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Numerical Calculation of the Neutral Fermion Gap in the $\nu = 5/2$ Fractional Quantum Hall State

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We present the first numerical computation of the neutral fermion gap, Δ_F, in the $\nu = 5/2$ quantum Hall state, which is analogous to the energy gap for a Bogoliubov-de Gennes quasiparticle in a superconductor. We find $\Delta_F \approx 0.027 \frac{e^2}{\varepsilon \ell_0}$, comparable to the charge gap. We also deduce an effective Fermi velocity v_F for neutral fermions from the low-energy spectra for odd numbers of electrons, and thereby obtain a correlation length $\xi_F = v_F/\Delta_F \approx 1.3\,\ell_0$. We comment on the implications of our results for experiments, topological quantum information processing, and electronic mechanisms of superconductivity.

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The $\nu = 5/2$ fractional quantum Hall state [1–3] has been the subject of intense experimental and theoretical investigation in recent years because it may support non-Abelian anyons and may serve as a platform for topological quantum information processing [4–6]. Theoretical [7–13] and experimental [14–17] evidence has been rapidly accumulating which is consistent with $\nu = 5/2$ being an Ising-type non-Abelian state, in the universality class of either the Moore-Read (MR) Pfaffian state [18, 19] or the anti-Pfaffian ($\overline{\text{Pf}}$) state [20, 21]. However, there are also some experiments [22–24] which are difficult, though not impossible, to interpret in a manner consistent with the simplest incarnations of either of these states. In addition, some theoretical studies [10, 25–28] highlight the sensitivity of numerical results to system size and the precise form of the Hamiltonian.

If the $\nu = 5/2$ fractional quantum Hall state is proven experimentally to be non-Abelian, then its potential use for topological quantum information processing is dependent on the size of the energy gaps Δ_a to different species of quasiparticles a. If the temperature T can be kept much less than these gaps Δ_a and inter-quasiparticle distances x kept much greater than the tunneling correlation lengths ξ_a, then the corresponding error rates will vanish as $e^{-\Delta_a/T}$ and e^{-x/ξ_a} and, hence, be negligible.

The smallest gap for charged quasiparticles is usually assumed to correspond to the minimally charged excitations of a state [46]. For the MR and $\overline{\text{Pf}}$ states, the minimal charge $\pm e/4$ quasiparticles also carry non-Abelian Ising topological charge σ. It is natural to interpret the gap corresponding to the temperature dependence of the longitudinal resistance, $\rho_{xx} \sim e^{-\Delta_{\text{trans}}/2T}$, as the energy gap $\Delta_c \equiv \Delta_{e/4} + \Delta_{-e/4}$ for a charge $\pm e/4$ quasihole-quasiparticle pair, which is thereby deduced from experiments to be $\Delta_c \equiv \Delta_{\text{trans}} \approx 0.5$K in the highest-mobility samples [29]. Numerical studies of small numbers of electrons interacting through Coulomb interactions at $\nu = 5/2$ with Landau level mixing, finite thickness, and disorder neglected find $\Delta_c \approx 0.025 - 0.029 \frac{e^2}{\varepsilon \ell_0}$ (which is $3.2 - 3.7$ K at 6.5T) [9, 30].

However, bulk electrical transport is not sensitive to the energy gap of electrically neutral excitations, such as the neutral fermion that carries Ising topological charge ψ in the MR and $\overline{\text{Pf}}$ states. Consequently, the neutral fermion gap, Δ_F, has not been measured (though it could, in principle, be determined from thermal transport measurements or, as we discuss below, from interferometry measurements). Δ_F has previously not been theoretically calculated, either.

The MR and $\overline{\text{Pf}}$ states are the quantum Hall analogues of spin-polarized $p_x + ip_y$ superconductors [18, 19, 31]. Charge $e/4$ quasiparticles σ correspond to flux $hc/2e$ vortices; neutral fermions ψ correspond to Bogoliubov-de Gennes quasiparticles in the superconductor. In most superconductors, these two gaps have completely different scales and are not considered on the same footing. However, in the $\nu = 5/2$ state, there is only a single energy scale $e^2/\varepsilon\ell_0$, so these gaps can be comparable. Thus far, however, only Δ_c has been computed. In this paper, we compute Δ_F. This is the appropriate quantity to use when comparing the gap in the $\nu = 5/2$ state to the gaps in other superconductors, and when drawing lessons for non-phonon mechanisms of superconductivity from this state.

