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Quasiparticle relaxation of superconducting qubits in the presence of flux

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Quasiparticle tunneling across a Josephson junction sets a limit for the lifetime of a superconducting qubit state. We develop a general theory of the corresponding decay rate in a qubit controlled by a magnetic flux. The flux affects quasiparticles tunneling amplitudes, thus making the decay rate flux-dependent. The theory is applicable for an arbitrary quasiparticle distribution. It provides estimates for the rates in practically important quantum circuits and also offers a new way of measuring the phase-dependent admittance of a Josephson junction.

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Long coherence times of superconducting qubits rely on the decoupling of the order parameter quantum oscillations from other low-energy degrees of freedom. Quasiparticles provide an intrinsic set of states capable of exchanging energy with the qubit degree of freedom. In equilibrium, quasiparticle populations should get completely depleted at low temperatures, rendering the corresponding qubit relaxation channel ineffective. In practice, however, some nonequilibrium quasiparticles are observed [1, 2]. The question whether the relaxation driven by quasiparticles is comparable to the extrinsic mechanisms of qubit relaxation has remained open.

The theory of quasiparticle relaxation was addressed in [3] for a charge qubit, whose computational space consists of two values of charge of the same parity (even or odd) stored in a Cooper pair box. The elementary process of relaxation essentially amounts to the well-studied quasiparticle poisoning [4, 5]: a quasiparticle entering the Cooper pair box changes the parity of the state. Later, the theory [3] was modified to estimate the effect of quasiparticles in a "transmon" device [6], where the dominant energy scale comes from the Josephson inductance. Quantum fluctuations of phase in a transmon are relatively small, while the uncertainty of charge in the qubit states is significant. The advantage of the transmon is its low sensitivity to charge noise. Further consideration of the relaxation induced by quasiparticles in superconducting qubits was developed in Ref. 2. The properties of qubits such as the phase and flux qubits [7], the transmon, and the newly developed fluxonium [8] can be tuned by threading a magnetic flux through the device.

Here, we predict that the relaxation rates induced by quasiparticle tunneling depend on the magnetic flux threading the qubit. The theory we develop allows for any quasiparticle distribution and any magnitude of quantum fluctuations of the phase of the order parameter; thus, it provides relaxation rates for the entire spectrum of superconducting qubits. We also show that a spectroscopic measurement on a device designed to have small phase fluctuations may enable one to measure the enigmatic phase-dependence of the Josephson junction admittance [9].

We consider the system consisting of a Josephson junction closed by an inductive loop, see Fig. 1. The Hamiltonian $\hat{H}$ governing the low-energy dynamics of the system can be divided into three parts

$$\hat{H} = \hat{H}_\varphi + \hat{H}_{\rm qp} + \hat{H}_T \, . \tag{1}$$

The first term takes the form of the inductively shunted Josephson junction

$$\hat{H}_\varphi = 4E_C\left(\hat{N} - n_g\right)^2 - E_J\cos\hat{\varphi} + \frac{1}{2}E_L\left(\hat{\varphi} - 2\pi\frac{\Phi_e}{\Phi_0}\right)^2 , \tag{2}$$

where $\hat{N} = -id/d\varphi$ is the number operator of Cooper pairs passed across the junction, n_g is the dimensionless gate voltage, Φ_e is the external flux threading the loop, and $\Phi_0 = h/2e$ is the flux quantum. With appropriate choices for the parameters characterizing the qubit – charging energy E_C, Josephson energy E_J, and inductive energy E_L – and of the biases n_g and Φ_e, Eq. (2) can describe a single-junction qubit and also a multi-junction one, so long as an array of junctions with energies $E_J^a \gg E_J$ is treated as an effective inductance [10].

