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Saturation of Gyrokinetic Turbulence Through Damped Eigenmodes

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In the context of toroidal gyrokinetic simulations it is shown that a hierarchy of damped modes is excited in the nonlinear turbulent state. These modes exist at the same spatial scales as the unstable eigenmodes that drive the turbulence. The larger amplitude subdominant modes are weakly damped and exhibit smooth, large scale structure in velocity space and in the direction parallel to the magnetic field. Modes with increasingly fine scale structure are excited to decreasing amplitudes. In aggregate damped modes define a potent energy sink. This leads to an overlap of the spatial scales of energy injection and peak dissipation, a feature that is in contrast with more traditional turbulent systems.

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In high Reynolds number fluid turbulence, as modeled by the Navier-Stokes equation, energy is injected at large scales and conservatively transferred by nonlinear interactions through a broad inertial range to a dissipation range at small scales [1]. Saturation is achieved when the rate of energy injection at large scales is balanced by the rate of energy dissipation at small scales. Saturation theories for plasma microturbulence typically involve variations on this theme. Indeed, such an energy cascade occurs in both physical space and velocity space by the rate of energy dissipation at small scales. Saturation is achieved when the rate of energy injection at large scales is balanced by the Navier-Stokes equation, energy is injected at large scales.

In the study of toroidal gyrokinetic simulations it is shown that a hierarchy of damped modes is excited in the nonlinear turbulent state. These modes exist at the same spatial scales as the unstable eigenmodes that drive the turbulence. The larger amplitude subdominant modes are weakly damped and exhibit smooth, large scale structure in velocity space and in the direction parallel to the magnetic field. Modes with increasingly fine scale structure are excited to decreasing amplitudes. In aggregate damped modes define a potent energy sink. This leads to an overlap of the spatial scales of energy injection and peak dissipation, a feature that is in contrast with more traditional turbulent systems.
eigenmodes that can provide a means of energy dissipation provided they are driven to finite amplitude by nonlinear interactions. For each wavenumber, numerical discretization allows for \( N = n_x \times n_{v_{||}} \times n_{\mu} \) degrees of freedom, where the \( n \)'s denote the number of grid points in each coordinate. The unstable eigenmode defines only one of these degrees of freedom; the remaining degrees of freedom provide an energy sink at large spatial scales.

We seek to characterize the nonlinear state by decomposing the gyrokinetic distribution function for selected wavenumbers \((k_x,k_y)\) as a superposition of modes, \( g_{k_x,k_y}(z,v_{||},\mu,t) = \sum_n f_{k_x,k_y}^{(n)}(z,v_{||},\mu)h_{k_x,k_y}^{(n)}(t) \). \( 1 \)

The structure \( f^{(1)}(z,v_{||},\mu) \) corresponds to the unstable eigenmode, but its time amplitude, \( h^{(1)}(t) \), rather than exhibiting its linear behavior \( e^{-i(\omega + iv) t} \), fluctuates as determined by a balance between the linear drive and the stabilizing influence of nonlinear interactions. The other modes are also defined by fixed mode structures \( f^{(n)}(z,v_{||},\mu) \) and fluctuate according to their respective time amplitudes \( h^{(n)}(t) \) in such a way that a superposition of all the modes exactly reproduces the total distribution function at each moment in time. In contrast with the unstable mode, the time amplitudes \( h^{(n)}(t) \) of damped modes fluctuate according to a balance between nonlinear drive and linear damping, the latter of which dissipates energy from the system, thereby facilitating saturation of the turbulence. This decomposition is constructed by performing a proper orthogonal decomposition \( [7] \) (POD) on data from a standard nonlinear gyrokinetic simulation. This provides a means to examine separately the contribution of individual modes, stable or unstable, to the saturation of the turbulence.