The neutral fermion gap is also a relevant quantity in determining the effectiveness of topological protection in Ising-type quantum Hall states, should they exist in nature. The transfer of Ising ψ charge between quasiparticles, e.g. through tunneling, alters the non-local state shared by the quasiparticles. It is, thus, responsible for splitting the degenerate non-local states and causing errors in the encoded information [32]. Similarly, the neutral fermion gap directly determines the visibility of non-Abelian statistical signatures in interference experiments (see [33–35], and references therein), since tunneling of the neutral ψ charge (between bulk quasiparticles and between bulk quasiparticles and the edge) suppresses interference terms. In this light, it is of paramount importance to study this quantity. In this letter, we produce numerical estimates of the neutral fermion gap and correlation length for the $\nu = 5/2$ quantum Hall state.

In order to model the $\nu = 5/2$ state, we assume that both spins of the lowest Landau level are filled and inert and focus on the second Landau level, which has $\nu = 1/2$. We perform numerical calculations with N_e electrons on the sphere at the flux values $N_\phi = 2N_e - 3$ at which the MR ground-state would occur for N_e even. We study small systems ($N_e \leq 15$ electrons) by exact diagonalization and larger systems ($13 \leq N_e \leq 26$ electrons) by the density-matrix renormalization group (DMRG), as in [9, 13, 36]. We work in the simplified situation of electrons interacting through Coulomb interactions, neglecting finite layer-thickness [10], Landau-level mixing [11, 12, 27], and disorder. These certainly play a role in real devices, and a more realistic calculation, including these effects, will be discussed elsewhere [41]. For the remainder of this paper, however, we will use the term "$\nu = 5/2$ state" to refer to this idealized model.

In order to compute the energy gap for an electrically-neutral quasiparticle, we need to compare the usual ground-state to a configuration that forces the system to have a neutral excitation of non-trivial topological charge. By increasing the electron number by 1 and the flux by 2, we maintain charge neutrality. However, an electron together with two flux quanta is a fermion (i.e. a "composite fermion"). Thus, an incompressible state at $\nu = n + 1/2$ will generically have a neutral fermionic excitation, and its energy can be computed by comparing the lowest energy of a state with electron number N_e and flux N_ϕ, which we will denote by $E\left(N_\phi, N_e\right)$ and the lowest energy $E\left(N_\phi + 2, N_e + 1\right)$ of a state with electron number $N_e{+}1$ and flux $N_\phi{+}2$. Since a state with an odd number N_e of electrons should be understood as a state with a neutral fermionic excitation, we will reserve the term *ground state* for the lowest energy state for N_e even and $N_\phi = 2N_e - 3$ and use "lowest energy state" for generic values of N_e, N_ϕ.

In order to isolate the energy of a neutral fermion we must subtract off the N_e dependence of the *ground state* energy:

$$\Delta_F\left(N_e\right) \equiv \frac{(-1)^{N_e}}{2}\left[E\left(N_\phi + 2, N_e + 1\right) + E\left(N_\phi - 2, N_e - 1\right) - 2E\left(N_\phi, N_e\right)\right] \quad (1)$$

In the regime in which $E\left(N_\phi, N_e\right)$ scales linearly with N_e, the neutral fermion gap $\Delta_F\left(N_e\right)$ will be constant. It is instructive to contrast Eq. 1 with the expression for the charge gap, $\Delta_c\left(N_e\right) = \frac{1}{2}\left[E\left(N_\phi + 1, N_e\right) + E\left(N_\phi - 1, N_e\right) - 2E\left(N_\phi, N_e\right)\right]$. In Eq. 1, we compare the energies of systems with the same charge-flux relation so that the net charge of all excitations is zero while Δ_c compares the energies of states with fluxes offset by one so that the net charge of all excitations is νe. One should also not confuse the neutral fermion gap Δ_F with the "neutral gap," which is the energy gap above the ground state at a fixed N_ϕ.

For pure Coulomb interactions in the second Landau level, we have computed the ground state energies for even numbers of electrons up to $N_e = 26$ and the lowest state energies for odd numbers of electrons up to $N_e = 17$. In a recent calculation, Lu *et al.* [37] have computed these energies up to $N_e = 18$ electrons by exact diagonalization; our energies are in agreement with theirs. In Fig. 1, we show the values of the neutral fermion gap $\Delta_F(N_e)$, computed using Eq. 1, as a function of inverse system size $1/N_e$ for up to $N_e = 17$ electrons. As may be seen from Fig. 1, the neutral fermion gap fluctuates considerably, which is a sign of finite-size effects. If we were to use a purely linear fit, then we would find $\Delta_F \equiv \lim_{N_e\to\infty} \Delta_F(N_e) \approx 0.028$; if we were to fit the gap to a constant, we would find $\Delta_F \approx 0.023$. However, the errors in these fits, determined from the maximum fluctuation away from the average, are large (though Δ_F is clearly non-zero). Therefore, more care is needed in order to perform an $N_e \to \infty$ extrapolation.

