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Phys. Rev. Lett. **106**, 084501 — Published 22 February 2011

DOI: 10.1103/PhysRevLett.106.084501

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(Dated: January 20, 2011)

Viscous compressible flow around a sphere is considered in the limit of zero Reynolds and Mach numbers. An exact expression for the force on the sphere undergoing arbitrary motion with compressibility effects is presented. Quasi-steady, inviscid unsteady, and viscous unsteady force components are identified. Numerical results are in excellent agreement with the theory. The present formulation offers an explicit expression for the unsteady force in the time domain and can be considered as a generalization of the Basset-Boussinesq-Oseen equation to compressible flow.

PACS numbers: 47.15G-,47.55.-t,47.40.-x,47.11.-j

INTRODUCTION

The unsteady force on a particle in accelerated motion was first analyzed by Stokes [1]. Later Basset [2], Boussinesq [3], and Oseen [4] independently examined the time-dependent force on a sphere in a quiescent viscous incompressible fluid. The resulting equation of motion the so-called BBO equation, can be written as

$$m_p \frac{d\mathbf{v}}{dt} = -6\pi a\mu \mathbf{v} - \frac{1}{2} m_f \frac{d\mathbf{v}}{dt} - 6a^2 \rho \sqrt{\pi\nu} \int_{-\infty}^{t} \frac{1}{\sqrt{t-\xi}} \left.\frac{d\mathbf{v}}{dt}\right|_{t=\xi} d\xi \,, \tag{1}$$

where m_p, $\mathbf{v}(t)$, and a are the particle mass, velocity, and radius. ρ, m_f, μ, and ν are the density, displaced-mass, dynamic viscosity, and kinematic viscosity of the fluid. The three terms on the right-hand side are the quasi-steady (Stokes) drag, inviscid unsteady (added-mass), and viscous unsteady (Basset history) forces, respectively. The BBO equation has been extended to non-uniform creeping flows by Maxey and Riley [5] and Gatignol [6].

Our primary goal is to extend the BBO equation to compressible flows. The first work relevant to our goal appears to be that of Zwanzig and Bixon [7] (also see Metiu *et al.* [8]), who investigated the velocity-correlation function of an atom immersed in a compressible visco-elastic liquid. Temkin and Leung [9] and Guz [10] have presented solutions that are essentially identical except for differences due to simplifying assumptions and some typographical mistakes.

The purpose of this work is to present an explicit expression for the time-dependent force on a spherical particle undergoing arbitrary unsteady motion on the acoustic time scale such that compressibility effects are important. Attention is restricted to the zero Reynolds- and Mach-number limits so that non-linear effects can be ignored. We use previously derived solutions of the linearized compressible Navier-Stokes equations in the Fourier/Laplace domains to determine the force on a particle in response to a delta-function acceleration in the time domain. This force response is then used to construct an expression for the time-dependent force on a particle undergoing arbitrary motion. The resulting expression can be interpreted as the generalization of the BBO equation to compressible flows. We show that compressibility causes the inviscid unsteady force to assume an integral representation first derived by Longhorn [11]. We obtain the effect of compressibility on the viscous unsteady force. The theoretical results are compared with direct numerical simulations of the compressible flow around an accelerating particle. Finally, we present the generalized BBO equation that can be used to track particles in compressible flow.

PROBLEM FORMULATION

We consider the unsteady motion of a particle in a quiescent compressible Newtonian fluid. We consider the limit of Re $\to 0$ and M $\to 0$ such that the perturbation field generated by the particle motion is governed by the linearized compressible Navier-Stokes equations. Here, M and Re are suitably defined Mach and Reynolds numbers. The continuity and momentum equations reduce to the form given by Zwanzig and Bixon [7],

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0 \,, \tag{2}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' - \mu \nabla^2 \mathbf{u}' - (\mu_b + \frac{1}{3}\mu) \nabla\nabla \cdot \mathbf{u}' = 0 \,. \tag{3}$$

