Electron Star Birth: A Continuous Phase Transition at Nonzero Density

Sean A. Hartnoll and Pavel Petrov

Phys. Rev. Lett. **106**, 121601 — Published 23 March 2011

DOI: 10.1103/PhysRevLett.106.121601

Electron star birth: A continuous phase transition at nonzero density

Sean A. Hartnoll and Pavel Petrov
Center for the Fundamental Laws of Nature,
Department of Physics, Harvard University,
Cambridge, MA 02138, USA

We show that charged black holes in Anti-de Sitter spacetime can undergo a third order phase transition at a critical temperature in the presence of charged fermions. In the low temperature phase, a fraction of the charge is carried by a fermion fluid located a finite distance from the black hole. In the zero temperature limit the black hole is no longer present and all charge is sourced by the fermions. The solutions exhibit the low temperature entropy density scaling $s \sim T^{2/z}$ anticipated from the emergent IR criticality of recently discussed electron stars.

PACS numbers:

'Electron stars', planar fluids of charged fermions in Anti-de Sitter space (AdS), are a compelling holographic framework to study metallic quantum criticality [1–3]. Key features of electron stars are emergent criticality at low energy [1, 2] and the presence of a 'smeared' Fermi surface [3]. Metallic criticality is difficult to study with conventional field theoretic techniques as the many gapless excitations of the Fermi surface must be included in a strongly interacting IR fixed point [4].

Zero temperature electron stars [2] are charged 'solitonic' gravitational configurations without a black hole horizon, fermionic analogues of zero temperature holographic superconductors [5]. Their existence is aided by the gravitational well of the AdS asymptopia. At sufficiently high temperatures we might expect the electron star to collapse to a Reissner-Nordstrom black hole (RN). This would be analogous to the first order transition to a black hole undergone by zero temperature stars in AdS with spherical, rather than planar, symmetry above a critical mass [6]. This letter will show that electron stars undergo a third order transition to a charged black hole above a critical temperature. Cooling through the transition, the electron star birth, gives a continuous phase transition in the nonzero density dual field theory characterised by the appearance of a smeared Fermi surface.

We will study the dynamics of a 3+1 dimensional ideal fluid of charged relativistic fermions coupled to electromagnetism and gravity with a negative cosmological constant. The Lagrangian may be written

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F_{ab}F^{ab} + p(\mu, s)\,. \qquad (1)$$

This is the Schutz form of the action in terms of the fluid pressure p [7], generalised to allow the fluid to be charged [2]. The local chemical potential $\mu = |d\phi + \alpha d\beta + \theta ds + A|$, where $\{\phi, \alpha, \beta\}$ are fluid potential variables, s the local entropy density and θ the thermasy. This action leads to the ideal fluid equations of motion, see e.g. [2, 7].

The fluid Lagrangian (1) is a coarse grained description in which fermion physics is subsumed into locally defined thermodynamic quantities. In astrophysics this is the Tolman-Oppenheimer-Volkoff description [8] while in condensed matter physics it is known as the Thomas-Fermi approximation [9]. The fluid Lagrangian (1) is a correct description of the system when [2] $e^2 \sim \kappa/L \ll 1$. Without loss of generality we will take the fermions to have unit charge and mass m. For detailed discussions of the connection between microscopic and fluid descriptions of gravitating fermions, see e.g. [6, 10].

The spacetime metric and Maxwell field take the form

$$\frac{1}{L^2}ds^2 = -f dt^2 + g dr^2 + \frac{1}{r^2}\left(dx^2 + dy^2\right)\,,\ A = \frac{eL}{\kappa} h dt\,.$$

A crucial role is played by the local chemical potential

$$\mu_{\text{loc.}} = A_{\hat{t}} = \frac{A_t}{L\sqrt{f}} = \frac{e}{\kappa}\frac{h}{\sqrt{f}}\,. \qquad (2)$$

This chemical potential determines the local thermodynamic quantities of the fermion fluid. Before specifying these we need to clarify the role of nonzero temperature. Periodically identifying the Euclidean time circle has two consequences. Firstly is that for regularity of the spacetime, recall that we are in planar coordinates, we must have a finite size black hole horizon in the interior. Secondly, the local fermion fluid equation of state is at finite temperature. This describes a black hole surrounded by a fluid in thermal equilibrium with the Hawking radiation. In our bulk classical limit the effects of Hawking radiation should be negligible, while the black hole remains present. In this limit we can treat the fermions as a zero temperature fluid in a black hole background. We will make this statement precisely below.

