonly interpreted in the framework of the Luttinger liquid theory [2, 3], according to which the conductance is not affected by either the electron-electron interactions in the wire or the temperature [4]. On the other hand, recent experiments show temperature-dependent corrections to the conductance [5–9], in apparent contradiction with the theory [4]. While the origin of the corrections remains the subject of debate [10], much of the recent theoretical interest focused on the physics of one-dimensional fermions not captured by the Luttinger liquid theory [11–15].

The concept of the Luttinger liquid applies to generic low-energy properties of systems of interacting one-dimensional fermions. The key feature of this theory is that the elementary excitations can be viewed as non-interacting bosons with an acoustic spectrum:

$$H = \sum_q \hbar v|q| b_q^\dagger b_q + \frac{\pi\hbar}{2L_0}[v_N(N-N_0)^2 + v_J J^2], \quad (1)$$

Ref. [2]. Here b_q is the operator destroying a boson with momentum $\hbar q$, the velocities v, v_N, and v_J may depend on the electron-electron interactions, L_0 is the system size (periodic boundary conditions are assumed). In addition to the boson occupation numbers N_q, the state of the system is described by two integers, N and J. The latter can be related to the total numbers of the right- and left-moving electrons in the system, $J = N^R - N^L$, and $N = N^R + N^L$; N_0 is the total number of electrons in some reference state.

It is important to note that within the Luttinger liquid approximation the numbers N^R and N^L of the right- and left-movers are conserved, i.e., electron-electron interactions do not cause backscattering of electrons. As a result, if the interacting quantum wire coupled to non-interacting leads is described by the Luttinger liquid model [4], the electrons entering the wire from the leads always pass through it, and the interactions do not affect the conductance. On the other hand, if the interactions are treated as a small perturbation to the non-interacting fermion picture, they do lead to backscattering of electrons and reduce the conductance [13–15]. The backscattering occurs via three-electron scattering processes, which are not included in the Luttinger liquid theory.

The corrections to the quantized conductance [5–9] are observed in the regime of low electron density, where the electron-electron interactions are strong. Thus the theory [14, 15] based on the picture of weakly interacting electrons cannot be applied to the experiments directly, and the generalization to the case of arbitrary interaction strength is needed. In this paper we develop such a theory using the Luttinger liquid picture as a starting point and accounting for the electron backscattering processes.

We start by noting that the Luttinger liquid Hamiltonian (1) describes the stable low-energy fixed point of a one-dimensional system of interacting fermions. The full low-energy Hamiltonian consists of (1) and a number of perturbations describing interactions between the excitations. Although formally irrelevant, these terms are responsible for establishing the thermal equilibrium in the system. The equilibrium distribution of the bosonic excitations is dictated by conservation laws and independent of the detailed form of the irrelevant perturbations,

$$N_q = \frac{1}{e^{\hbar(v|q|-uq)/T} - 1}. \quad (2)$$

Due to the conservation of energy and momentum, the distribution function is characterized by two parameters, temperature T and velocity u. Relaxation towards the equilibrium distribution (2) is due to the irrelevant perturbations which have the form of higher-order terms in bosonic variables, and the respective relaxation time is expected to scale as a power of temperature, $\tau_0 \propto T^{-\alpha}$.

The boson distribution (2) can be derived from the general principle that for systems where energy and momentum are conserved the probability of realization of a given microscopic state i depends on its energy E_i and momentum P_i as $e^{-(E_i - uP_i)/T}$. The momentum of a Lut-

tinger liquid is given by [2]

$$P = p_F J + \sum_q \hbar q\, b_q^\dagger b_q, \tag{3}$$

where $p_F = \pi\hbar N/L_0$ and the first term represents the momentum of a filled Fermi surface with $J = N^R - N^L$ extra electrons at the right Fermi point. The presence of J in the momentum (3) means that its distribution $e^{-(\Delta J^2 - u p_F J)/T}$ and the boson occupation numbers (2) are controlled by the same parameter u. (Here $\Delta = \pi\hbar v_J/2L_0$.) In particular, the thermodynamic average of J is given by $J = Nu/v_J$. Taking into account the relation [2] between J and the electric current, $I = ev_J(J/L_0)$, we conclude that at full equilibrium the velocity u of the boson gas coincides with the drift velocity $v_d = I/en$, where $n = N/L_0$ is the electron density. The conclusion $u = v_d$ is obvious in the presence of Galilean invariance, in which case the boson gas should be at rest in a frame moving with the drift velocity.