In Eqs. (2) and (3), properties associated with the quiescent fluid are denoted by the subscript 0, perturbation quantities are denoted by the superscript $'$, $\mathbf{u}$ is the velocity, p is the pressure, and μ_b is the bulk viscosity. Because temperature fluctuations are neglected, the viscosities are constant and the speed of sound

$$c_0 = \sqrt{(\partial p/\partial \rho)_s} = \sqrt{p'/\rho'} \tag{4}$$

can be used as a closure relation. These linearized equations have been solved analytically by Zwanzig and Bixon [7]. who obtained an explicit expression for the force on the particle in the frequency domain. Given a general particle motion with velocity $v(t)$, the solution of Eqs. (2)–(4) in Laplace space can be written as

$$\mathcal{F}(s) = -m_f s\, G(r_1, r_2) \mathcal{L}(v) \tag{5}$$

where $\mathcal{F}(s) = \mathcal{L}(F(t))$ and $\mathcal{L}(v)$ are the Laplace transforms of the time dependent force $F(t)$ and rectilinear particle velocity $v(t)$, respectively, and $m_f = 4\pi\rho_0 a^3/3$ is the mass of fluid displaced by the particle. The transfer function $G(r_1, r_2)$ is given by

$$G(r_1, r_2) = \frac{(9 + 9r_1 + 2r_1^2)(1 + r_2) + (1 + r_1)r_2^2}{r_1^2(1 + r_2) + (2 + 2r_1 + r_1^2)r_2^2} \,, \tag{6}$$

where

$$r_1(s) = \frac{as/c_0}{\sqrt{1 + (\mu_b/\mu + 4/3)\,\nu s/c_0^2}} \quad \text{and} \quad r_2(s) = a\sqrt{\frac{s}{\nu}}\,. \tag{7}$$

SOLUTION FOR IMPULSIVE MOTION

Since the problem is linear, the force on a particle undergoing arbitrary rectilinear motion $v(t)$ can be expressed as a convolution integral

$$F(t) = \int_{-\infty}^{t} \frac{dv}{d\xi}\, F_\delta(t-\xi)\, d\xi\,, \tag{8}$$

where $F_\delta(t)$ is the force response to a delta-function acceleration (i.e., corresponding to a unit step change in particle velocity). Using Eq. (5), $F_\delta(t)$ can be expressed as

$$\mathcal{F}_\delta(s) = -m_f G(r_1, r_2)\,. \tag{9}$$

An explicit Laplace inverse transform of Eq. (9) and, therefore, a closed-form expression for $F_\delta(t)$ is not readily available. Before constructing the time-domain solution, we first analyze the limiting case of incompressible flow.

The incompressible limit is obtained by letting $c_0 \to \infty$ in Eq. (9) to obtain

$$\mathcal{F}_{\delta,\mathrm{inc}}(s) = -m_f \frac{9 + 9r_2 + r_2^2}{2r_2^2}\,, \tag{10}$$

where r_2^2 can be interpreted as the Laplace variable corresponding to time non-dimensionalized by the viscous time scale a^2/ν. The corresponding expression in the time domain is

$$F_{\delta,\mathrm{inc}}(t) = -6\pi a\mu H(t) - \frac{1}{2} m_f \delta(t) - 6a^2\rho_0\sqrt{\frac{\pi\nu}{t}}H(t)\,, \tag{11}$$

where $H(t)$ is the Heaviside step function. The above expression is identical to the BBO equation for a delta-function acceleration, cf. Eq. (1).