To study the role of damped modes in saturation we track energy injected into or removed from the turbulence using diagnostics related to the conserved (in the absence of drive and dissipation) energy-like quantity \( E = \int dv_{||} \int dz B_0^2 \pi n_0 T_0 |\phi|^2/F_0 + \int dz D(k_{\bot},z)|\phi|^2 \), where \( B_0 \) is the equilibrium magnetic field, \( \phi \) is the electrostatic potential, \( n_0 \) and \( T_0 \) are the background density and temperature, \( D \) is a function of \( z \) and the perpendicular wavenumbers. The energy evolves according to

\[
\frac{\partial E_k}{\partial t} \bigg|_{N.C.} = Q_k + C_k
\]

where \( Q = \int dv_{||} \int dz 2\pi n_0 T_0 B_0/L_T (v_{||}^2 + \mu B_0) g^* i k_y \phi \) is a term proportional to the heat flux and includes the turbulent drive (\( \phi \) is the gyro-averaged potential, and \( L_T \) is the temperature gradient scale length), \( C \) represents collisional dissipation and, in a simulation, whatever artificial dissipation (e.g., hyper-diffusive terms) is included in the code. The subscript N.C. indicates that this equation describes only the non-conservative energy evolution, i.e., processes that inject or dissipate net energy from the fluctuations (as opposed to processes like the \( E \times B \) nonlinearity that move energy from one scale to another in a conservative fashion).

The GENE code \( [9] \) is used to simulate ITG driven turbulence defined by the Cyclone Base Case parameters \( [10] \) of safety factor \( q = 1.4 \), magnetic shear \( s = 0.8 \), inverse aspect ratio \( \epsilon = r/R = 0.18 \), equilibrium ratios of density and temperature \( n_0/n_e = T_i/T_e = 1.0 \), and background gradients \( R/L_T = 6.9, R/L_n = 2.2 \) where \( R \) is the major radius. The perpendicular box size is \( (L_x,L_y) = (126\rho_i,126\rho_i) \), and the number of grid points is \( 32 \times 48 \times 8 \) for the \((z,v_{||},\mu)\) coordinates respectively. The perpendicular spatial resolution consists of 128 grid points in the \( x \) direction giving \( k_x,\text{max} = 3.12 \), and \( 64 \) \( k_y \) Fourier modes for \( k_y,\text{max} = 3.15 \). We deviate from the Cyclone Base Case by using a linearized Landau-Boltzmann collision operator rather than exclusively artificial dissipation. The collision frequency is \( \nu(R/\nu_T) = 3.0 \times 10^{-3} \) which is much less than the dynamic time scales of the system (e.g., the most unstable mode at \( k_y \rho_i = 0.3 \) has a growth rate \( \gamma(R/\nu_T) = 0.267 \), and frequency \( \omega(R/\nu_T) = 0.783 \) so that \( \nu/\omega \sim 10^{-2} \). In these runs \( C_k \) consists mostly of collisional dissipation but also includes contributions from fourth order hyper-diffusive dissipation in the \( z \) and \( v_{||} \) coordinates.

To illustrate the spatial scale dependence of the energy balance we first consider separately the drive term \( Q_k \) and the dissipation term \( C_k \) in Eq. (2). Figure 1 shows \( Q_k \) and \( C_k \) from the saturated state of a simulation, averaged over the parallel coordinate and time. In Fig. 1A \( k_x \) dependence is shown and \( k_y \) is summed; in Fig. 1B \( k_y \) dependence is shown and \( k_x \) is summed. There is a significant amount of dissipation at all scales, including \( k_y = 0.0 \) and high \( k_x \). However, the largest range of peak dissipation corresponds with the same scales where the energy drive peaks. As described in detail below, the drive \( Q_k \) is dominated by the unstable modes, while
the dissipation $C_k$ is dominated by the stable modes. Contrast this with the corresponding scenario in high Reynolds number Navier-Stokes turbulence for which the drive is localized at large scales, the dissipation is localized at small scales and there is a broad inertial range of intermediate scales with neither drive nor dissipation.