The quasiparticle term $H_{\rm qp}$ is the sum of the BCS quasiparticle Hamiltonians for the left and right leads

$$\hat{H}_{\rm qp} = \sum_{j=L,R} \hat{H}^j_{\rm qp} \, , \quad \hat{H}^j_{\rm qp} = \sum_{n,\sigma} \epsilon^j_n \hat{\alpha}^{j\dagger}_{n\sigma}\hat{\alpha}^j_{n\sigma} . \tag{3}$$

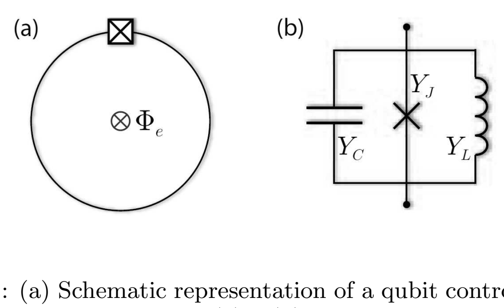

FIG. 1: (a) Schematic representation of a qubit controlled by a magnetic flux, see Eq. (2). (b) Effective circuit diagram with three parallel elements – capacitor, Josephson junction, and inductor – characterized by their respective admittances.

Here $\hat{\alpha}^j_{n\sigma}(\hat{\alpha}^{j\dagger}_{n\sigma})$ is the quasiparticle annihilation (creation) operator, $\sigma =\uparrow,\downarrow$ accounts for spin, and the quasiparticle energies are $\epsilon^j_n = \sqrt{(\xi^j_n)^2 + (\Delta^j)^2}$, with ξ^j_n and Δ^j being the single-particle energy level n in the normal state of lead j, and the gap parameter in that lead, respectively. Finally, the tunneling term $\hat{H}_T$ describes quasiparticle tunneling across the junction,

$$\hat{H}_T = t\sum_{n,m,\sigma}\left(e^{i\frac{\hat{\varphi}}{2}}u^L_n u^R_m - e^{-i\frac{\hat{\varphi}}{2}}v^R_m v^L_n\right)\hat{\alpha}^{L\dagger}_{n\sigma}\hat{\alpha}^R_{m\sigma} + \text{H.c.} \quad (4)$$

The electron tunneling amplitude t here is related to the junction conductance $g = 4\pi e^2\nu^L\nu^R t^2/\hbar$. We work in the tunneling limit $t \ll 1$ and assume identical densities of states per spin direction in the leads, $\nu^L = \nu^R = \nu_0$. The Bogoliubov amplitudes u^j_n, v^j_n can be taken real, since Eq. (4) already accounts explicitly for the phases of the order parameters in the leads via the gauge-invariant phase difference [9] in the exponentials. Accounting for Josephson effect and quasiparticles dynamics by Eqs. (2)-(4) is possible as long as the qubit energies and characteristic energies of quasiparticles (measured from Δ) are small compared to Δ [3]. In this low-energy limit, we may approximate $u^j_m \simeq v^j_n \simeq 1/\sqrt{2}$. Then the operators $e^{\pm i\hat{\varphi}/2}$ in Eq. (4) which describe transfer of charge $\pm e$ across the junction, combine to $\sin(\hat{\varphi}/2)$. The superposition of the tunneling amplitudes containing $e^{i\hat{\varphi}/2}$ and $e^{-i\hat{\varphi}/2}$ is a manifestation of interference between tunneling of particle- and hole-like excitations, possible in the presence of the Cooper pair condensate. We stress that the phase $\hat{\varphi}$ is an operator subject to quantum fluctuations, as determined by $\hat{H}_\varphi$, not an externally controlled parameter as in the "classical" Josephson junction [9]. Moreover, the non-linear nature of phase-quasiparticle coupling is in stark contrast with the linear coupling between phase and the electromagnetic environment.