COMPRESSIBILITY EFFECT ON INVISCID UNSTEADY FORCE

We isolate the three terms (quasi-steady, inviscid unsteady, and viscous unsteady forces) on the right-hand side of Eq. (1) and investigate the effect of compressibility. First, we consider the compressibility effect on the inviscid unsteady force. The inviscid limit is obtained by substituting $\nu = 0$ in Eq. (9) to get

$$\mathcal{F}_{\delta,iu} = -m_f \frac{1 + r_1}{2 + 2r_1 + r_1^2}\,, \tag{12}$$

where r_1 can be interpreted as the Laplace variable corresponding to time non-dimensionalized by the acoustic time scale a/c_0. The corresponding expression in the time domain is

$$F_{\delta,iu}(\tau) = -m_f \frac{c_0}{a} e^{-\tau} \cos\tau\, H(\tau)\,, \tag{13}$$

where $\tau = c_0 t/a$. The effect of compressibility on the inviscid unsteady force can be established by comparing Eq. (13) with the second term on the right-hand side of Eq. (11). The finite speed of sound destroys the instantaneous relationship between acceleration and force. Furthermore, compressibility regularizes the singular delta-function kernel to a smooth oscillatory exponential decay. From a physical perspective, this can be explained by the compression and rarefaction waves that emanate from the accelerated particle which propagate outward at finite speed. However, due to the exponential-decay term in Eq. (13), the compressibility effect is significant only for $\tau \lesssim 10$.

The above inviscid unsteady force was first obtained by Longhorn [11] using the acoustic approximation of the velocity potential equation and is thus valid only in the zero Mach-number limit. The right-hand side of Eq. (13) can be considered to be the response kernel for a delta-function acceleration for $\mathrm{M} \to 0$. Note that $\int_0^\infty e^{-\tau}\cos\tau\, d\tau = 1/2$, and thus over times much longer than the acoustic time scale, the net impulse on the particle reduces to the correct limit as that given by the incompressible added-mass force. The corresponding kernels for finite Mach numbers can be obtained through numerical simulations, see Parmar *et al.* [12].

ASYMPTOTIC BEHAVIORS OF COMPRESSIBLE VISCOUS UNSTEADY FORCE

We now examine the effect of compressibility on the viscous unsteady force. To study the force at arbitrary times, we resort to numerical inversion of Eq. (9). With V denoting the scale of the velocity variation, we define the Reynolds and Mach numbers as $\mathrm{Re} = \rho_0 V a/\mu$ and $\mathrm{M} = V/c_0$. Thus we can write

$$\frac{F_\delta(\tau)}{m_f c_0/a} = -\mathcal{L}^{-1}\left(G(R_1, R_2)\right) , \tag{14}$$

where $\mathcal{L}^{-1}$ denotes the Laplace inverse with respect to the non-dimensional time $\tau = c_0 t/a$ and

$$R_1 = \frac{S}{\sqrt{1 + (\mu_b/\mu + 4/3)\,\mathrm{Kn}' S}} \quad \text{and} \quad R_2 = \sqrt{\frac{S}{\mathrm{Kn}'}} , \tag{15}$$

where $S = as/c_0$ is the non-dimensional Laplace variable and $\mathrm{Kn}' = \mathrm{M}/\mathrm{Re} = \mu/\rho_0 c_0 a$ denotes a modified Knudsen number. The continuum assumption is commonly taken to imply $\mathrm{Kn} < 0.01$, where $\mathrm{Kn} = \sqrt{\gamma\pi/2}\,\mathrm{M}/\mathrm{Re}$, and thus we are interested in the force response for $\mathrm{Kn}' \lesssim O(10^{-2})$.

Although an explicit expression for $F_\delta(\tau)$ valid for arbitrary τ is not available, four different asymptotic regimes can be identified: (i) *Regime I:* Very short time, defined as $\tau \ll \mathrm{Kn}' \ll 1$, (ii) *Regime II:* Intermediate short time, defined as $\mathrm{Kn}' \ll \tau \ll 1$, (iii) *Regime III:* Intermediate long time, defined as $1 \ll \tau \ll 1/(\mathrm{Re}\,\mathrm{M})$, (iv) *Regime IV:* Very long time, defined as $1 \ll 1/(\mathrm{Re}\,\mathrm{M}) \ll \tau$.