The energy density, charge density and pressure of the fermion fluid are therefore determined by the local potential (2) via the zero temperature equation of state. We introduce dimensionless hatted quantities which are rescaled by factors of $\{e, L, \kappa\}$ in such a way that no such factors will appear in the equations of motion:

$$\rho = \tfrac{1}{L^2\kappa^2}\hat{\rho},\ \ \sigma = \tfrac{1}{eL^2\kappa}\hat{\sigma}\,,\ p = \tfrac{1}{L^2\kappa^2}\hat{p}\,,\ \hat{m}^2 = \tfrac{\kappa^2}{e^2}m^2\,.$$

The dimensionless thermodynamic variables are then

$$\hat{\rho} = \hat{\beta}\int_{\hat{m}}^{\frac{h}{\sqrt{f}}} \epsilon^2\sqrt{\epsilon^2 - \hat{m}^2}d\epsilon\,,\ \hat{\sigma} = \hat{\beta}\int_{\hat{m}}^{\frac{h}{\sqrt{f}}} \epsilon\sqrt{\epsilon^2 - \hat{m}^2}d\epsilon, \qquad (3)$$

and $-\hat{p} = \hat{\rho} - \frac{h}{\sqrt{f}}\hat{\sigma}$. Here $\hat{\beta}$ depends on the ratio of the Maxwell and Newton couplings and is order one in our regime $e^2 \sim \kappa/L$ [2]. The equations of motion following from the action (1) are [2]

$$\frac{1}{r}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{4}{r}\right) + \frac{g h \hat{\sigma}}{\sqrt{f}} \;=\; 0\,, \tag{4}$$

$$\frac{f'}{rf} - \frac{h'^2}{2f} + g(3+\hat{p}) - \frac{1}{r^2} \;=\; 0\,, \tag{5}$$

$$h'' + \frac{g\hat{\sigma}}{\sqrt{f}}\left(\frac{r h h'}{2} - f\right) \;=\; 0\,. \tag{6}$$

From (3), a nonvanishing density of fermions at a particular radius requires that the mass be lower than the local chemical potential $\hat{m} < h/\sqrt{f}$. In the RN solution

$$f = \frac{1}{r^2} - \left(\frac{1}{r_+^2} + \frac{\hat{\mu}^2}{2}\right)\frac{r}{r_+} + \frac{\hat{\mu}^2}{2}\frac{r^2}{r_+^2}\,, \quad h = \hat{\mu}\left(1 - \frac{r}{r_+}\right)\,,$$

and $g = 1/(r^4 f)$, one sees that $h/\sqrt{f}$ is always bounded with a maximum in between the horizon at $r = r_+$ and boundary at $r = 0$. Thus if $\hat{m}$ is too large, the fluid density vanishes everywhere, $\hat{\rho} = \hat{\sigma} = \hat{p} = 0$, and RN is the only solution. As the mass is lowered it will eventually equal the local chemical potential at a critical radius

$$\left.\frac{h}{\sqrt{f}}\right|_{r=r_c} = \hat{m}\,, \qquad \left.\frac{d}{dr}\frac{h}{\sqrt{f}}\right|_{r=r_c} = 0\,. \tag{7}$$

These two equations determine the critical radius at which the star is born, dimensionlessly expressed as r_c/r_+, and the critical temperature of the black hole over the rescaled chemical potential $T_C/\hat{\mu}$. The temperature is $T = 1/(4\pi c)\left|df/dr\right|_{r=r_+}$. At this point $c = 1$, we have included this factor for later convenience.

Figure 1 plots the critical radius and critical temperature as a function of the fermion mass. The electron star forms at increasingly low temperatures, and closer to the black hole horizon, as the fermion mass is increased. Beyond $\hat{m} = 1$ the star is never formed, consistent with the observation [2] that $\hat{m} < 1$ is necessary for the zero temperature electron star to exist. The plots in figure 1 do not depend on the fluid equation of state (3). In fact, the conditions (7) simply characterise the radius at which a charged point particle can remain stationary, with gravitational and electromagnetic forces balancing.

