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Quantum optomechanics of a Bose-Einstein Antiferromagnet

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We investigate the cavity optomechanical properties of an antiferromagnetic Bose-Einstein condensate, where the role of the mechanical element is played by spin-wave excitations. We show how this system can be described by a single rotor that can be prepared deep in the quantum regime under realizable experimental conditions. This system provides a bottom-up realization of dispersive rotational optomechanics, and opens the door to the direct observation of quantum spin fluctuations.

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Cavity optomechanics (COM) has witnessed considerable advances in recent years, promising significant contributions in several topics of broad interest including the production and manipulation of macroscopic quantum states, the investigation of the crossover between quantum and classical regimes in systems of increasing size, and the realization of a wide range of applications from high-precision metrology to quantum information science and motion-mediated nonlinear photonics [1–3].

The first such systems featured optomechanical couplings with a linear dependence on the displacement of a vibrating mechanical element such as a mirror [1]. More recently, systems featuring quadratic optomechanical coupling have also been realized [3]. This advance was quite important since, in principle, quadratic optomechanical coupling allows the non-destructive measurement of the phonon number state of the oscillator, as well as observation of the discrete quantum jumps that accompany changes in phonon number [3, 4]. In contrast, these measurements are impossible with only linear optomechanical coupling.

A parallel development in COM was the realization of COM systems formed by ultracold atomic gases trapped inside driven high-Q resonators [5], the role of the macroscopic mechanical element now being played by a collective excitation of the atoms. These systems can be thought of as “bottom-up” realizations of COM, since in practice one can readily prepare a degenerate atomic gas in its quantum ground state, and thus the motion of the fictitious mechanical element is initially in the quantum regime [5]. This is in contrast to the more conventional “top-down” COM systems described above where despite considerable progress it remains in general an ongoing challenge to cool the optomechanically-coupled mechanical mode into the quantum regime.

This Letter considers a bottom-up situation where the role of the mechanical element is played by spin-wave excitations of a macroscopic mode of an antiferromagnetic Bose-Einstein condensate (BEC), whose dynamics can be mapped to a quantum rotor model [6–9]. Although there have been multiple investigations of the cavity dynamics

of a spin ensemble [10–14], as well as an atom-chip-based realization of quadratic-coupling COM [4], this is to our knowledge the first study of bottom-up COM where the atomic gas has spin degrees of freedom and the cavity confinement leads to the ability to resolve the quantum regime of an antiferromagnetic spin gas.

Especially pertinent to the present study is Ref. [12], which provides the first link that we know of between the fields of spin systems and COM control in the literature. That paper studies the dynamics of a large quantum spin in an optical resonator and finds spin bistability and squeezing of light by establishing an analogy with a harmonic oscillator coupled *linearly* to a single cavity mode. In contrast, the present Letter describes an experimentally realizable process for the dramatic cooling of a collective spin-wave excitation mode by establishing an analogy with a torsional oscillator coupled *quadratically* to a single cavity mode. As already mentioned the form of the optomechanical coupling is crucial in determining which phenomena can be observed in such a system.

We consider an atomic spin-1 Bose condensate characterized by antiferromagnetic spin-exchange collisions, and described by the Hamiltonian ($\hbar = 1$)

$$H_R = \frac{c_2}{2N} F^2 - q\psi_0^\dagger \psi_0. \quad (1)$$

Here c_2 is the antiferromagnetic spin coupling, N is the total particle number, and $F = \psi_i^\dagger F_{ij} \psi_j$ is the total spin operator, where F_{ij} are the spin-1 matrices and ψ_i ($i = \pm, 0$) are the bosonic annihilation operators for each spin component. The additional coupling, $q = (\mu_B B)^2 / (4\Delta_{\text{hf}})$, is the quadratic Zeeman shift, with external magnetic field B and hyperfine splitting Δ_{hf} .

Recently, it has been shown that there is an exact mapping of the Hamiltonian (1) to a Hamiltonian describing a single quantum rotor [8, 15],

$$H_R = \frac{L^2}{2I} + V(\theta),$$
$$V(\theta) = q(N + 3/2) \sin^2 \theta + \frac{q^2 N}{8c_2} \sin^2(2\theta), \quad (2)$$

where the conjugate operators θ and L are the rotor's effective angular displacement and angular momentum, respectively, and the effective moment-of-inertia is $I = N/c_2$. An effective position space basis is specified by defining the eigenstates of the Cartesian components $|X\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, $|Y\rangle = (|+\rangle + |-\rangle)/i\sqrt{2}$, and $|Z\rangle = |0\rangle$, where $\{|\pm\rangle, |0\rangle\}$ is the single-particle spin eigenbasis.