Establishment of full thermodynamic equilibrium requires the presence of backscattering processes which change the numbers of the right- and left-moving electrons N^R and N^L and transfer the corresponding momentum to the boson excitations. Such processes are not accounted for in the Luttinger liquid theory and, as we discuss below, occur at a time scale τ much longer than the time τ_0 required for the formation of the boson distribution (2). Therefore, at time scales $t \gg \tau_0$ equilibration processes may be described by relaxation of the velocity u of the boson gas towards v_d,

$$\dot u = -\frac{u - v_d}{\tau}. \tag{4}$$

Due to the time dependence of u, the total momentum of the bosons P_b changes as well. The rate of this change is easily obtained from Eqs. (2) and (3), $\dot P_b = (\pi L_0 T^2/3\hbar v^3)\dot u$. Conservation of the total momentum (3) of the Luttinger liquid implies that the number of right-moving electrons changes at the rate $\dot N^R = \dot J/2 = -\dot P_b/2p_F$, resulting in

$$\dot n^R = \frac{\pi}{6\hbar}\,\frac{T^2}{v^3 p_F}\,\frac{u - v_d}{\tau}. \tag{5}$$

Here n^R is the density of the right-moving electrons in the system.

The expressions (4) and (5) describe equilibration of a system of one-dimensional electrons at any interaction strength. Microscopic theory of the phenomenological parameter τ can be developed in two limiting cases. In the limit of weakly interacting electrons the backscattering processes involve a hole diffusing through the bottom of the band due the three-particle collisions [13, 15]. Because of small occupation probabilities of the hole states, the backscattering rate follows the Arrhenius law, $\tau^{-1} \propto e^{-D/T}$, with D given by the Fermi energy. In the limit of strong interactions, the electrons form a Wigner crystal structure. The excitations are phonons in this crystal, which at low energies are equivalent to the bosons in the Luttinger liquid. The total (quasi-) momentum of the phonons can change via umklapp processes, whose rate also follows the Arrhenius law, with D given by the Debye frequency of the Wigner crystal [16].

The above examples demonstrate that the backscattering processes involve states at energies of the order of the bandwidth. Such processes are neglected in the renormalization group schemes describing the low-energy properties of Luttinger liquids. Although at arbitrary interaction strengths these processes have not been studied microscopically, we expect the scattering rate to follow the Arrhenius law with the activation energy of the order of the bandwidth, $D \sim vp_F$. As a result, at low temperatures we have $\tau \gg \tau_0$.

In this paper we study the conductance of a long uniform quantum wire adiabatically connected to ideal metal leads, starting with the case of spinless electrons. We assume that the length of the wire L satisfies the condition $L \gg v\tau_0$, but can be comparable with $v\tau$. The electrons in the leads are assumed to be non-interacting, and their distributions are given by the standard Fermi functions with slightly different chemical potentials, $\mu_l - \mu_r = eV \ll T$. Inside the wire electron-electron interactions are significant, Luttinger liquid is formed, and the state of the system is described by the occupation numbers of the bosonic excitations.

It is convenient to divide the wire into small segments of length ΔL chosen such that $v\tau_0 \ll \Delta L \ll L \sim v\tau$. Because $\Delta L \ll v\tau$, the backscattering of electrons in each segment is negligible, the numbers of the right- and left-movers $\Delta N^{R,L}$ are conserved, and the Luttinger liquid is described by the Hamiltonian (1). Since $\Delta L \gg v\tau_0$, the bosonic excitations are in equilibrium with each other, with the distribution given by Eq. (2). The absence of global equilibrium in the wire subjected to a small voltage bias V is reflected in the fact that the distribution (2) has a nonvanishing boost velocity $u \propto V$. Thus the state of the Luttinger liquid in each segment of the wire is described by the temperature T and velocity u of the bosonic excitations and by the densities $n^{R,L} = \Delta N^{R,L}/\Delta L$ of the right- and left-moving electrons. Although each small segment is in equilibrium, the electron backscattering may give rise to different equilibrium states for segments at distances of order $v\tau$ from each other. Thus all four parameters T, u, n^R, and n^L may depend on the position in the wire.

Equilibration of the electron system results in the finite backscattering rate (5). In the dc regime the total number N^R of the right-movers in the wire does not depend on time, which means that the currents j_l^R and j_r^R of the right-moving electrons at the left and right ends of the wire are different, $j_r^R = j_l^R + \dot N^R$. Using the expression $I = e(j_r^R + j_r^L)$ for the electric current at the

right end of the wire, we rewrite the above condition as $I = e(j_l^R + j_r^L) + e\dot{N}^R$. The currents j_l^R and j_r^L enter the wire from the leads and are easily computed using the respective Fermi distributions of non-interacting electrons. This yields

$$I = G_0 V + e\dot{N}^R, \tag{6}$$

where $G_0 = e^2/h$ is the standard Landauer conductance of a non-interacting wire.