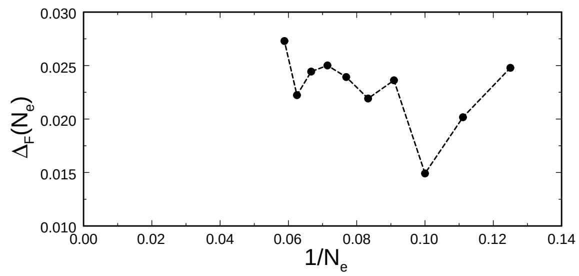

FIG. 1: The neutral fermion gap, as defined in Eq. (1), for the $\nu = 5/2$ quantum Hall state as a function of inverse system size $1/N_e$.

To this end, we note that if the system is gapped, then we can write $E\left(N_\phi, N_e\right)$ in the form

$$E\left(N_\phi, N_e\right) = \mathcal{E}N_e + E_{\text{even, odd}} + O(e^{-a\sqrt{N_e}}) \quad (2)$$

for N_e even or odd, respectively. The leading terms are the same for even and odd N_e because the energy per particle $\mathcal{E}$ must be the same in the thermodynamic limit. The constant terms $E_{\text{even,odd}}$ are due to the internal order of the phase and the genus of the system [45], as well as the energy cost of the (collectively) neutral quasiparticle(s) for N_e odd. Corrections to these first two terms are exponentially small in the linear size of the system ($\sim \sqrt{N_e}$) since the system has a gap; here, a is a constant inversely proportional to the correlation length.

Substituting Eq. 2 into Eq. 1, we find

$$\Delta_F\left(N_e\right) = E_{\text{odd}} - E_{\text{even}} + O(e^{-a\sqrt{N_e}}) \quad (3)$$

and thus $\Delta_F = E_{\text{odd}} - E_{\text{even}}$, further justifying our definition of the neutral fermion gap. We can, however, use Eq. 2 to extract $E_{\text{odd}} - E_{\text{even}}$ more directly by simply fitting the numerical data with functions of this form, and it allows us to exploit the larger system sizes for which we have computed the ground state energies for even N_e. In Fig. 2, we plot $E\left(N_\phi, N_e\right)/N_e$ vs. $1/N_e$, and fitting to Eq. 2 (divided by N_e) but replacing, for simplicity, the $O(e^{-a\sqrt{N_e}})$ term by a single term $ce^{-a\sqrt{N_e}}$, we find $\mathcal{E} = -0.3634$, $E_{\text{even}} = -0.5381$, and $E_{\text{odd}} = -0.5114$. For N_e even, we find $c = -0.7876$ and $a = 0.6675$, while for N_e odd we find $c = -1.4700$ and $a = 0.8287$. Thus, we can reliably extract the thermodynamic limit of the neutral fermion gap by taking the difference between the $1/N_e$ terms in the expressions for $E\left(N_\phi, N_e\right)/N_e$. We find $E_{\text{odd}} - E_{\text{even}} \approx 0.027$ (in units of $e^2/\varepsilon\ell_0$).

One advantage of using this method of extracting the neutral fermion gap is that it is easier to diagnose potential difficulties with the $N_e \to \infty$ extrapolation. For instance,

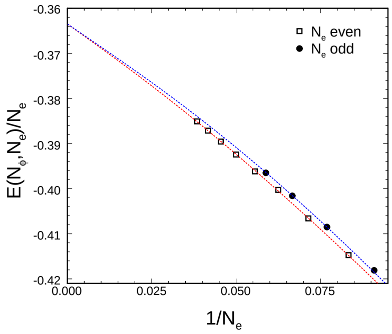

FIG. 2: The $N_e \to \infty$ extrapolation of $E\left(N_\phi, N_e\right)/N_e$ corresponding to the $\nu = 5/2$ state for N_e even (squares) and N_e odd (dots) using Eq. 2, from which we find $\Delta_F \approx 0.027$.

one potential pitfall is aliasing. If one of the systems studied is actually in the ground state of a different phase, then $E\left(N_\phi, N_e\right)/N_e$ would not sit on the expected (nearly-linear) curve. As may be seen from the figure, the data points deviate negligibly from the fitting curves, so this is not the case for the system sizes we study. At any rate, the most serious potential aliases occur at $N_e < 10$, which we do not consider for the extrapolation in Fig. 2