In general, the qubit Hamiltonian in Eq. (2) has a discrete low-energy spectrum. The tunneling term, Eq. (4), couples the qubit to the quasiparticles; therefore, a transition between initial, $|i\rangle$, and final, $|f\rangle$, qubit states becomes possible when a quasiparticle is excited during a tunneling event. The transition rate $\Gamma_{i\to f}$ can be calculated using Fermi's Golden Rule,

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\sum_{\{\lambda\}_{\rm qp}}\langle\!\langle\left|\langle f,\{\lambda\}_{\rm qp}|\hat{H}_T|i,\{\eta\}_{\rm qp}\rangle\right|^2 \times\delta\left(E_{\lambda,{\rm qp}} - E_{\eta,{\rm qp}} - \hbar\omega_{if}\right)\rangle\!\rangle_{\rm qp}\,, \quad (5)$$

where $\hbar\omega_{if}$ is the energy difference between the qubit states, $E_{\eta,\rm qp}$ and $E_{\lambda,\rm qp}$ are the total energies of the quasiparticles in their respective initial $\{\eta\}_{\rm qp}$ and final $\{\lambda\}_{\rm qp}$ states; double angular brackets $\langle\!\langle\ldots\rangle\!\rangle_{\rm qp}$ denote averaging over the initial quasiparticle states whose occupation is determined by the distribution functions $f^j(\xi^j_n) = \langle\!\langle\hat{\alpha}^{j\dagger}_{n\uparrow}\hat{\alpha}^j_{n\uparrow}\rangle\!\rangle_{\rm qp} = \langle\!\langle\hat{\alpha}^{j\dagger}_{n\downarrow}\hat{\alpha}^j_{n\downarrow}\rangle\!\rangle_{\rm qp}$ ($j = L,R$) assumed to be independent of spin. In terms of the matrix elements of the operator $\sin(\hat{\varphi}/2)$ we find

$$\Gamma_{i\to f} = \left|\langle f|\sin\frac{\hat{\varphi}}{2}|i\rangle\right|^2 S_{\rm qp}\left(\omega_{if}\right). \quad (6)$$

In deriving Eq. (6) we have made a simplifying assumption of equal gaps, $\Delta_L = \Delta_R = \Delta$. More importantly, we concentrate on the case of low-lying excitations, assuming the characteristic energy δE of the quasiparticles (determined by the distribution functions; in thermal equilibrium, $\delta E = k_B T$) and the energy of the qubit transition $\hbar\omega_{if}$ are small, $\hbar\omega_{if}\,, \delta E \ll 2\Delta$. That enables us to factorize the transition rate $\Gamma_{i\to f}$ into a product of terms which depend separately on the qubit dynamics and quasiparticle kinetics. The latter determines the normalized quasiparticle current spectral density $S_{\rm qp}$,

$$S_{\rm qp}(\omega) = \frac{8E_J}{\pi\hbar}\int_0^\infty dx\,\frac{1}{\sqrt{x}\sqrt{x+\hbar\omega/\Delta}}\Big[f^L_E\left((1+x)\Delta\right) \times\left(1 - f^R_E\left((1+x)\Delta + \hbar\omega\right)\right) + (L\leftrightarrow R)\Big], \quad (7)$$

where $\omega > 0$ [11] and we used the relation $E_J = g\Delta/8g_K$; $g_K = e^2/h$ is the conductance quantum. The integrand in $S_{\rm qp}$ equals, up to a factor, the rate of transitions of a quasiparticle between the initial and final states, properly weighted with their occupation probabilities (their energies are $(1+x)\Delta$ and $(1+x)\Delta + \hbar\omega$, respectively). These probabilities are expressed in terms of the quasiparticle energy distribution functions, $f^j_E(\epsilon) = \left(f^j(\xi) + f^j(-\xi)\right)/2$, with $\epsilon = \sqrt{\xi^2+\Delta^2}$ and $j = L,R$.