We focus on the weak magnetic field limit $q \ll c_2$ where the dynamics of the quantum rotor are dominated by the first term of $V(\theta)$, and are thus localized around the poles of the Bloch sphere. By expanding $V(\theta)$ to quadratic order about the minimum $\theta = 0$ we find that the quantum rotor model of Eq. (2) is further reduced to one that describes a one-dimensional harmonic torsional oscillator characterized by θ and L_z , where $[\theta, L_z] = i$ [7, 8], and θ is the polar angle of the total spin state. In that limit the approximate potential is

$$V(\theta) \simeq \frac{1}{2}I\omega_\theta^2\theta^2, \quad \omega_\theta^2 = 2qc_2 \left[1 + \frac{3}{2}\frac{1}{N} + \frac{q}{c_2} \right], \quad (3)$$

for which the ground state wave-function is [8]

$$\Psi_0(\theta) = \sqrt{\frac{1}{\pi\theta^2}} \exp\left(-\frac{\theta^2}{2\theta^2}\right), \quad \bar{\theta} = \sqrt{\frac{c_2}{2qN^2}}. \quad (4)$$

One finds however that the harmonic torsional oscillator description is only valid when $\bar{\theta} \ll 1$. Combining this requirement with the initial weak-field condition, the parameter region where harmonic approximation of the quantum rotor is valid is

$$1 \ll c_2/q \ll 2N^2. \quad (5)$$

When this condition is met, any finite field will localize the state of the spinor condensate about $\theta = 0$ regardless of its initial state [8, 17]. In this so-called ‘‘Josephson regime’’ [9] the rotor model can be used to predict several purely quantum mechanical phenomena such as small-number effects ($c_2/q = 20$, $N = 200$ or $\bar{\theta} \simeq 0.02$) [7], observation of spin collapse and revival ($\bar{\theta} = 0.1$) [8], and nematic-mixing dynamics [16]. These effects are difficult to observe experimentally because for coupling strengths characteristic of antiferromagnetic BEC, mean-field effects will dominate unless one uses relatively small condensates, $N \sim$ tens to hundreds of atoms [7]. This requirement is in conflict with the requirement of relatively large condensates needed to characterize them through the usual probe of absorption imaging. As a result most studies of spinor BECs have focused on mean-field physics [17].

However, when an antiferromagnetic BEC is trapped in a driven high- Q optical cavity, the atom-photon coupling is greatly enhanced, and can thus be used as a sensitive experimental probe for the BEC. It follows that, under presently available experimental conditions (see,

for example, Ref. [14]), one can measure the quantum dynamics of small condensates, and as we will show, even observe the quantum regime in large condensates. More specifically, such cavity confinement allows one to enter the quantum regime where an antiferromagnetic spin-1 BEC is described by the quantum rotor model.

When trapped inside a unidirectional ring cavity the antiferromagnetic BEC is described by the Hamiltonian

$$H = H_R + H_C + H_L + H_{D_\theta} + H_\gamma, \quad (6)$$

where $H_R = L_z^2/(2I) + (1/2)I\omega_\theta^2\theta^2$ is the quantum rotor Hamiltonian, H_C describes the intracavity field and its interaction with the rotor, $H_L = -i\kappa_L(a - a^\dagger)$ describes the laser pumping with amplitude κ_L , and H_{D_θ} and H_γ account for mechanical and input noise, respectively.

The effective optomechanical coupling arises within H_C . In the rotating frame of the pump laser

$$H_C = (\omega_C - \omega_L)a^\dagger a + U_0(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-)a^\dagger a, \quad (7)$$

where ω_L and ω_C are the frequencies of the pump laser and the empty cavity resonance, respectively. The second term of Eq. (7) describes the far off-resonant coupling of the cavity field to the BEC, where $U_0 = g^2/(\omega_L - \omega_a)$ is the single-photon light shift with dipole coupling g and atomic transition frequency ω_a . Assuming that the cavity field is π -polarized, the atoms in the $F = 1$ ground-state manifold couple to the $F' = 1$ excited-state manifold. The selection rules for the corresponding transitions are $\Delta m_F = 0$, with the exception that the $|F = 1, m_F = 0\rangle \rightarrow |F' = 1, m_{F'} = 0\rangle$ transition is forbidden, which means that atoms in the spin-0 state do not couple to the cavity field [13].