We now use conservation of momentum and energy to find $\dot{N}^R$ in Eq. (6) in the linear response regime $I \to 0$. Conservation of the momentum (3) implies that backscattering of a single right-mover, $\Delta N^R = -1$, is accompanied by transfer of momentum $2p_F$ to the bosons. On the other hand, according to Eq. (1) at $J \to 0$ the total energy E of the bosons should remain unchanged. Thus, the momenta transferred to the right- and left-moving bosons must be equal, i.e., $\Delta P^{R,L} = p_F$ and $\Delta E^{R,L} = \pm v p_F$. We therefore conclude that backscattering of right-moving electrons changes the energy of the right-moving bosons at a rate

$$\dot{E}^R = -v p_F \dot{N}^R. \tag{7}$$

In the absence of backscattering all the bosons in the system are in thermal equilibrium with the leads, whose temperatures are assumed to be equal. Thus the energy current in the wire vanishes. Backscattering transfers energy from the left- to the right-moving bosons at the rate (7), resulting in the net energy current

$$j_E = \dot{E}^R. \tag{8}$$

Equations (6)–(8) express conservation laws for particle number, momentum, and energy. To make further progress we need to evaluate the energy current j_E.

In the linear response regime $J \to 0$ and only the bosons contribute to the energy current in the Luttinger liquid. The latter are distributed according to Eq. (2). At non-vanishing velocity u their energy current is

$$j_E = \frac{\pi}{3}\frac{T^2}{\hbar v}\, u. \tag{9}$$

The parameters u and T of the boson distribution (2) may depend on position x along the wire. In the linear regime, $I \to 0$, one expects $u \propto I$ and $dT/dx \propto I$. Then from conservation of energy, $dj_E/dx = 0$, and Eq. (9) we conclude that u is position independent.

Combining Eqs. (7)–(9) we find the following relation between the backscattering rate $\dot{N}^R$ and velocity u,

$$\dot{N}^R = -\frac{\pi}{3}\frac{T^2}{\hbar v^2 p_F}\, u. \tag{10}$$

This equation expresses the energy and momentum conservation laws in a wire connected to ideal leads and must be satisfied regardless of the equilibration rate τ. On the other hand, if the equilibration rate is known, the backscattering rate $\dot{N}^R$ can also be related to the boson gas velocity u via Eq. (5). Comparing (5) and (10) we conclude that in a wire of length L the boson gas velocity u is related to the drift velocity $v_d = I/en$ as

$$u = \frac{L}{l_{\rm eq} + L}\, v_d, \quad l_{\rm eq} = 2v\tau. \tag{11}$$

Substituting Eqs. (11) and (10) into (6), we find the conductance of the wire

$$G = G_0 \left(1 - \frac{\pi^2}{3}\frac{T^2}{v^2 p_F^2}\frac{L}{l_{\rm eq} + L}\right). \tag{12}$$

The second term in Eq. (12) represents a correction to the Luttinger liquid result [4] originating from equilibration of bosonic excitations. The result (12) is valid for any strength of electron-electron interactions. At weak interactions v coincides with the Fermi velocity $v_F = p_F/m$, and Eq. (12) agrees with the results of Refs. [14, 15]. In short wires, $L \ll l_{\rm eq}$, the correction to the quantized conductance is proportional to the equilibration rate $\tau^{-1} \propto e^{-D/T}$ and the length of the wire, $\delta G \propto L e^{-D/T}$. At $L \gg l_{\rm eq}$, the correction saturates at $\delta G \sim (e^2/h)(T/v p_F)^2$, but remains small within the range of applicability of our theory, $T \ll v p_F$.

Our result (12) applies to systems where electron spins are polarized by an external magnetic field. On the other hand, the temperature-dependent corrections to conductance [5–9] are observed in unpolarized wires, and spins are believed to play a significant role in this phenomenon. We now generalize our discussion to the case of electrons with spin. Compared to Eq. (1), the Hamiltonian of the Luttinger liquid with spins [3] has twice the number of modes. Most importantly, there are two branches of bosonic excitations, corresponding to waves of charge and spin densities and propagating at different velocities, v_ρ and v_σ, respectively. In addition, instead of the numbers of right- and left-moving electrons $N^{R,L} = (N \pm J)/2$, the state of each system is described by the numbers of such electrons for each spin direction, $N_\uparrow^{R,L}$ and $N_\downarrow^{R,L}$.