The preceding considerations are completely general. We now momentarily interpret our results in terms of a putative Ising-type system at this filling fraction, where the lowest energy state on the sphere with N_e odd electrons must have nontrivial quasiparticles whose total topological charge is ψ [45]. The two simplest possibilities are that such a state either has a neutral ψ quasiparticle, or a charge $e/4$ σ quasihole and $-e/4$ σ quasiparticle pair that fuses into a ψ. Note that possible mismatches between allowed topological charges and stable quasiparticle species are a feature of all topological states. For instance, in the $\nu = 1/3$ Laughlin state [38], the charge $2e/3$ quasihole carries an allowed value of topological charge, but it is not an energetically stable excitation (for Coulomb interactions); if we attempt to create one, it will decay into two charge $e/3$ quasiholes. Similarly, we must consider the possibility that a neutral ψ quasiparticle will simply decay into a charge $\pm e/4$ σ quasihole-quasiparticle pair that fuses into the ψ channel. In this case, $\Delta_F = E_{\text{odd}} - E_{\text{even}}$ would be identified with Δ_c and would provide a lower bound for Δ_ψ. However, since we find $\Delta_F \approx 0.027$ and previous studies [13] obtained $\Delta_c \approx 0.029$, we tentatively conclude that the neutral fermion is stable (at least against this decay channel) and has

$$\Delta_F = E_{\text{odd}} - E_{\text{even}} \approx 0.027. \tag{4}$$

Stronger evidence supporting this interpretation comes from the good fit of our data to the N_e odd case of Eq. 2. If the neutral fermion were unstable, there would be a $-1/32\sqrt{N_e}$ term in the odd electron number energy, resulting from the Coulomb interaction energy between the $\pm e/4$ charges [30].

For purposes of comparison, we note that a similar computation of the neutral fermion gap for the $\nu = 1/3$ Laughlin state [38] would give the value zero because the even- and odd-electron number ground state energies lie on the same line [13]; since it is not a paired state, there is no qualitative difference between even and odd electron numbers. On the other hand, the $\overline{\text{Pf}}$ state, the $(3,3,1)$ state [39] and the Bonderson-Slingerland (BS) states [40] have neutral fermionic excitations whose gaps can be computed by the method explained in this paper. In the absence of Landau-level mixing, Δ_F is expected to be precisely the same for the $\overline{\text{Pf}}$ state as it is for the MR state; preliminary calculations are consistent with this, as we report elsewhere [41]. In the case of the $k \geq 3$ Read-Rezayi states [42], there are neutral excitations that are non-Abelian and, therefore, cannot be obtained by simply altering the electron number and flux.

Although the neutral fermion gap has not been previously calculated, a related quantity has recently been calculated, namely the splitting between the two degenerate states that occur for four $e/4$ σ quasiparticles [43]. This splitting, $\Delta E(r)$, decays with distance r between the σ quasiparticles as $\Delta E(r) \sim f(r)\, e^{-r/\xi}$ for large r. Here, $f(r)$ is an oscillatory function and ξ is the characteristic length scale for the decay. If we interpret this splitting as the energy associated with inter-quasiparticle tunneling of neutral fermions, then we expect $\xi = \xi_F = v/\Delta_F$, where v is the velocity of a neutral fermion. If the $\nu = 5/2$ state is interpreted as a paired state with small gap, then v would be the Fermi velocity v_F of the underlying Fermi-liquid-like metallic state. In such a case, the Fermi velocity could be deduced by studying the spectrum of a single neutral fermion as follows. For odd N_e, the energy spectrum will not have a gap above the lowest energy state (in the thermodynamic limit) since there will be one unpaired neutral fermion above the Fermi energy, and this fermion can be excited to any other state above the Fermi energy. In a BCS mean-field theory, the energy spectrum for odd N_e will be bounded below by the curve $E_L = \sqrt{\epsilon_L^2 + \Delta_F^2} + E_{\text{gs}} \approx \frac{1}{2\Delta_F}\epsilon_L^2 + \Delta_F + E_{\text{gs}}$, where E_{gs} is the ground state energy for $N_e - 1$ electrons, ϵ_L is a single-particle energy relative to the Fermi energy for a state with angular momentum L. We take $\epsilon_L = \frac{v_F}{N_\phi \ell_0}[L(L+1) - L_0(L_0+1)]$, where L_0 is the highest occupied angular momentum orbital. Thus, for $L \approx L_0$, the excitation energies are expected to be quadratic in $L - L_0$:

$$E_L \approx \frac{1}{2\Delta_F}\left(\frac{v_F(2L_0+1)}{N_\phi \ell_0}\right)^2 (L - L_0)^2 + \text{const.} \tag{5}$$