With the normalization condition $\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_- + \psi_0^\dagger\psi_0 = N$, the mapping $-q\psi_0^\dagger\psi_0 \rightarrow V(\theta)$, and in the harmonic approximation H_C can be reexpressed as

$$H_C \simeq (-\Delta + \xi_\theta\theta^2)a^\dagger a, \quad (8)$$

where $\Delta = \omega_L - (\omega_C + U_0N)$ is the static part of the pump-cavity detuning. The dynamic part of the pump-cavity detuning is readily identified as the quadratic optomechanical coupling term $H_{OM} = \xi_\theta\theta^2a^\dagger a$, with

$$\xi_\theta = U_0N \left[1 + \frac{3}{2}\frac{1}{N} + \frac{q}{c_2} \right]. \quad (9)$$

The most salient feature of H_{OM} is the quadratic dependence on θ , which, in principle, allows one to perform quantum non-demolition measurements of the rotational energy of the oscillator as well as to observe the discrete quantum jumps that accompany a change in roton number [3, 4], thereby providing a unique probe of the quantum state of an antiferromagnetic BEC. We remark that, as in other studies of systems featuring quadratic optomechanical coupling [3, 4, 18], one can adjust the

system parameters, in this case tune the magnetic field, to realize higher order couplings. For example

$$H_{\theta^4} = H_{\theta^2} - \beta\theta^4, \quad \beta \simeq (q - U_0 a^\dagger a)N/3. \quad (10)$$

where H_{θ^2} is the quadratic coupling Hamiltonian discussed above.

We now turn to the system dynamics of H_{θ^2} , which we study by considering the corresponding Heisenberg-Langevin equations

$$\begin{aligned} \dot{\theta} &= L_z/I, \quad \dot{L}_z = -I\omega_\theta^2\theta - 2\xi_\theta a^\dagger a\theta - \frac{D_\theta}{I}L_z + \varepsilon_\theta^{\text{in}}, \\ \dot{a} &= -i(-\Delta + \xi_\theta\theta^2)a - \gamma a + \kappa_L + \sqrt{2\gamma}a^{\text{in}}. \end{aligned} \quad (11)$$

Here γ is the cavity damping rate; a^{in} is the operator describing the input noise, characterized by a zero mean and Markovian correlations $\langle a^{\text{in}}(t)a^{\text{in},\dagger}(t') \rangle = \delta(t-t')$, and $\varepsilon_\theta^{\text{in}}$ accounts for noise from a thermal reservoir and/or incoherent atomic scattering, with $\langle \varepsilon_\theta^{\text{in}} \rangle = 0$ and

$$\begin{aligned} \langle \varepsilon_\theta^{\text{in}}(t)\varepsilon_\theta^{\text{in}}(t') \rangle &= \\ D_\theta \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\omega \left[1 + \coth\left(\frac{\omega}{2k_B T}\right) \right], \end{aligned}$$

where D_θ is the intrinsic damping constant for the quantum rotor and k_B is Boltzmann's constant.

In the mean-field approximation, the steady-state solutions of the Heisenberg-Langevin equations are

$$\theta_s = L_{z,s} = 0, \quad a_s = \frac{\kappa_L}{\gamma - i\Delta}. \quad (12)$$

An important feature of these solutions is the lack of multistability, a common hallmark of many cavity optodynamical systems, including both the linear optorotational coupling case [19] and the general cavity-spin coupling case [10–13]. The lack of a multistable region plays a pivotal role in the enhanced cooling and trapping of the effective quantum rotor: it allows one to achieve stronger cooling and trapping by increasing the pump laser power without having to worry about instabilities arising due to the emergence of additional stable states. Of course, when the harmonic motion condition of Eq. (5) is severely violated, such as in the limit of high magnetic fields ($q > c_2$) treated in Ref. [13], strong bistability tends to occur.