In what follows, we examine the behavior of $F_\delta(\tau)$ in the first three asymptotic regimes. The very short time behavior of $F_\delta(\tau)$ can be obtained by considering the following limit in the Laplace space: $1 \ll \mathrm{Kn}'|S| \ll |S|$. Correspondingly, the transfer function can be simplified and the Laplace inverse gives the time domain force response in *Regime I* as

$$F_\delta(\tau) \sim -\left(\frac{4}{9} + \frac{2}{9}\sqrt{\frac{\mu_b}{\mu} + \frac{4}{3}}\right) 6a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}} H(\tau) . \tag{16}$$

Comparing with the third term on the right-hand side of Eq. (11), which can be written as $-6a^2\rho_0 c_0\sqrt{\pi\mathrm{Kn}'/\tau}$, it can been seen that compressibility modifies the viscous unsteady force at very short times by a factor that depends on μ_b/μ. For $\mu_b = 0$, compressibility reduces the unsteady force by $4(1 + 1/\sqrt{3})/9 \approx 0.70$. The correction factor to the viscous unsteady force increases with increasing bulk viscosity. Interestingly, for $\mu_b/\mu = 59/12 \approx 4.92$ the correction factor is equal to unity and Eq. (16) becomes identical to that in the incompressible case.

From the definition of the intermediate short time ($\mathrm{Kn}' \ll \tau \ll 1$), the following condition can be placed on the Laplace variable: $\mathrm{Kn}'|S| \ll 1 \ll |S|$. Then $G(R_1, R_2)$ can be simplified and the dominant contribution for intermediate short time yields in *Regime II*

$$F_\delta(\tau) \sim -m_f \frac{c_0}{a} e^{-\tau}\cos\tau - \frac{8}{3} a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}} H(\tau) . \tag{17}$$

The first term is same as $F_{\delta,iu}(\tau)$ given by Eq. (13). Comparing the second term with the third term on the right-hand side of Eq. (11), it can been seen that the viscous unsteady force at intermediate short times is reduced by a factor of $4/9 \approx 0.44$ because of compressibility effects. Note that this reduction is independent of μ_b/μ. As will be seen below in Fig. 1, where the results of the numerical Laplace inversion are shown, *Regime II* can be observed only if $\mathrm{Kn}' \ll 1$. With increasing Kn', the duration of the *Regime II* reduces and vanishes entirely for $\mathrm{Kn}' \approx 10^{-2}$. Thus the overall effect of compressibility on the short-time behavior of the viscous unsteady force is not as pronounced as for the inviscid unsteady force. The $\tau^{-1/2}$ decay observed in the incompressible case persists and only the magnitude of the viscous unsteady force is modified.

The intermediate long-time behavior can be obtained in a similar manner by considering an asymptotic expansion for $|S| \to 0$ and carrying out the Laplace inverse we obtain in *Regime III*

$$F_\delta(\tau) \sim -6\pi a\mu H(\tau) - 6a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}} . \tag{18}$$

Comparing with Eq. (11), both the quasi-steady and the viscous unsteady forces are recovered and found to be unaffected by compressibility effects. Strictly speaking, the above solution for the linearized perturbation Navier-Stokes equations is valid for $\tau \gg 1$ and the additional limit of $\tau \ll 1/(\mathrm{Re}\,\mathrm{M})$ arises only from the neglect of the nonlinear terms. The time scale on which nonlinear effects become significant can be estimated as follows. In deriving the linearized form of the compressible Navier-Stokes equations, the assumption that the inertial terms are negligible compared to the viscous terms implies that the length scale $L \ll \nu/V$. If we take the length scale to grow by diffusion as $\sqrt{\nu t}$, the assumption of linearized compressible Navier-Stokes equations can be justified only for $t \ll \nu/V^2$. Expressed in terms of the acoustic time scale, this restriction becomes $\tau \ll 1/(\mathrm{Re}\,\mathrm{M})$. Note that the above argument applies in an incompressible flow also, and the nonlinear effects can be shown to become important for $\tau_c \gg 1/\mathrm{Re}$, where τ_c is time non-dimensionalized by the convective time scale a/V. This is consistent with past results for incompressible flow that the Basset-history kernel is valid only for $\tau_c \ll 1/\mathrm{Re}$ even at low Reynolds numbers (see Mei and Adrian [13]). Thus, owing to nonlinearity, the very long time force behavior in *Regime IV* will be both Reynolds- and Mach-number dependent in a complex manner and will not be addressed here.

