Complex Critical Exponents for Percolation Transitions in Josephson-Junction Arrays, Antiferromagnets, and Interacting Bosons

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Complex critical exponents for percolation transitions in Josephson-junction arrays, antiferromagnets, and interacting bosons

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Abstract

We show that the critical behavior of quantum systems undergoing a percolation transition is dramatically affected by their topological Berry phase $2\pi\rho$. For irrational ρ, we demonstrate that the low-energy excitations of diluted Josephson-junctions arrays, quantum antiferromagnets, and interacting bosons are spinless fermions with fractal spectrum. As a result, critical properties not captured by the usual Ginzburg-Landau-Wilson description of phase transitions emerge, such as complex critical exponents, log-periodic oscillations and dynamically broken scale-invariance.

A fundamental aspect of the Ginzburg-Landau-Wilson (GLW) description of phase transitions is scale invariance, which relies on the absence of characteristic length and energy scales at criticality, leading to the concept of universality [1, 2]. For instance, near a quantum critical point (QCP), if a physical observable $O(T)$ transforms for an arbitrary scale transformation $b > 0$ according to $O(T) = b^{-x} O(b^z T)$, then we obtain a power-law temperature dependence $O(T) \propto T^{x/z}$, with universal critical exponent x/z. However, if scaling is valid only for powers of a discrete value b_0, it follows that $O(T) = T^{x/z} Q(\ln T)$, with $Q(t)$ a periodic function of period $z \ln b_0$. Fourier expansion of $Q(t)$ yields:

$$O(T) = \sum_{n=-\infty}^{\infty} \alpha_n T^{x/z + i2\pi n/(z \ln b_0)} \tag{1}$$

with constant coefficients α_n. Thus, the system is characterized by a family of non-universal complex exponents. An invariant scale b_0, leading to this *discrete scale invariance,* is found in several critical systems that either are out of equilibrium or have an underlying built-in hierarchical structure (for a review, see [3]).

In this Letter, we show that complex critical exponents and log-periodic behavior appear in a variety of disordered systems close to a percolation QCP, such as Josephson-junction (JJ) arrays, quantum antiferromagnets (QAF) and interacting bosons. Rather than being related to non-equilibrium properties or to the fractality of the percolating cluster, in these systems the invariant scale b_0 emerges naturally in their low-energy excitation spectrum for certain values of their Berry phase $2\pi\rho$.

By calculating their specific heat and compressibility at the percolation threshold, we show that, for rational ρ, large clusters have the lowest excitation energies, giving rise to usual power-law behavior below a crossover temperature T^*, which varies in a pronounced non-monotonic way with respect to ρ (see Fig. 1). For irrational ρ, the low-energy properties are governed instead by degenerate clusters of intermediate sizes, leading to the breakdown of continuous scale invariance ($T^* \to 0$). Remarkably, the sizes and energies of these resonating clusters depend solely on the continued-fraction expansion of ρ. As a result, when ρ is a quadratic irrational, the periodicity of its continued-fraction expansion gives rise to an invariant scale b_0 and, consequently, to complex critical exponents.

To introduce a model for all the systems discussed above [2], consider an array of grains characterized by an XY order parameter $\Psi_j = |\Psi_0| \exp(i\theta_j)$, with phase θ_j and fixed amplitude $|\Psi_0|$. The array is diluted on a regular lattice of dimension $d > 1$, characterized by