The observed $k$-dependence of the dissipation is due to the excitation of a hierarchy of damped modes in the nonlinear state. POD analysis elegantly characterizes this hierarchy of modes. POD uses the singular value decomposition [11] (SVD) of a matrix to create an optimal orthonormal basis for fluctuation data. In this application, each column of the input matrix consists of a time slice (at every 50 time steps) of the nonlinearly evolved gyrokinetic distribution function for a selected wavevector. The non-spectral coordinates $(z, v_i, \mu)$ are unraveled to one dimension, e.g., as the data would be stored in computer memory. The singular values, $s_n$, define the amplitudes of the $n^{th}$ modes. The left singular vectors are the POD modes - basis vectors that are orthonormal with regard to the scalar product $\langle f^{(n)}f^{(m)} \rangle = \int J(z)dz dv_i d\mu$, where $J(z)$ is a Jacobian (these correspond to $f^{(n)} (z, v_i, \mu)$ in Eq. (1)). The right singular vectors are time traces of the the amplitudes of the corresponding POD modes (these, multiplied by the singular values, correspond to $h^{(n)} (t)$ in Eq. (1)).

It is observed that, for wavevectors with a strongly unstable eigenmode, the $n = 1$ POD mode is very similar to the unstable linear eigenmode; they exhibit nearly identical mode structures and the scalar product between the two is $\sim 0.9$. As a result, for much of this study we will conceptually equate the $n = 1$ POD mode with the corresponding unstable linear eigenmode.

In order to elucidate the energy drive and dissipation processes in the instability range, we will examine in detail the POD analysis of the wavevector of peak transport, $k_y \rho_1 = 0.2$, $k_x \rho_1 = 0.0$. These results are representative of other important energy-containing wavevectors in the spectrum. The POD singular values decay rapidly in mode number, $n$, up to $n \approx 100$. At that point and beyond the spectrum exhibits exponential decay as seen in Fig. 2A. Further insight can be gained by calculating $Q_k$ and $C_k$ for each mode $f^{(n)} (z, v_i, \mu)$. In these calculations $Q_k + C_k$ can, in a sense, be conceptualized as growth (or damping) rates since the modes are normalized and contain no amplitude dependence. The mode-by-mode values of $C_k$ are all negative and increase strongly in amplitude with mode number, as seen in Fig. 2C. As expected, the first POD mode produces a large positive value of $Q_k$. The remaining modes are associated with amplitudes of $Q_k$ which decrease with mode number and have seemingly random signs, i.e., their $\phi^T \mu$ phase angles are randomly distributed around zero. This is shown in Fig. 2B. These results differ from fluid models, where the damped eigenmodes are stable due to a large and systematic effect on cross correlations like $\phi^T [4]$ (in contrast with the modes described here, which are damped due to collisional dissipation).

The contribution of the $n > 1$ modes to the energy balance can be separated from that of the unstable ($n = 1$) mode by decomposing the distribution function as $g = f^{(1)} h^{(1)} + f^{(res)} h^{(res)}$ where the residual distribution function, $f^{(res)} h^{(res)}$, is the sum of the $n > 1$ POD modes and represents all fluctuations not associated with the unstable mode. It is found that the energy drive $Q_k$ is dominated by the unstable mode, whereas the dissipation $C_k$ is dominated by the residual distribution function. This can be seen by examining a selection of wavevectors in the region of instability centered around the peak of the spectrum: $(k_x \rho_1 = 0.0, k_y \rho_1 = [0.05, 0.2, 0.3, 0.4])$, and $k_y \rho_1 = 0.2, k_x \rho_1 = [0.1, 0.2, 0.4])$. For the sum of these wavevectors, the residual drive, $Q_k (f^{(res)} \int |h^{(res)} (t)|^2 dt$, accounts for only 7% of the total energy drive, but the residual dissipation $C_k (f^{(res)} \int |h^{(res)} (t)|^2 dt$ accounts for 63% of the total dissipation. Both the dissipation associated with the $n = 1$ mode and the residual dissipation peak at $k_y \rho_1 = 0.2, k_x \rho_1 = 0.0$ and decrease as $k_\perp$ increases. To summarize: unstable eigenmodes (collectively represented here by $f^{(1)}_{k_x, k_y}$) drive the turbulence. Nonlinear interactions excite linearly damped modes (represented here by $f^{(res)}_{k_x, k_y}$) at the same perpendicular scales $(k_x, k_y)$ as the driving instability. The excitation of these modes causes the dissipation to peak at large perpendicular scales. The calculation in this paragraph, in conjunction with Fig. 1, establish the main claim of this letter; Fig. 1 shows that the scale range of energy drive and dissipation overlap, and this calculation demonstrates that, in this same scale range, the dissipation is dominated by modes other than the unsta-
FIG. 3: Mode structures for a selection of POD modes at the peak of the nonlinear spectrum, \( k_x \rho_i = 0.2, k_z \rho_i = 0.0 \). The mode structures of the electrostatic potential are plotted on the top row for modes 1, 2, 10, and 100 and the \( v_i || \) dependence is shown on the bottom row for same POD modes at \( \mu = 0.18 \) and \( z = 0 \). Fine scale structure develops in both coordinates as \( n \) increases.