Similarly to the case of a linearly coupled rotating mirror [19], the *quadratic* optomechanical coupling of the quantum rotor results in an increased effective trapping frequency. We show this by replacing each system operator with its first-order fluctuation expansion $\hat{O}(t) \equiv \mathcal{O}_s + \delta\hat{O}(t)$, writing the cavity field fluctuation in terms of its quadrature operators $\delta X_1 = (\delta a + \delta a^\dagger)/\sqrt{2}$, $\delta X_2 = (\delta a - \delta a^\dagger)/i\sqrt{2}$, and then linearizing the Heisenberg-Langevin equations with respect to fluctuations to obtain

$$\begin{aligned} \dot{v}(t) &= Rv(t) + \Lambda(t), \quad v(t) = (\delta\theta, \delta L_z, \delta X_1, \delta X_2)^T, \\ \Lambda(t) &= (0, \varepsilon_\theta^{\text{in}}, \sqrt{2\gamma}X_1^{\text{in}}, \sqrt{2\gamma}X_2^{\text{in}}), \end{aligned} \quad (13)$$

where

$$R = \begin{pmatrix} 0 & 1/I & 0 & 0 \\ -I\omega_\theta^2 - 2\xi_\theta|a_s|^2 & -D_\theta/I & 0 & 0 \\ 0 & 0 & -\gamma & -\Delta \\ 0 & 0 & \Delta & -\gamma \end{pmatrix}, \quad (14)$$

and $X_1^{\text{in}} = (a^{\text{in}} + a^{\text{in},\dagger})/\sqrt{2}$ and $X_2^{\text{in}} = (a^{\text{in}} - a^{\text{in},\dagger})/i\sqrt{2}$ are the quadrature operators for the input-field fluctuations. It is important to note that a steady state solution is only stable if the real part of each of the corresponding eigenvalues of R is non-positive, which can be easily confirmed by using the Routh-Hurwitz criterion [20].

After Fourier transform and subsequent algebraic manipulation one solves for the first-order fluctuations

$$\delta\theta[\omega] = \frac{\varepsilon_\theta^{\text{in}}}{I(\omega_\theta^2 - \omega^2) + 2\xi_\theta|a_s|^2 - i\omega D_\theta}, \quad (15)$$

$$\delta X_{1,2}[\omega] = \frac{\sqrt{2\gamma}\{(\gamma - i\omega)X_{1,2}^{\text{in}}[\omega] \pm \Delta X_{2,1}^{\text{in}}[\omega]\}}{\Delta^2 + (\gamma - i\omega)^2}. \quad (16)$$

Equation (15) is in the form of a linear response

$$\delta\theta[\omega] = \chi(\omega)\varepsilon_\theta^{\text{in}}[\omega],$$

where the noise operator $\varepsilon_\theta^{\text{in}}$ plays the role of the external perturbation, and with

$$\chi^{-1}[\omega] = I(\omega_{\text{eff}}^2 - \omega^2) - i\Gamma_{\text{eff}}\omega, \quad (17)$$

where $\chi(\omega)$ is the susceptibility of a fictitious mechanical rotor characterized by the damping factor $\Gamma_{\text{eff}} = D_\theta$ and the resonant frequency $\omega_{\text{eff}} = \eta\omega_\theta$, where we have defined the enhancement factor

$$\eta = \sqrt{1 + \frac{U_0}{q} \frac{\kappa_L^2}{\Delta^2 + \gamma^2}}. \quad (18)$$

In contrast to the linear-coupling case [19], η is independent of frequency, maximized at $\Delta = 0$, and can be as large as 10^3 or even higher for typical parameter values. Stiffer trapping and even more efficient cooling can be expected for small values of q , a key factor for the detection of quantum fluctuation effects [7–9].

The effect of this enhancement on the mean roton number can be found from the first-order correlation functions of the system. From Eq. (11) we find

$$\begin{aligned} \frac{d}{dt} \langle \theta^2 \rangle &= \frac{1}{I} \langle L_z\theta + \theta L_z \rangle, \\ \frac{d}{dt} \langle L_z^2 \rangle &= [-I\omega_\theta^2 - 2\xi_\theta|a_s|^2] \langle L_z\theta + \theta L_z \rangle \\ &\quad - 2\frac{D_\theta}{I} \langle L_z^2 \rangle + 2I\frac{D_\theta}{I} (n + 1/2)\omega_\theta, \\ \frac{d}{dt} \langle L_z\theta + \theta L_z \rangle &= \frac{2}{I} \langle L_z^2 \rangle + 2(-I\omega_\theta^2 - 2\xi_\theta|a_s|^2) \langle \theta^2 \rangle \\ &\quad - \frac{D_\theta}{I} \langle L_z\theta + \theta L_z \rangle, \end{aligned} \quad (19)$$