NUMERICAL EVALUATION OF VISCOUS UNSTEADY FORCE

In the following, we extract the compressible form of the viscous unsteady force at arbitrary times using numerical Laplace inversion and compare it with its incompressible form. We isolate the viscous unsteady force from the overall force expression given in Eq. (14) by subtracting the quasi-steady contribution and the inviscid unsteady force given in Eq. (13).

$$\frac{F_{\delta,vu}(\tau)}{m_f c_0/a} = -\left(\mathcal{L}^{-1}\left(G(R_1, R_2)\right) - \frac{9}{2}\mathrm{Kn}' - e^{-\tau}\cos\tau\right) . \tag{19}$$

We recast this viscous unsteady response to delta function acceleration in the following form

$$\frac{F_{\delta,vu}(\tau)}{m_f c_0/a} = -\frac{9}{2}\sqrt{\frac{\mathrm{Kn}'}{\pi\tau}}\, C(\tau)\,, \tag{20}$$

where $C(\tau)$ is a compressible correction function, defined as the ratio of $F_{\delta,vu}(\tau)$ relative to the incompressible form of the viscous unsteady force.

Figure 1 shows $C(\tau)$ plotted against τ. In *Regime I* ($\tau \ll \mathrm{Kn}' \ll 1$) the correction function approaches 0.70 at very short times. Also, we observe that $C(\tau) \to 1$ as $\tau \to \infty$ as expected. At intermediate short times (which exist only for $\mathrm{Kn}' \ll 10^{-2}$) the correction function takes a constant value of 0.44. A more complex compressibility effect can be observed at transitional times between the different asymptotic regimes. The decrease from the constant value at very short time to the new constant value at intermediate short time occurs in a monotonic fashion. At about $\tau = O(10^{-2}\mathrm{Kn}')$, $C(\tau)$ starts to deviate from its limiting value of 0.70 and decreases to 0.44 at about $\tau = O(\mathrm{Kn}')$. The transition from *Regime II* to *Regime III* that occurs at $\tau \approx O(1)$ is more complex. At $\tau = O(10^{-2})$, $C(\tau)$ increases rapidly irrespective of Kn' toward a peak value of about 1.45 before decreasing in a strongly damped oscillatory manner toward unity. Thus, the compressibility correction to the viscous unsteady force is bounded between 0.44 and 1.45 for $\mu_b = 0$. The sensitivity of $C(\tau)$ to the bulk viscosity is also shown in Fig. 1.

INVISCID AND VISCOUS UNSTEADY FORCE KERNELS AND NUMERICAL CONFIRMATION

Based on results presented in the previous sections, we write

$$\frac{F_\delta}{m_f c_0/a} = \frac{1}{m_f c_0/a}\left(F_{\delta,qs} + F_{\delta,iu} + F_{\delta,vu}\right) = -\frac{9}{2}\frac{\nu}{a c_0} - e^{-\tau}\cos\tau - \frac{9}{2}\sqrt{\frac{\mathrm{Kn}'}{\pi}}\frac{C(\tau)}{\sqrt{\tau}}\,, \tag{21}$$

where $F_{\delta,qs}$ is the quasi-steady force in response to a delta-function acceleration. While the normalized inviscid unsteady force depends only on τ, the normalized viscous unsteady force also depends on both Kn' and μ_b/μ through $C(\tau)$. The above force response to a delta function acceleration can be used to define inviscid and viscous unsteady force kernels as

$$K_{iu}(\tau) = e^{-\tau}\cos\tau\,, \quad K_{vu}(t) = \frac{C(c_0 t/a)}{\sqrt{t}} = C(c_0 t/a) K_B(t)\,, \tag{22}$$

where $K_B(t)$ is the Basset history kernel.