Cooling below the critical temperature we will find two solutions $\{r_1, r_2\}$ to the equation $\mu_{\text{loc.}} = m$ defining the boundary of the star. The electron star broadens into a thin shell. The solution at $T < T_C$ has three components:

1. *Inner region*, $r > r_2$. The solution is RN, but $\hat{\mu} \to \hat{\mu}_0$, not directly the dual field theory chemical potential.
2. *Intermediate region*, $r_2 > r > r_1$. Here we must solve the equations (4) to (6) with $\hat{\rho}, \hat{\sigma}, \hat{p}$ nonzero. The quantities $\{f, g, h, h'\}$ must be matched onto the inner and exterior regions at r_2 and r_1.
3. *Exterior region*, $r_1 > r$. Again RN, although now

$$f = c^2\left(\frac{1}{r^2} - \hat{M}r + \tfrac{1}{2}r^2\hat{Q}^2\right)\,,\; g = \frac{c^2}{r^4 f}\,,\; h = c\left(\hat{\mu} - r\hat{Q}\right)\,.$$

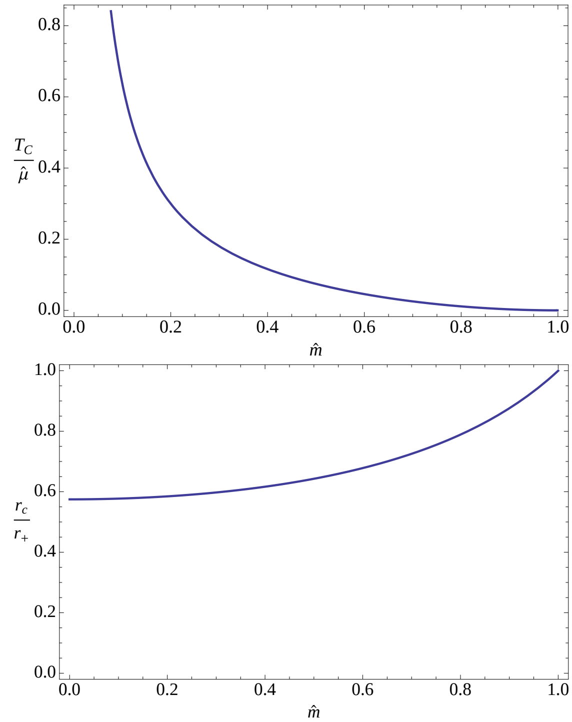

FIG. 1: Critical temperature (top) and radius (bottom) of electron star birth, as a function of fermion mass.

The factor c, determining the normalisation of time, was included in the definition of the temperature above.

The solutions can be parametrised by $r_+\hat{\mu}_0$ in the inner region. Via matching at r_2 and r_1, this initial condition integrates forward to determine the values of the dual field theory quantities $\{c, \hat{M}, \hat{Q}, \hat{\mu}\}$. The solutions can then be labelled by the physical dimensionless ratios $T/\hat{\mu}$ or T/T_C. Figure 2 shows an example of how the fermion density $\hat{\sigma}$ builds up and extends to the horizon

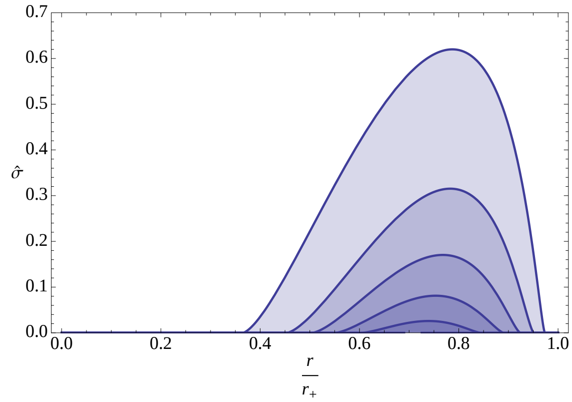

FIG. 2: Radial density profile of an electron star as a function of temperature. Curves show five temperatures between $0.07\,T_C$ and T_C, with $\hat{\mu}$ held fixed. With $\hat{\beta} = 10$ and $\hat{m} = 0.7$.

$(r_2 \to r_+)$ as the temperature is lowered. In reading this plot one should keep in mind [2] that the total charge in the fermion fluid will be $\hat{Q}_{\text{ferm.}} = \int_{r_1}^{r_2} \hat{\sigma}(s)\sqrt{g(s)}/s^2\, ds$, including the spatial volume element $\sqrt{g}/r^2$.