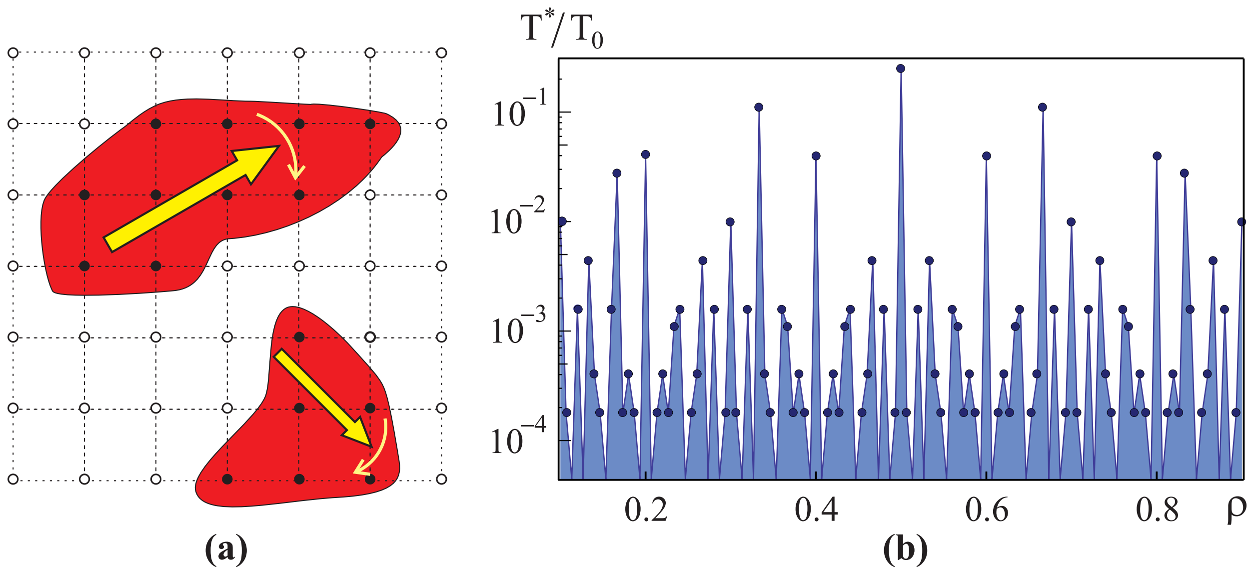

Figure 1: (a) In the diluted quantum system, the global phase (arrow) of clusters of connected grains (dark/red) coherently precesses due to the Berry phase $2\pi\rho$. (b) Strong variation of the crossover temperature T^* below which scaling with real-valued exponents holds (blue points, $T_0 \sim U$).

a quenched random-site variable ϵ_j that takes the values 0 and 1 with probabilities P and $(1-P)$, respectively. We consider the Hamiltonian [2, 4, 5]:

$$H = U \sum_i \epsilon_i \left(n_i - \rho\right)^2 - \sum_{ij} \epsilon_i \epsilon_j J_{ij} \cos\left(\theta_i - \theta_j\right), \tag{2}$$

where $n_i = -i\frac{\partial}{\partial\theta_i}$. ρ can be externally controlled and causes the phase to precess in time according to $\partial\theta_j/\partial t = 2U\rho$ (see Fig. 1). In JJ-arrays [4, 5], Ψ_j denotes the superconducting order parameter, J_{ij} is the Josephson coupling, U is the charging energy and ρ, related to the AC-Josephson effect, can be changed by an external gate voltage. Ψ_j can also represent planar quantum rotors, associated with the low-energy modes of QAF [2], where ρ is proportional to a perpendicular external magnetic field. For systems of interacting bosons[6, 7], which can be realized in optical experiments with cold atoms [8], $2U\rho$ corresponds to the chemical potential μ.

The effects of percolative dilution on all these systems have been the subject of various experimental and numerical investigations [8–12]. Here, we focus on the critical properties at the percolation threshold P_c, where the density of clusters with s connected occupied sites varies as $N(s) \propto s^{-\tau}$, with $2 < \tau \equiv d/D_f + 1 \leq 2.5$ and D_f the fractal dimension of the percolating cluster [13]. At low temperatures $T \ll |J_{ij}|$ and deep in the ordered state of the clean system ($U < |J_{ij}|$), the relative phase between grains inside each cluster is fixed, implying that the entire cluster is characterized by a global phase [14–16]. At P_c, contributions to the total specific heat of a single cluster arise from the coherent precession of its global phase, C, and from the excitations of internal collective (spin-wave) modes that change the relative phase between grains, C_{sw}. As we will show below, C_{sw} is subleading; thus, similar to the behavior in superparamagnets, each size-s cluster can be treated effectively as a big single rotor, with the corresponding action:

$$\mathcal{A}_s = -\frac{s}{4U}\int_0^\beta d\bar{\tau}\left(\frac{\partial\theta\left(\bar{\tau}\right)}{\partial\bar{\tau}} - i\mu\right)^2, \tag{3}$$

which describes the coherent phase-precession due to both quantum fluctuations and the Berry phase $2\pi\rho$. Notice that, unlike the case of SU(2) spins, the Berry phase of quantum rotors has a topological character, since the imaginary part $\mathcal{A}_{\mathrm{Berry}} = is\rho\int_0^\beta d\bar{\tau}\frac{\partial\theta}{\partial\bar{\tau}}$ of $\mathcal{A}_s$ is independent on the time evolution of $\theta\left(\bar{\tau}\right)$, enabling us to solve our problem using sums over winding numbers. Shifting the imaginary time $\bar{\tau} \to \bar{\tau}/s$ in (3) eliminates the pre-factor s at the expense of a cluster size dependent temperature $T \to sT$ and, most importantly, Berry phase $\rho \to s\rho$. This yields the free energy scaling $F_s\left(\rho, T\right) = s^{-1}F_1\left(s\rho, sT\right)$, from which we can derive scaling relations for the heat capacity $C_s\left(T\right) = -T\partial^2 F_s/\partial T^2$ and the compressibility $\kappa_s\left(T\right) = -\partial^2 F_s/\partial\mu^2$. Here, the suffix 1 ($s$) refers to quantities on a single site (single cluster). Thus, macroscopic quantities can be calculated by averaging over all clusters, i.e. $\mathcal{O}\left(\rho, T\right) = \sum_s N\left(s\right)\mathcal{O}_s\left(\rho, T\right)$.