As may be seen in Fig. 3, the lowest excitation energies for $N_e = 9, 11, 13, 15$ appear to follow a parabola. A linear extrapolation of the v_F values obtained from these spectra according to Eq. 5 gives $v_F \approx 0.021\, e^2/\varepsilon$, which leads to $\xi_F \approx 0.8\, \ell_0$. However, the parabolic fit is quite poor for $N = 13$; the other three system sizes are consistent with $v_F \approx 0.035\, e^2/\varepsilon$, or $\xi_F \approx 1.3\, \ell_0$. We note, for comparison, that Baraban *et al.* [43] find a length scale $\xi \approx 2.3\ell_0$, although their calculation is for much larger system sizes and for trial wavefunctions, rather than the Coulomb ground state.

Our results imply that a quantum computer based on a possible Ising-type state at $\nu = 5/2$ should be operated at temperatures much lower than Δ_F, which is ≈ 3.4K for a magnetic

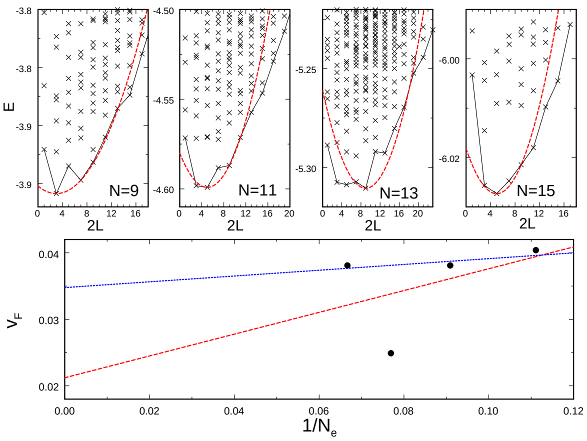

FIG. 3: The low-energy spectrum of the $\nu = 5/2$ state for $9, 11, 13, 15$ electrons with parabolic fits of the lowest-lying states to Eq. 5, from which we extract the effective velocities plotted in the lower panel.

field $B = 6.5$T. This implies that, at 35mK, the error rate due to thermally-excited neutral fermions is $\sim e^{-\Delta_F/T} \approx 10^{-44}$ if the computational anyons are further than $\xi_F \approx 130$Å from each other and the edge. However, there may be other potential sources of error. Furthermore, as a result of disorder, the neutral fermion gap may locally be smaller than Δ_F, thus increasing the error rate. In the experiments of Willett *et al.* [16], the inter-quasiparticle distances are probably comparable to ξ_F; this implies that the error rate may be large and there is probably significant splitting between the 2^{n-1} states expected for $2n$ quasiparticles [44]. By measuring the time over which the signal through an interferometer remains stable, it should be possible to measure the error rate and, thereby, Δ_F. In addition, bulk thermal transport may be dominated by thermally-excited neutral fermions. Although charge $e/4$ quasiparticles may have a smaller energy gap (approximately half that for a neutral fermion), they will be much more strongly localized by disorder than neutral fermions.

Finally, we note that $\Delta_F \approx 0.027\,\frac{e^2}{\varepsilon \ell_0}$ is small compared to the Coulomb energy. Since one might argue that the gap is small because of the proximity to competing phases, such as the striped phase [8], we consider the neutral fermion gap for a Hamiltonian in which the only interaction is the (repulsive) three-body interaction for which the MR wavefunctions are the exact ground states [8, 19]. For this Hamiltonian, the ground state energy is precisely zero for N_e even, so $E_{\text{even}} = 0$. Thus, we must only compute the lowest state energies for N_e odd. We note that these states occur at different values of the angular momentum than for Coulomb interactions, perhaps because the precise shape of the Fermi surface (which may be quite irregular for some N_e for these system sizes) is different for these two Hamiltonians – a non-universal but quantitatively important effect. A simple linear extrapolation of these energies gives $\Delta_F = E_{\text{odd}} \approx 0.45$, if the coefficient of the three-body interaction is 1. Thus, there is nothing wrong in principle with the naïve idea that the superconducting gap can be comparable to the Coulomb energy scale for an electronic pairing mechanism, so long as there are no nearby competing phases to suppress it.

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[45] On the torus, an odd number of electrons can be accommodated without creating quasiparticles. This agrees with the fact that the curvature is zero, so there are no constant terms in Eq. 2.
[46] This is generally a good assumption, given the Coulombic energy cost of forming quasiparticles with larger charge.