The charge density of the boundary field theory, read off from the exterior solution, is $\hat{Q} = \hat{Q}_{\text{BH}} + \hat{Q}_{\text{ferm.}}$, where the charge density carried by the black hole is $\hat{Q}_{\text{BH}} = \hat{\mu}_0/r_+$. The presence of the fermionic charge density is the defining bulk characteristic of the electron star. The fraction of charge carried by the fermionic fluid, $(\hat{Q} - \hat{Q}_{\text{BH}})/\hat{Q}$, is a sort of bulk order parameter. The ratio is zero above the critical temperature and, given that at zero temperature all the charge is carried by fermions [2], should tend to unity at low temperatures. Figure 3 shows precisely this phenomenon. The rate of charge transfer does not depend monotonically on the fermion mass.

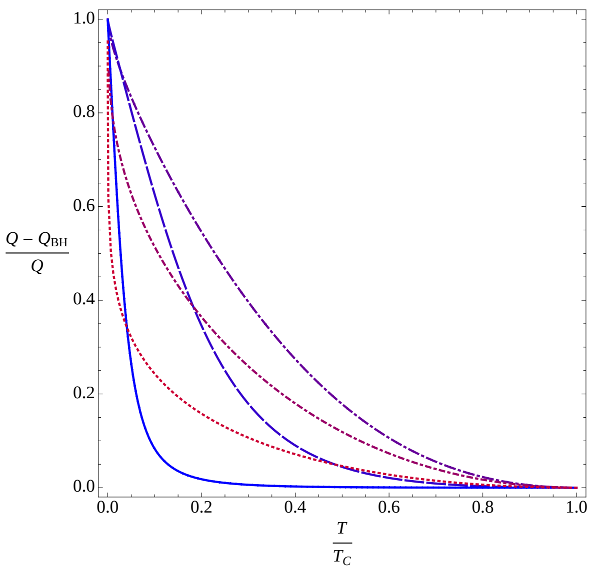

FIG. 3: The fraction of charge carried by the fermion fluid as a function of temperature. All curves have $\hat{\beta} = 10$ while from blue to red (solid to dotted) $\hat{m} = \{0.01, 0.07, 0.3, 0.55, 0.75\}$. In quoting T/T_c, the chemical potential is kept fixed.

Our observations so far suggest that the electron star birth is a continuous phase transition in the system. To make this claim precise, we must compute the behaviour of the free energy across the transition. The free energy density of the theory is most easily obtained from the thermodynamic relation $\hat{\Omega} = \hat{M} - \hat{\mu}\hat{Q} - \hat{s}T$. Here $\hat{s}$ is the (rescaled) Bekenstein-Hawking entropy density given by $\hat{s} = \frac{8\pi}{4r_+^2}$. It was shown in [2] that the thermodynamic relation held for zero temperature electron stars by showing that the Lagrangian (1) was a total derivative on shell. In the presence of a nonzero temperature horizon, there is an additional contribution to the on shell action at the horizon that contributes the necessary $\hat{s}T$ term.

Figure 4 compares the free energy of two electron stars with RN in the absence of a fermion fluid. The first observation is that the free energy of the stars is indeed lower than that of RN below the transition temperature at which the star is born. Secondly, the transition is

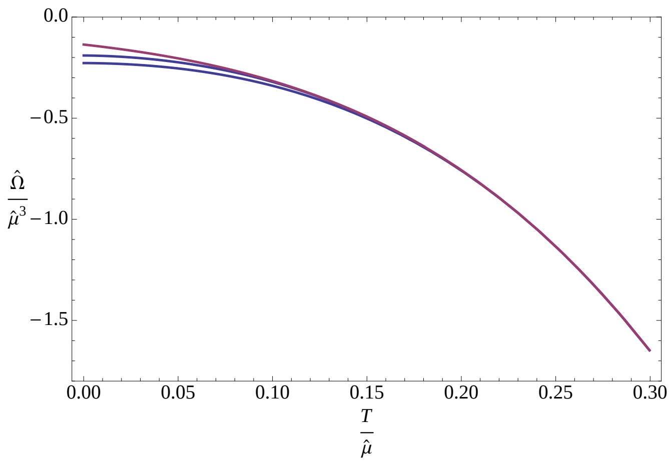

FIG. 4: Free energy of RN (top, red) and of two electron stars (blue) as a function of temperature. The electrons stars have $\hat{\beta} = 10$ and $\hat{m} = 0.2$ (lower) and $\hat{m} = 0.3$ (upper).

extremely soft. Fitting to high precision numerics, we find the transition is third order, so that $\Delta\Omega \sim (T_C - T)^3$. This fact can be understood analytically as follows.