We notice that $S_{\rm qp}$ is related to the real part of the quasiparticle contribution to the "classical" [9] Josephson junction admittance at zero phase difference,

$$S_{\rm qp}(\omega) - S_{\rm qp}(-\omega) = \frac{\omega}{\pi}\frac{1}{g_K}{\rm Re}\,Y_{\rm qp}(\omega), \quad (8)$$

at arbitrary ratio $\hbar\omega/\delta E$ and for any quasiparticle distribution function. In the rest of this Letter we consider the "high-frequency" limit [12] $\hbar\omega/\delta E \gg 1$ (but still $\hbar\omega \ll 2\Delta$), and equal populations on the two sides of the junction, $f^L = f^R$. Then we can simplify Eq. (8),

$$S_{\rm qp}(\omega) = \frac{\omega}{\pi}\frac{1}{g_K}{\rm Re}\,Y_{\rm qp}(\omega)\,, \qquad \omega > 0\,, \quad (9)$$

and express ${\rm Re}\,Y_{\rm qp}$ in terms of the quasiparticle density

$$n_{\rm qp} = 2\sqrt{2}\nu_0\Delta\int_0^\infty\frac{dx}{\sqrt{x}}f_E((1+x)\Delta) \quad (10)$$

(written using the same approximations as above) as

$${\rm Re}\,Y_{\rm qp}(\omega) = \frac{1}{2}x_{\rm qp}g\left(\frac{2\Delta}{\hbar\omega}\right)^{3/2}, \quad x_{\rm qp} = \frac{n_{\rm qp}}{2\nu_0\Delta}. \quad (11)$$

Here $x_{\rm qp}$ is the quasiparticle density normalized by the density of Cooper pairs; in thermal equilibrium, $x_{\rm qp} =$

$\sqrt{2\pi k_B T/\Delta}\, e^{-\Delta/k_B T}$. Under the above stated assumptions, Eq. (11) applies to an otherwise arbitrary distribution function, and hence to nonequilibrium conditions.

The structure of $S_{\rm qp}$ is identical to the corresponding one in the Mattis-Bardeen formula [13], when generalized to the case of non-equilibrium distributions [14]. The connection between the Mattis-Bardeen theory, which describes absorption of electromagnetic waves impinging on the surface of a "dirty" superconductor, and the theory of qubit relaxation caused by quasiparticles in a Josephson junction was noticed in [2]. At the microscopic level, dissipation in the two settings is caused by quasiparticle transitions occurring without momentum conservation (both systems lack translational invariance). The difference between the two problems is in the form of the perturbation causing the transitions. In our case, the quasiparticle transitions are due to the coupling to the qubit degree of freedom. The matrix element $\langle f|\sin(\hat\varphi/2)|i\rangle$ in Eq. (6) characterizes that coupling and it is sensitive to flux [15].

We stress that Eq. (6) holds for any single-junction qubit, its properties being encoded in the wavefunctions $|i\rangle$, $|f\rangle$ entering the matrix element. For a qubit comprising multiple junctions the tunneling Hamiltonian Eq. (4) and hence Eq. (6) are given by a sum over junctions, with $\hat\varphi$ being replaced by the phase difference across each junction. In a multi-junction qubit, the states $|i\rangle$ and $|f\rangle$ depend on all the phase differences (up to the constraint set by fluxoid quantization [16]).

We focus first on the case of a weakly anharmonic system, which already reveals a non-trivial flux dependence of relaxation. Its low-lying states, as the example of the transmon shows [6], can be used as qubit states. We assume $E_L \neq 0$ to eliminate n_g by a gauge transformation [17]. In the transmon $E_L = 0$, but the n_g-dependent corrections are exponentially small [6]; hence, the results below can be applied to a single-junction transmon by setting $\varphi_0 = 0$. In the limit $E_C \ll E_J, E_L$, Eq. (2) describes an ideal LC circuit with a junction in parallel. The phase across the junction is determined, up to small fluctuations, by the external magnetic flux: by minimizing the potential energy part of Eq. (2), we find that the phase φ_0 at the minimum satisfies