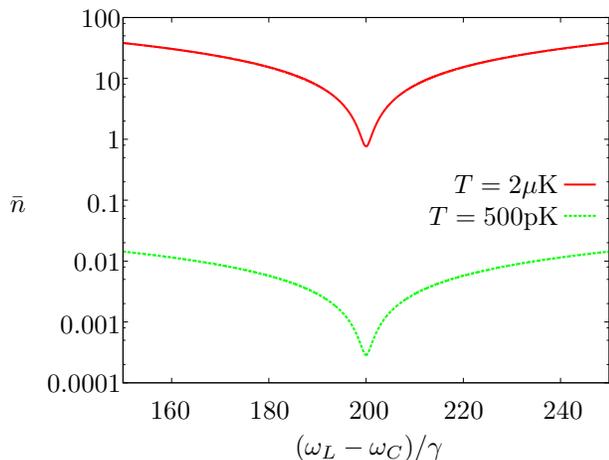


FIG. 1: Average roton occupation number \bar{n} as a function of scaled static detuning for $T = 2 \mu\text{K}$ and 500 pK , $U_0 = 2\pi \times 100 \text{ Hz}$, $\gamma = 2\pi \times 50 \text{ kHz}$, $\kappa_L = 2\pi \times 3 \text{ MHz}$.

where the term featuring $n = [\exp(\omega_\theta/k_B T) - 1]^{-1}$, the mean thermal excitation number at temperature T , accounts for a shift of the first-order correlations due to contact with a thermal reservoir. The steady state solution to Eq. (19) is stable and exhibits non-vanishing expectations $\langle \theta_z^2 \rangle_{\text{ss}}$ and $\langle L_z^2 \rangle_{\text{ss}}$. The steady-state energy of the quantized rotor is written $E_Q = \langle L_z^2 \rangle_{\text{ss}}/2I + I\omega_\theta^2 \langle \theta_z^2 \rangle_{\text{ss}}/2$. Using the steady-state solution of Eq. (19) one finds $E_Q = (n + 1/2)\omega'_\theta$, where $\omega'_\theta = (\omega_\theta/2)[1 + \eta^{-2}]$. Setting $E_Q = \bar{n}\omega_{\text{eff}}$ then gives the roton occupation number

$$\bar{n} = \frac{(n + 1/2)\omega'_\theta}{\omega_{\text{eff}}} = \left(n + \frac{1}{2}\right) \frac{\eta^2 + 1}{2\eta^3}. \quad (20)$$

The large trap frequency enhancement η resulting from the COM coupling enables one to reach the otherwise very elusive quantum regime $\bar{n} < 1$. Without that effect, a typical temperature of $T \sim 2 \mu\text{K}$ [21] gives $n \sim 4 \times 10^3$ for $q/c_2 = 10^{-3}$, indicative of large thermal excitations of the rotor. The COM enhancement, with typical values of $U_0 \sim 2\pi \times 100 \text{ Hz}$, $c_2 \sim 2\pi \times 20 \text{ Hz}$ (for ^{23}Na atoms), $N \sim 10^5$ [5], $\gamma \sim 2\pi \times 50 \text{ kHz}$, and $\kappa \sim 2\pi \times 3 \text{ MHz}$, gives $\bar{n}_{\text{min}} < 1$ for all suitable values of q . For very low temperatures $T \sim 500 \text{ pK}$ [22], we have $\bar{n}_{\text{min}} \sim 10^{-4}$ (see Fig. 1), illustrating how the COM technique opens up the possibility to observe a single-roton state [2] or even quantum jumps of a spin gas [2–4].

In summary, we have investigated the optomechanics of an antiferromagnetic BEC by using a formal analogy with a torsional oscillator that is quadratically coupled to a single cavity mode. The resulting bistability-free effect, the noise spectra of both the atoms and the transmitted photons, and the quantized energy of the atoms were discussed. That proposal provides an ideal nondestructive tool for the control of quantum spin dynamics [6–8], and facilitates the experimental study of the deep quantum

regime of the rotor model, which has been previously unattainable. Furthermore, since the system we describe is within the reach of current experimental capabilities, our results can be of immediate use to researchers studying such diverse problems as spin-based nano-mechanical devices [23], squeezed rotors [18], hybrid spin-mirror entanglement, and high-precision control of spinor atoms, polar molecules, or even rotating BECs [24].

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Note added: After this manuscript was submitted, another paper [23] presented a related theoretical analysis on a torsional model of a cavity-spin system.

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