Below the critical temperature the fluid has width $\Delta r = r_2 - r_1$. Solving the conditions $\mu_{\text{loc.}}(r_{1,2}) = m$ perturbatively at small $T - T_C$, using the background RN geometry, gives to leading order $\Delta r/r_+ = \#\,(1 - T/T_C)^{1/2}$. Here and below $\#$ refers to a complicated but computable mass dependent number. The backreaction of the fluid on the black hole solution is subleading. The leading order difference in free energies is simply the contribution of the fluid to action (1):

$$\Delta\hat{\Omega} = -\int_{r_1}^{r_2} \hat{p}\,\sqrt{fg}/r^2 dr = -\#\,\hat{\beta}\,T_C^3\,(1 - T/T_C)^3\,. \quad (8)$$

The scaling follows immediately from that for $\Delta r/r_+$ together with the facts that, from (3), we have $\hat{p} \sim \delta\hat{\mu}_{\text{loc.}}^{5/2}$ for $\hat{\mu}_{\text{loc.}} = \hat{m} + \delta\hat{\mu}_{\text{loc.}}$ and that $\delta\hat{\mu}_{\text{loc.}} = \mathcal{O}\left((\Delta r)^2\right)$. This second statement in turn follows from the fact that the local chemical potential may be approximated, again simply using the RN geometry, by the parabola $\hat{\mu}_{\text{loc.}} = \hat{m} + \#(1 - T/T_C)^{1/2}\Delta r/r_+ - \#(\Delta r/r_+)^2$ just below the transition temperature. To determine the coefficient in (8), one should use the expression (3) for the pressure together with the parabolic expression for $\hat{\mu}_{\text{loc.}}$ and perform the integral. This can be done either numerically or analytically. Some illustrative values for the coefficient $\#$ in (8) are

$$\begin{array}{c|c|c|c} \hat{m} = 0.1 & \hat{m} = 0.3 & \hat{m} = 0.5 & \hat{m} = 0.8 \\ \# = 0.0139 & \# = 0.7489 & \# = 3.529 & \# = 27.02 \end{array}\,.$$

We have checked that these values agree to within around one part in a hundred with the values obtained by fitting the output of a full numerical integration of the equations of motion (using MATHEMATICA's `NDSolve` with

`WorkingPrecision` set to 30). This confirms that (8) is correct. There is a third order phase transition.

A second interesting thermodynamic observable is the entropy density. An emergent IR criticality with dynamical critical exponent z implies that at low temperatures, $T \ll \hat{\mu}$, the entropy density scales as $\hat{s} \sim T^{2/z}$. This scaling was previously exhibited in a different holographic system in [11]. Figure 5 shows the entropy as a function of temperature for RN together with three electron stars.

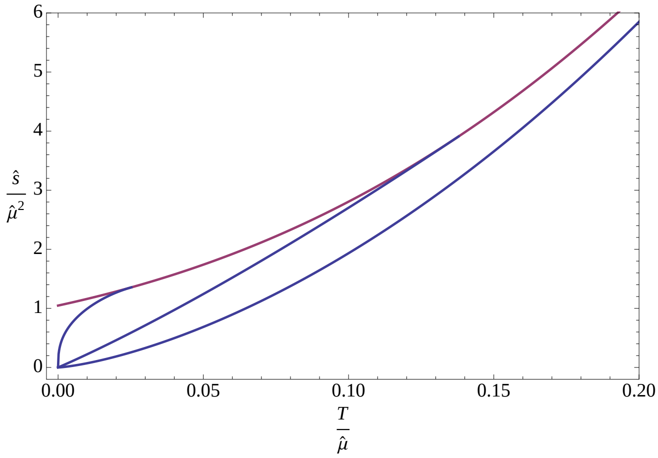

FIG. 5: Temperature dependence of entropy density. Red line (top) is RN. Blue lines (lower) are electron stars with $\hat{\beta} = 20$. From left to right, $\hat{m} = 0.7, 0.36, 0.1$. Fitting to a power law at low temperatures gives $z \approx 5.4, 2, 1.5$, respectively.

The RN entropy density tends to a constant at zero temperature ($z = \infty$) while those of the electron stars tend to zero. Fitting the numerical output to a power law, we can read off z. The values obtained, quoted in the caption of figure 5, agree with the zero temperature results of [2] to within one part in a hundred.