$$E_J \sin\varphi_0 + E_L(\varphi_0 - 2\pi\Phi_e/\Phi_0) = 0. \tag{12}$$

Near this minimum, the system behaves as a weakly anharmonic oscillator. Anharmonicity and quality factor Q determine the operability of the system as a qubit [7]. In the present case, the operability condition can be written as $Q/n_w \gg 1$, where n_w is the number of levels in the anharmonic well. For large Q the system can therefore be operated as a qubit despite its weak anharmonicity. To the leading order in $E_C/(E_L + E_J\cos\varphi_0)$ [18] and for low-lying levels $n \ll n_w$, we can neglect anharmonic corrections when evaluating the matrix element in Eq. (6) [19]. Using standard results for the harmonic oscillator, its value for transitions between two neighboring levels is approximated as:

$$\left|\langle n-1|\sin\frac{\hat\varphi}{2}|n\rangle\right|^2 = n\frac{E_C}{\hbar\omega_{10}}\frac{1+\cos\varphi_0}{2}, \tag{13}$$

with $\omega_{10} = \sqrt{8E_C(E_L + E_J\cos\varphi_0)}/\hbar$. In deriving Eq. (13), we replaced $\sin(\hat\varphi/2)$ by its linear expansion, which sets the limit on the excitation level, $n \ll \hbar\omega_{10}/E_C$. For higher n quantum fluctuations of phase are significant and transitions between distant levels proliferate, due to the nonlinearity of the operator $\sin(\hat\varphi/2)$.

Concentrating on $n \ll \hbar\omega_{10}/E_C$, we use $E_C = e^2/2C$ and Eqs. (6), (9), and (13) to find

$$\Gamma_{n\to n-1} = \frac{n}{C}\mathrm{Re}\, Y_{\rm qp}(\omega_{10})\frac{1+\cos\varphi_0}{2}\,. \tag{14}$$

For the transition between the two lowest states ($n = 1 \to 0$) and in the absence of magnetic flux ($\varphi_0 = 0$), Eq. (14) reduces to the expression for the rate of Ref. 2.

Using now Eq. (14) with $n = 1$, we find the inverse of the transition Q-factor at arbitrary φ_0,

$$\frac{1}{Q_{10}} = \frac{\Gamma_{1\to 0}}{\omega_{10}} = \frac{1}{\pi g_K}\mathrm{Re}\, Y_{\rm qp}(\omega_{10})\frac{E_C}{\hbar\omega_{10}}\frac{1+\cos\varphi_0}{2}, \tag{15}$$

Note that φ_0 and hence the transition frequency ω_{10} depend on the external flux Φ_e, see Eq. (12). We stress that the flux dependence presented here is specific to the single-junction case; results for multiple-junction qubits obtained with the developed method will be presented elsewhere. In the limit $E_J \gg E_L$, the flux dependence can be neglected and the last factor on the right hand side of Eq. (15) reduces to ≈ 1. Qubit Q-factors measured at temperatures ~ 20 mK are in the range 10^4-10^5 [20]. Using typical parameters for Al-based qubits ($\Delta \sim 2\times 10^{-4}$ eV, $\omega_{10}/2\pi \sim 1-10$ GHz) we find that to reproduce the experimental Q-factors with Eq. (15) and a thermal equilibrium-like quasiparticle density, we must assume an effective quasiparticle temperature an order of magnitude larger than the base temperature. This points either to the quasiparticles being in a non-equilibrium state whose origin is at present unclear, or to the prevalence of extrinsic relaxation mechanisms.