In order to characterize the POD modes and describe their role in the energetics, we show in Fig. 3 mode structures for selected POD modes for the wavevector of peak transport, \( k_x \rho_i = 0.2, k_z \rho_i = 0.0 \). On the top row of Fig. 3, the parallel mode structures for the electrostatic potential are plotted (for POD modes 1, 2, 10, and 100), and on the bottom row the \( v_i \) dependence is shown (for the same POD modes at \( \mu = 0.18, z = 0 \)). As mentioned above, the \( n = 1 \) POD modes are very similar to the unstable linear eigenmodes. The second most important structure, the \( n = 2 \) POD mode, is also very similar to a linear eigenmode. This linear eigenmode is the most weakly damped stable mode, having a damping rate an order of magnitude smaller than the growth rate of the ITG mode. Figure 3 also demonstrates the fine scale structure that develops in the \( z \) and \( v_i \) coordinates. The scale lengths in the \( z \) and \( v_i \) coordinates both decrease as \( n \) increases. As a result, these modes become increasingly dissipative due to the higher-order derivatives in the dissipation operators. The development of fine scale structure in \( v_i \) is consistent with aspects of linear phase mixing [12]. There must also be a nonlinear excitation mechanism involved in the process since the linear system produces only the unstable mode (all other eigenmodes decay exponentially). An effort to better understand this phenomenon will be an aspect of future work.

The results presented in this letter do not contradict the numerically observed power law \( k_\perp \rho_i \) spectra reported in the literature [2, 13, 14]. From a dissipation range analysis [15] a spectrum goes as \( k^{-\alpha} \exp(c(k/k_d)^{-\beta}) \) where \( k_d \) is the wavenumber at which damping and nonlinear decorrelation rates are equal, \( c \) is a positive constant, and \( 0 < \beta < 2/3 \), provided the damping increases with wavenumber \( k \) more slowly than the nonlinear decorrelation rate. This condition appears to be satisfied for the data presented here. This spectrum transitions to a regime dominated by the power law behavior for high \( k \). An analysis of high-\( k_\perp \) spectra for simulations very similar to those presented in this letter is provided in Ref. [14] where power law spectra agree quite well with those described in Ref. [2].

The following is a plausible saturation scenario: Through collisional dissipation, damped modes dissipate a significant portion of the injected energy at the same spatial scales as the instability operates \((k_\perp \rho_i \lesssim 1)\). This is accompanied by a spatial cascade carrying energy to smaller perpendicular scales \((k_\perp \rho_i > 1)\). At these smaller scales, the remaining dissipation occurs and processes such as nonlinear perpendicular phase mixing dominate.

In summary, we have shown that in ITG driven turbulence modeled by the gyrokinetic equations, dissipation occurs at all scales, peaking in the wavenumber range of the instability drive. The dissipation is associated with a very large number of damped eigenmodes excited to finite amplitude by nonlinearity.

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