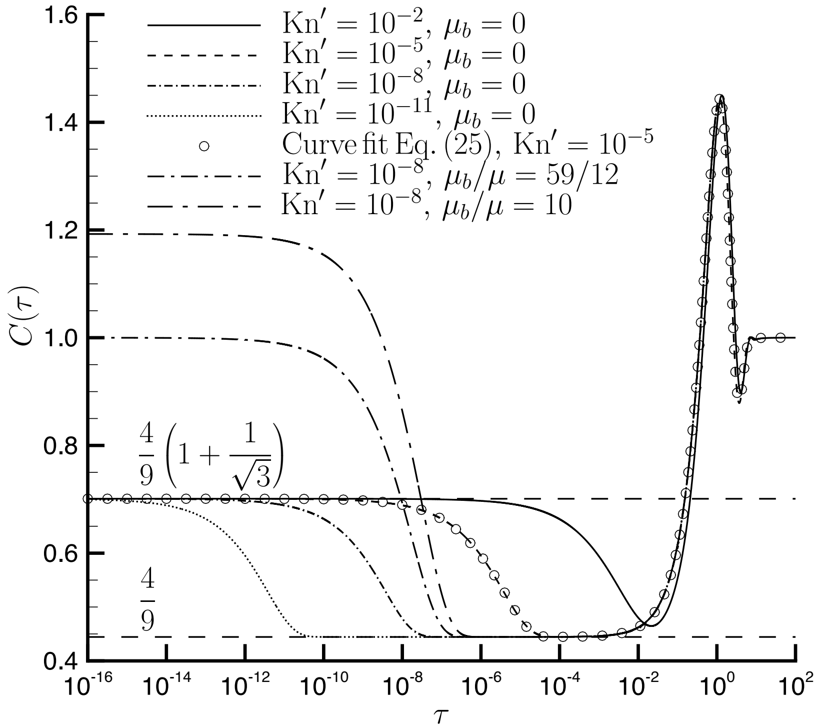

FIG. 1. The behavior of $C(\tau)$ that accounts for the compressibility effect on the viscous unsteady force.

We have carried out numerical simulations for $\mu_b = 0$, wherein the spherical particle is initially stationary in a quiescent fluid and impulsively accelerated to a final steady state. To extract the unsteady force, we subtract the quasi-steady standard drag. The results of the simulations are shown in Fig. 2, where the non-dimensional unsteady force is plotted as a function of the acoustic time $\tau = c_0 t/a$. The agreement between the theory and the simulations is excellent. The simulations capture accurately the $\tau^{-1/2}$ decay for both $\tau \ll 1$ and $\tau \gtrsim O(10)$. At intermediate times, the influence of inviscid unsteady force becomes significant and the simulations capture this well.

In the first three simulations, we have $1/(\mathrm{Re}\,\mathrm{M}) = \{10^4, 10^3, 10^2\}$, respectively, and thus the agreement between the nonlinear simulations and the linear theory is good over the entire range of the computed time interval, as expected. We have also simulated a case of $\mathrm{M} = 10^{-1}$ and $\mathrm{Re} = 10$, corresponding to $\mathrm{Kn}' = 10^{-2}$. Again the agreement is excellent at small times. However, since $1/(\mathrm{Re}\,\mathrm{M}) = 1$ for this case, the effect of nonlinearity becomes important for $\tau \approx O(1)$ and the results of the simulation show a faster decay than the $\tau^{-1/2}$ behavior predicted by the linear theory.