Beside causing dissipation, quasiparticle tunneling leads to a shift in the resonant frequency of the circuit. In the regime under consideration, the resonant frequency is the zero of the total admittance $Y(\omega)$ which for the parallel elements representing the qubit (Fig. 1b) is $Y = Y_C + Y_L + Y_J$, with $Y_C = i\omega C$, $Y_L = 1/i\omega L$, and the junction admittance Y_J being the sum [9] of a purely inductive Josephson term and a quasiparticle term,

$$Y_J(\omega,\varphi) = \frac{(1-2x^{\rm A}_{\rm qp})}{iL_J\omega}\cos\varphi + Y_{\rm qp}(\omega)\frac{1+\cos\varphi}{2}\,. \tag{16}$$

Here $L_J = \hbar/\pi g\Delta$ is the conventionally defined Josephson junction inductance, $x_{\rm qp}^{\rm A} = f_E(\Delta)$ has the meaning of the population of Andreev levels [21] ($x_{\rm qp}^{\rm A} = e^{-\Delta/k_BT}$ in thermal equilibrium), and

$$Y_{\rm qp}(\omega) = -\frac{2}{iL_J\omega}\left[\frac{x_{\rm qp}}{\pi}\sqrt{\frac{\Delta}{i\hbar\omega}} - x_{\rm qp}^{\rm A}\right] \tag{17}$$

is the quasiparticle admittance at zero phase difference in the high-frequency limit [its real part agrees with Eq. (11)]. Both free ($x_{\rm qp}$) and bound ($x_{\rm qp}^{\rm A}$) quasiparticles affect Y_J. The frequency shift they cause, measured from the frequency in the absence of quasiparticles, is found by solving $Y_C + Y_L + Y_J = 0$ at linear order in $x_{\rm qp}$, $x_{\rm qp}^{\rm A}$:

$$\delta\omega = \frac{i}{2C}Y_{\rm qp}(\omega_{10})\frac{1+\cos\varphi_0}{2} - \frac{\pi g\Delta}{C\hbar\omega_{10}}x_{\rm qp}^{\rm A}\cos\varphi_0\,. \tag{18}$$

The imaginary part of frequency shift reproduces – when Eq. (17) for the admittance is used – half the dissipation rate $\Gamma_{1\to 0}$ calculated above, while the real part in the high-frequency limit equals

$$\mathrm{Re}\,\delta\omega = \frac{1}{2}\frac{\omega_p^2}{\omega_{10}}\left[x_{\rm qp}^{\rm A}\left(1-\cos\varphi_0\right) - \frac{x_{\rm qp}}{2\pi}\sqrt{\frac{2\Delta}{\hbar\omega_{10}}}\left(1+\cos\varphi_0\right)\right], \tag{19}$$

where $\omega_p = \sqrt{8E_JE_C}/\hbar$ is the junction plasma frequency. The relation Eq. (18) for $\delta\omega$ in the weakly anharmonic regime opens new ways to study experimentally the effect of quasiparticles on the Josephson junction admittance. It may elucidate the phase dependence of its dissipative part, where experimental data are still controversial. Measuring $\mathrm{Im}\,\delta\omega$ along with $\mathrm{Re}\,\delta\omega$, Eqs. (18)-(19), may also shed light on the nature of nonequilibrium quasiparticle distributions: unlike the dissipative part, $\mathrm{Re}\,\delta\omega$ depends on both $x_{\rm qp}^{\rm A}$ and $x_{\rm qp}$. Experimental efforts to verify the flux and quasiparticle density dependence of $\delta\omega$ are in progress in our group and elsewhere [22].

The dependence of relaxation on flux is also sensitive to the states involved in the transition. The interaction between phase and quasiparticles is non-linear (the form of the non-linearity is associated with the discreteness of charge, as discussed above), thus allowing transitions between distant levels even in a harmonic-oscillator potential. At $E_C/\hbar\omega_{10} \ll 1$ these transitions are suppressed by the smallness of quantum fluctuations of the phase. For example, $\Gamma_{2\to 0}$ appears only in the second order in fluctuations of $\hat\varphi$ around φ_0,

$$\Gamma_{2\to 0} = \frac{2\omega_{10}}{\pi}\frac{1}{g_K}\mathrm{Re}\,Y_{\rm qp}(2\omega_{10})\left(\frac{E_C}{\hbar\omega_{10}}\right)^2\frac{1-\cos\varphi_0}{4}. \tag{20}$$

Here $\mathrm{Re}\,Y_{\rm qp}$ is given by Eq. (11). Notice the difference in the φ_0 dependence between Eqs. (14) and (20).