The discussion leading to Eq. (12) can then be readily modified as follows. The conductance G_0 of non-interacting wire in Eq. (6) is now $2e^2/h$. The number of right-movers N^R in Eqs. (5), (6), (7), and (10) should be understood as $N_\uparrow^R + N_\downarrow^R$, whereas E^R in Eqs. (7) and (8) is the total energy of right-moving bosons, in both charge and spin sectors. In the right-hand sides of Eq. (5) and (9) one has to replace $v^{-n} \to v_\rho^{-n} + v_\sigma^{-n}$. The most significant change appears in Eq. (7). When an electron is backscattered and momentum $p_F = \pi\hbar n/2$ is transferred to right-moving bosons, it is distributed unevenly between the charge and spin branches. Due to the relatively fast equilibration of bosons their distribution functions have the form (2) with different velocities

v_ρ and v_σ in the two sectors but the same u. Momentum change is accommodated by a small change of u. As a result, the fraction of the momentum p_F transferred into charge or spin branch is proportional to $v_{\rho,\sigma}^{-3}$, whereas the corresponding energies scale as $v_{\rho,\sigma}^{-2}$. We thus generalize Eq. (7) as

$$\dot{E}^R = -\frac{v_\rho^{-2} + v_\sigma^{-2}}{v_\rho^{-3} + v_\sigma^{-3}}\, p_F \dot{N}^R. \qquad (13)$$

As a result we recover the general form (12) of the dependence of conductance on the length of the wire with

$$v^2 = \frac{2(v_\rho^{-2} + v_\sigma^{-2})}{(v_\rho^{-1} + v_\sigma^{-1})(v_\rho^{-3} + v_\sigma^{-3})}, \quad l_{\rm eq} = 2\tau \frac{v_\rho^{-1} + v_\sigma^{-1}}{v_\rho^{-2} + v_\sigma^{-2}},$$

and $G_0 = 2e^2/h$. The presence of both v_ρ and v_σ in the expression for the correction to the quantized conductance $2e^2/h$ reflects the fact that the momentum of backscattered electrons is distributed among the bosonic excitations in both the charge and spin sectors.

The experimentally observed corrections to the quantized conductance [5–9] are most prominent at low electron densities. In this regime the Coulomb interactions between electrons are effectively very strong, resulting in $v_\rho \gg v_\sigma$. The conductance then takes the form

$$G = \frac{2e^2}{h}\left[1 - \frac{\pi^2}{6}\left(\frac{T}{v_\sigma p_F}\right)^2 \frac{L}{l_{\rm eq} + L}\right], \qquad (14)$$

with $l_{\rm eq} = 2v_\sigma\tau$. Importantly, at $v_\sigma \ll v_\rho$ the magnitude of the correction to the quantized conductance is controlled by the spin velocity.

Similarly to the case of spinless electrons, for long wires, $L \gg l_{\rm eq}$, the correction shows quadratic temperature dependence, $\delta G \sim (e^2/h)(T/v_\sigma p_F)^2$. Within the range of applicability of our theory, $T \ll v_\sigma p_F$, the correction remains small. In short wires, $L \ll l_{\rm eq}$, the temperature dependence of δG is controlled by that of the relaxation time, $\delta G \propto \tau^{-1}$. We are not aware of microscopic calculations of the relaxation time τ for strongly interacting electrons with spin. However, based on the analogy with the case of spinless electrons [16], we expect activated temperature dependence $\tau^{-1} \propto e^{-D_\sigma/T}$, with the activation energy of the order of the bandwidth of the spin excitations, $D_\sigma \sim v_\sigma p_F \ll v_\rho p_F$.

In a wire of fixed length the crossover between the regimes of long and short wires can be explored by changing the temperature T. Because of the exponential temperature dependence of τ, the condition $L \sim 2v_\sigma\tau$ is satisfied at $T \sim D_\sigma/\ln N$, where N is the number of electrons in the one-dimensional part of the device. At $T \ll D_\sigma/\ln N$ the activated behavior $\delta G \propto e^{-D_\sigma/T}$ should be observed, in agreement with the experiment [6]. The quadratic temperature dependence is expected in the relatively narrow range $D_\sigma/\ln N \ll T \ll D_\sigma$, and its experimental observation requires working with rather long wires. At the upper limit of this range $\delta G \sim e^2/h$. At this point the system crosses over into the so-called spin-incoherent regime. Study of this crossover would be very interesting from both the theoretical point of view and because of its possible relevance for the experiments [5–9] showing a shoulder-like feature in the conductance at $0.7 \times 2e^2/h$.

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