As $\tau \to 0$, while the inviscid kernel is equal to unity, the viscous kernel diverges as $1/\sqrt{\tau}$. At large times, the inviscid kernel decays exponentially, while the viscous kernel decays algebraically. Thus, both at short and long times, the viscous unsteady force dominates the inviscid unsteady force. At intermediate times, the inviscid unsteady force becomes important and it can be shown that only for $\mathrm{Kn}' > 5.96 \times 10^{-2}$ will the viscous unsteady force dominate the inviscid unsteady force at all times. This limiting value of Kn' must be viewed with caution, however, because the continuum assumption breaks down for $\mathrm{Kn}' > 10^{-2}$. At smaller values of Kn', there exists an intermediate range of time where the inviscid unsteady force will exceed the viscous unsteady force.

GENERALIZATION OF THE BBO EQUATION TO COMPRESSIBLE FLOWS

The above-presented results can be used to write a general expression for the force on a particle undergoing arbitrary time-dependent motion $\mathbf{v}(t)$ in a viscous compressible fluid. The generalization of the BBO equation to compressible flow can be expressed as

$$m_p \frac{d\mathbf{v}}{dt} = -6\pi a \mu \mathbf{v} - m_f \int_{-\infty}^{t} K_{iu}\left((t-\xi)\frac{c_0}{a}\right) \left.\frac{d\mathbf{v}}{dt}\right|_{t=\xi} d\left(\xi \frac{c_0}{a}\right) - 6a^2\rho_0\sqrt{\pi\nu} \int_{-\infty}^{t} K_{vu}(t-\xi) \left.\frac{d\mathbf{v}}{d\xi}\right|_{t=\xi} d\xi\,, \tag{23}$$

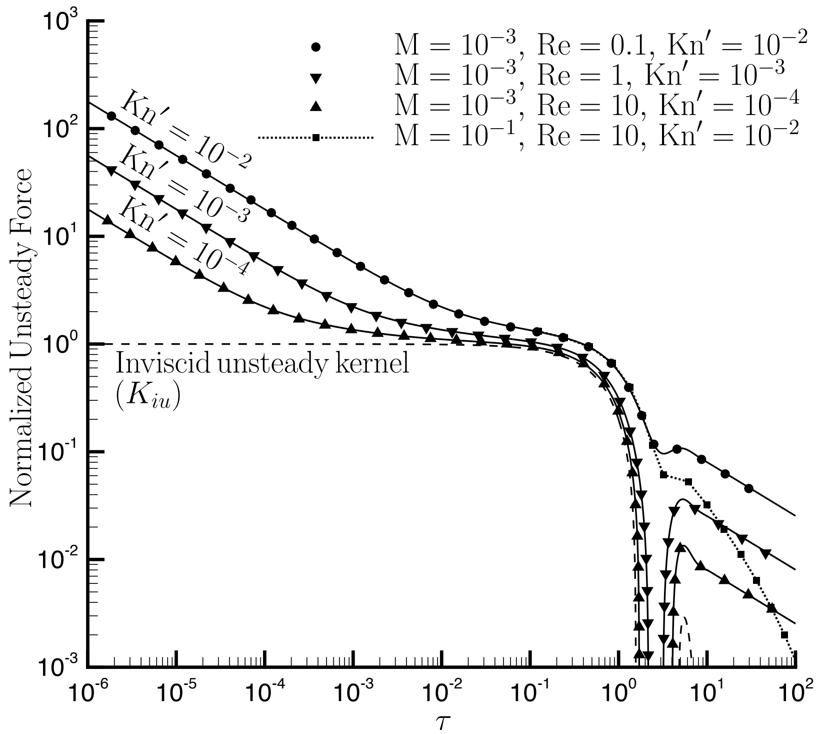

FIG. 2. Time evolution of the normalized unsteady force. Theoretical predictions (last two terms of Eq. (21)) are plotted as solid lines for $\mu_b = 0$. Corresponding simulation results for four different cases are shown as symbols.

where the inviscid and viscous unsteady force kernels are given in Eq. (22). The above equation is valid in the limit of zero Reynolds and Mach numbers. The significance of this extension is two-fold. First, it includes explicit expressions for the inviscid unsteady and viscous unsteady components of the force that reduce to their well-known counterparts in the incompressible limit. Second, the extension can be combined with other forces such as buoyancy and lift to give a complete equation of motion.