The effect of nonlinear in $\hat\varphi$ coupling of the qubit degree of freedom to quasiparticles is striking for a system with $E_J > E_L$ biased near half the flux quantum [23]. In the case of a flux qubit, the potential has a pronounced double-well shape and the qubit states $|0\rangle$ and $|1\rangle$ are the lowest tunnel-split eigenstates in this potential [7]. The rate $\Gamma_{1\to 0}$ vanishes at $\Phi_e = \Phi_0/2$ due to the destructive interference: at that point, the potential in Eq. (2) and function $\sin\varphi/2$ in Eq. (6) are symmetric around $\varphi = \pi$, while the qubit states $|0\rangle$, $|1\rangle$ are symmetric and antisymmetric, respectively. (The symmetry and its consequences are missed if the $\sin\hat\varphi/2$ interaction is replaced with the linear phase-quasiparticle coupling accepted in the environmental approach.) To evaluate $\Gamma_{1\to 0}$ at finite $\Phi_e - \Phi_0/2$ for a qubit governed by Eq. (2) – *i.e.*, a single junction connected to an (effective) inductor – we consider the case of tunnel splitting [6, 17]

$$\epsilon_0 = 4\sqrt{\frac{2}{\pi}}\hbar\omega_p\left(\frac{8E_J}{E_C}\right)^{1/4}e^{-\sqrt{8E_J/E_C}} \tag{21}$$

small compared to inductive and plasma energies, $\hbar\omega_p \gg E_L \gg \epsilon_0$. Then we can evaluate the matrix element in Eq. (6) in a tight-binding approximation, with the qubit states given by linear combinations of states localized in the two wells. Using Eq. (9), we arrive at

$$\Gamma_{1\to 0} = \frac{\omega_{10}}{\pi}\frac{1}{g_K}\mathrm{Re}\,Y_{\rm qp}(\omega_{10})\left(\frac{\epsilon_0}{4\pi E_J}\right)^2\left(1-\frac{\epsilon_0^2}{(\hbar\omega_{10})^2}\right), \tag{22}$$

where the transition frequency is related to the flux by $\omega_{10} = \sqrt{\epsilon_0^2 + \left[(2\pi)^2E_L(\Phi_e/\Phi_0 - 1/2)\right]^2}/\hbar$ and $\mathrm{Re}\,Y_{\rm qp}$ is given in Eq. (11).

Losses in other elements of the qubit give additional contributions to the relaxation rate in Eq. (22). For example, if the qubit comprises additional Josephson junctions, by generalizing Eq. (6) as previously described one can account for the quasiparticle losses in them. Unlike Eq. (22) those losses remain finite at $\Phi_e = \Phi_0/2$, but can be made small in a chain of $N \gg 1$ junctions of larger E_J, as they scale with $1/N$ due to small phase fluctuations in the chain (see also [10]).

In summary, we have presented a general approach to study the quasiparticle relaxation mechanism of superconducting qubits. It enables us to determine how the qubit decay rate depends on the magnetic flux used to tune the system properties. The method is applicable to any superconducting qubit and arbitrary quasiparticle population. For small phase fluctuations and transitions between neighboring levels, the decay rate can be expressed in terms of the real part of the "classical" [9] junction admittance, see Eq. (14), while the imaginary part of the admittance determines the shift in the qubit frequency [Eq. (19)]. This limit is applicable, *e.g.*, to a single-junction transmon. Decay rate and frequency shift have distinct dependencies on flux and quasiparticle distribution, so comparing the two quantities may give information on the (possibly) nonequilibrium state of quasiparticles.

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