Let us first revisit the results for $\rho = 0$, where universal power-law behavior was previously found [14]. The low-temperature specific heat of a cluster is given by $C_s\left(\rho = 0\right) \propto \exp\left(-U/sT\right)$, i.e. the typical excitation energy of a cluster decreases *monotonically* with its size, $\varepsilon_s = U/s$. Then, the low-energy behavior is dominated by large clusters and we can replace the sum over s by an integral, obtaining, for $T \ll U$,

$$C\left(\rho = 0, T\right) \propto T^{d/z_r}, \tag{4}$$

where the dynamic scaling exponent $z_r = D_f$ was introduced. This result was also found in detailed computer simulations [17, 18]. Consider now a finite Berry phase $\rho \neq 0$. Solving the problem of a single quantum rotor and using the scaling $T \to sT$, $\rho \to s\rho$ derived from the action (3), we find the spectrum of a single cluster $E_m = U\left(m - s\rho\right)^2/s$, with $m \in \mathbb{Z}$.

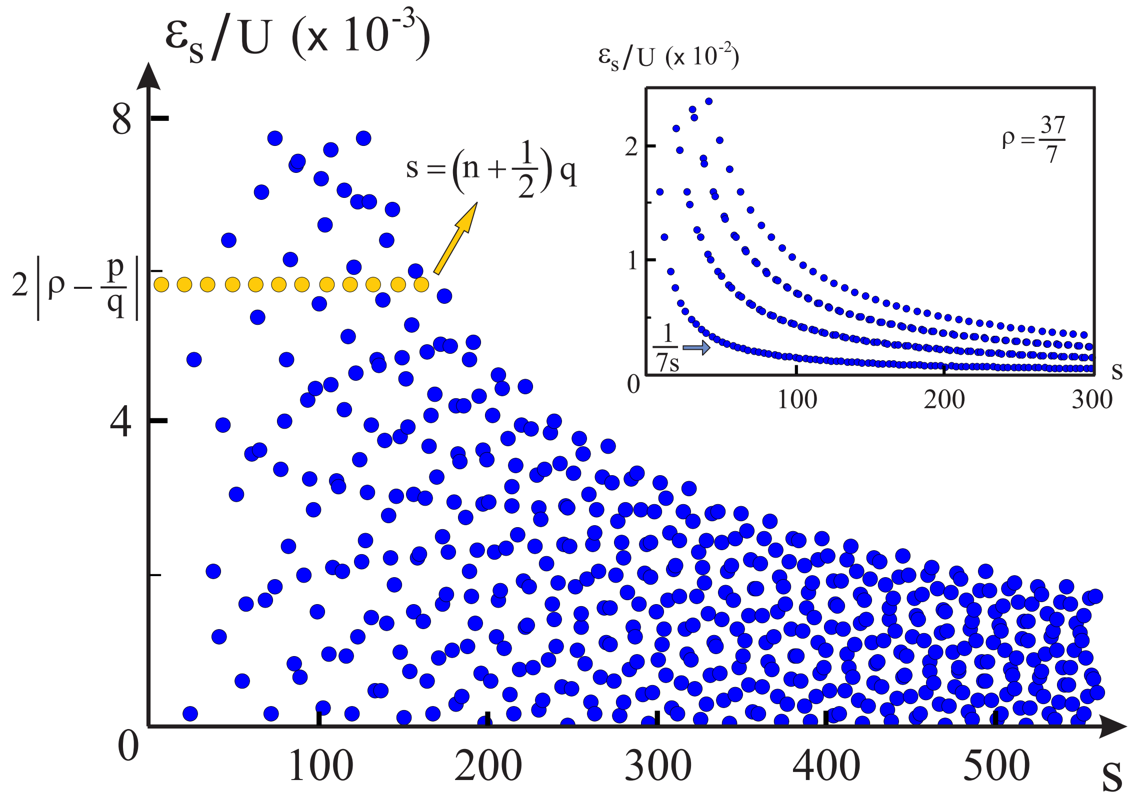

Figure 2: Excitation energy ε_s as function of the cluster size s for the irrational $\rho = \sqrt{7}$. The highlighted points correspond to energetically degenerate clusters associated to the Diophantine approximant $p/q = 37/14$, and are responsible for a jump in the integrated density of states. The inset shows the spectrum for the rational $\rho = 37/7$, characterized by four well-defined branches.