Both the inviscid and viscous compressibility corrections decay rapidly and can be ignored for $\tau \gtrsim 10$. Note that it was mentioned earlier that the long-time integral of the inviscid unsteady kernel $K_{iu}(\tau)$ reduces to the added-mass coefficient of 1/2. A similar result can be established for the viscous unsteady kernel $K_{vu}(\tau)$ given by Eq. (22). The long-time integration of $K_{vu}(\tau)$ can be shown to approach the incompressible limit,

$$\int_0^t K_{vu}(\xi)\, d\xi - \int_0^t K_{\mathrm{B}}(\xi)\, d\xi \to 0 \quad \text{for} \quad t \gg 1\,. \tag{24}$$

When the proposed equation of motion (23) is used in practice, an expression for $C(\tau)$ is required. The following curve-fit can be employed for this purpose, assuming that $\mu_b = 0$:

$$C(\tau) = \frac{4}{9} + \frac{4}{9\sqrt{3}} \frac{1}{1 + 2.38\left(\frac{\tau}{\mathrm{Kn}'}\right)^{0.57} e^{1.02\tau/\mathrm{Kn}'}} + 2\tau^{C_1} e^{-C_2\tau} \left\{\cos\left[C_3(\tau - C_4)\right] + \sin\left[C_3(\tau - C_4)\right]\right\} + \frac{5}{9}\frac{\tau^{C_5}}{\tau^{C_5} + C_6}\,, \tag{25}$$

where

$$C_1 = 0.96 + 1.71 \exp\left[0.51(\log \mathrm{Kn}' + 1.25)\right]\,, \quad C_2 = 1.14 + 0.22 \exp\left[0.57(\log \mathrm{Kn}' + 1.45)\right]\,, \tag{26}$$

$$C_3 = 0.87 - 0.26 \exp\left[0.50(\log \mathrm{Kn}' + 3.55)\right]\,, \quad C_4 = 0.25 - 1.38 \exp\left[0.60(\log \mathrm{Kn}' + 1.98)\right]\,, \tag{27}$$

$$C_5 = 3.38 + 7.82 \exp\left[0.72(\log \mathrm{Kn}' - 0.16)\right]\,, \quad C_6 = 5.09 + 5.71 \exp\left[0.89(\log \mathrm{Kn}' + 2.42)\right]\,. \tag{28}$$

The maximum error of the curve-fit is less than 1 percent (see Fig. 1).

Finally, it should also be pointed out the kernels presented in Eq. (22) combined with the above correction function are appropriate in the limit of zero Reynolds and Mach numbers. The finite Mach-number influence on the inviscid kernel has been addressed by Parmar *et al.* [12]. Similarly, the correction function $C(\tau)$ can be expected to depend on both Reynolds and Mach numbers.

CONCLUSIONS

We have obtained an explicit equation for the time-dependent force on a spherical particle undergoing arbitrary unsteady motion in a compressible flow. The resulting equation of motion is the generalization of the Basset-Boussinesq-Oseen equation to the compressible regime. The significance of this extension is that it includes explicit expressions for the quasi-steady, inviscid unsteady, and viscous unsteady components of the force, which reduce to their well-known counterparts in the incompressible limit. The effect of compressibility on the inviscid unsteady force is significant, while the effect on the viscous unsteady force is modest. The modification due to compressibility appears as a multiplicative correction factor $C(c_0 t/a)$ to the Basset history force, whose value is bounded, $4/9 \leq C(c_0 t/a) < 1.5$ (for zero bulk viscosity). The effect of compressibility on the inviscid unsteady and the viscous unsteady is significant only up to few acoustic times, say $c_0 t/a < 10$.

ACKNOWLEDGMENTS

The authors gratefully acknowledge support by the National Science Foundation under grant number EAR0609712 and AFOSR under grant number FA9550-10-1-0309. The authors also acknowledge the University of Florida High-Performance Computing Center for providing computational resources and support.

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