Two issues remain. First, the consistency of zero temperature fluids in black hole backgrounds. The Euclidean time circle is periodically identified with radius $1/T$. The local charge density in (3) therefore includes thermal excitations $\sigma = 2\pi^2 \beta \sum_{\pm} \pm \int \frac{d^3p}{(2\pi)^3} \left(1 + e^{(E \mp \mu_{\text{loc.}})/T_{\text{loc.}}}\right)^{-1}$. The local chemical potential is again (2). The local temperature $T_{\text{loc.}} = T/\sqrt{g_{tt}} = T/L\sqrt{f}$. From the definitions and our limit, $T_{\text{loc.}}/\mu_{\text{loc.}} = \mathcal{O}(\kappa/eL) \ll 1$. The local fermion fluid may be treated at zero temperature.

Finally, an order parameter of the boundary quantum field theory. The bulk microscopic fermion field is dual to a 'single trace' fermionic operator Ψ in the dual quantum field theory. It is useful to Fourier decompose this operator into creation operators $c^\dagger_{\omega,k}$ [3, 6]. A natural boundary quantity is the generalized fermion density $n = \int \frac{d\omega\, d^2k}{(2\pi)^3} \langle c^\dagger_{\omega,k} c_{\omega,k} \rangle$. This expectation value is computed as follows. Each bulk fermionic state corresponds to a solution of the Dirac equation in the background electron star spacetime. Although within our Thomas-Fermi description the local fermion density drops to zero at the outer boundary of the electron star, each occupied state will have a nonvanishing tail that reaches the boundary of the spacetime [13]. Squaring the coefficient of each $\{\omega, k\}$ mode will determine the expectation values via the usual holographic dictionary. It should be possible to perform this computation explicitly.

In our bulk semiclassical limit, the generalized fermion density is nonzero at low temperatures but vanishes above T_C. This order parameter does not break any symmetries. A third order transition is somewhat exotic, and deserves a better field theoretic understanding. Away from the semiclassical limit, thermal excitations imply that the fermion density is never exactly zero.

Acknowledgements. We had helpful discussions with Frederik Denef, Diego Hofman, Koenraad Schalm and Alireza Tavanfar. Research supported by DOE grant DE-FG02-91ER40654, FQXi foundation and Center for the Fundamental Laws of Nature at Harvard University.

Noted Added. The preprint [12] overlaps with our results, which agree with a revised version of that preprint.

[1] S. A. Hartnoll, J. Polchinski, E. Silverstein and D. Tong, JHEP **1004**, 120 (2010). [arXiv:0912.1061 [hep-th]].

[2] S. A. Hartnoll and A. Tavanfar, [arXiv:1008.2828 [hep-th]].

[3] S. A. Hartnoll, D. M. Hofman, A. Tavanfar, [arXiv:1011.2502 [hep-th]].

[4] H. v. Löhneysen, A. Rosh, M. Vojta and P. Wölfle, Rev. Mod. Phys. **79**, 1015 (2007). [arXiv:cond-mat/0606317 [cond-mat.str-el]].

[5] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, JHEP **0812**, 015 (2008). [arXiv:0810.1563 [hep-th]]; S. S. Gubser and A. Nellore, Phys. Rev. D **80**, 105007 (2009) [arXiv:0908.1972 [hep-th]]; G. T. Horowitz and M. M. Roberts, JHEP **0911**, 015 (2009) [arXiv:0908.3677 [hep-th]].

[6] J. de Boer, K. Papadodimas and E. Verlinde, JHEP **1010**, 020 (2010). [arXiv:0907.2695 [hep-th]]; X. Arsiwalla, J. de Boer, K. Papadodimas and E. Verlinde, [arXiv:1010.5784 [hep-th]].

[7] B. F. Schutz, Phys. Rev. D **2**, 2762 (1970).

[8] R. C. Tolman, Phys. Rev. **55**, 364 (1939); J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).

[9] L. H. Thomas, Math. Proc. Camb. Phil. Soc., **23**, 542 (1927); E. Fermi, Rend. Accad. Naz. Lincei **6**, 602 (1927).

[10] R. Ruffini and S. Bonazzola, Phys. Rev. **187**, 1767 (1969).

[11] K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, JHEP **1008**, 078 (2010). [arXiv:0911.3586 [hep-th]].

[12] V. G. M. Puletti, S. Nowling, L. Thorlacius and T. Zingg, arXiv:1011.6261 [hep-th].

[13] We are grateful to Koenraad Schalm for emphasizing the importance of the tail of the Dirac equation.