Therefore, the lowest excitation energy is (see also [6, 7]):

$$\varepsilon_s = \frac{U}{s}\left(1 - 2\left|\rho_s\right|\right), \tag{5}$$

where $\rho_s = s\rho - \left\lfloor s\rho + \frac{1}{2}\right\rfloor$ and $\lfloor x \rfloor$ is the integer part of x, i.e. $\left\lfloor x + \frac{1}{2}\right\rfloor$ is the integer closest to x, such that $|\rho_s| \leq 1/2$. Note that ρ_s depends on the droplet size in a highly non-monotonic way, reflecting the periodicity of the Berry phase (see Fig. 2). Now, not only large clusters yield small excitation energies, but also intermediate-size clusters with $|\rho_s| \lesssim 1/2$.

These low-energy excitations can be described by spinless fermions, yielding the well-known fermionic expressions $C_1(\omega, T) = (\omega/T)^2 n_F(\omega) n_F(-\omega)$ and $\kappa_1(\omega, T) = (1/T) n_F(\omega) n_F(-\omega)$ for the single site specific heat and compressibility, with $n_F(\omega)$ the Fermi function. In Fig. 3 we demonstrate this numerically by comparing $C(T)$ obtained from the exact energy spectrum and from the fermionic expression only. We can also show this result analytically: Consider, for definiteness, $0 \leq \rho \leq 1/2$. The spectrum of a single rotor, including degeneracies, can be generated from a model of effective interacting fermions and bosons, with occupation numbers $n_f = f^\dagger f$ and $n_b = b^\dagger b$, respectively

$$H_0 = U\left(n_b + (1-2\rho)\, n_f + \rho\right)^2. \tag{6}$$

For $\rho = 0$, one recovers the $N = 2$ super-symmetric description of a rotor [19]. Expanding for $\rho \simeq 1/2$, we obtain instead $H_0 \simeq \varepsilon_1 n_f + U n_b + U n_b^2 + 2\varepsilon_1 n_b n_f$. While the excitation energy of single bosons is U, the fermionic excitation energy is $\varepsilon_1 = U(1-2\rho) \ll U$. Thus, at sufficiently low temperatures, bosons are diluted and the interaction terms can be neglected, implying that free spinless fermions are the dominant excitations of the system. The limit $\rho \to 1/2$ also plays an important role in the excitation spectrum of droplets in the Bose-glass phase [6] and in the insulating phase of interacting bosons in a disordered chain [7].

Let us now consider $\rho = p/q$ to be rational, i.e. p and q are integers with no common divisors. Due to the periodicity of the Berry phase term, the number of distinct values of ρ_s is of the order of $q/2$, defining well separated branches in ε_s, all decaying as s^{-1}, with lowest branch $\varepsilon_s \simeq U(qs)^{-1}$ (see inset of Fig. 2). For $T \ll U/q^2$, the problem is virtually the same as for $\rho = 0$, leading to a heat capacity dominated by very large clusters as given in Eq.4 with the same exponent z_r, as shown in Fig. 3. For the compressibility, we obtain $\kappa(T) \propto T^{d/z_r - 1}$, with a Wilson ratio $(\kappa T)/C \approx 0.3$ for $d = 2$. Note, from Fig. 3, that the crossover temperature T^* changes as q^{-2} and is insensitive to p. Thus, systems with similar values of ρ can have very different T^* (see Fig. 1).

The natural question is: what happens in the regime $U/q^2 \ll T \ll U$ for very large q? For irrational ρ, $T^* \to 0$ and this regime prevails down to the lowest energies. Indeed, when ρ is irrational, the sequence ρ_s is uniformly distributed between $-1/2$ and $1/2$ [20], i.e. there are finite-size droplets with arbitrarily low excitation energy ε_s. It is convenient to introduce the averaged integrated density of fermionic states, $D(\omega) = \sum_s s^{-\tau}\theta(\omega - \varepsilon_s)$; from the periodicity of ε_s with respect to ρ (i.e. summing over winding numbers), we obtain, for $\omega \ll U$:

$$D(\omega) = \zeta(\tau - 1)\frac{\omega}{U} + \sum_{s,\lambda=\pm} \frac{f_{\text{saw}}(s x_\lambda)}{s^\tau}, \tag{7}$$

where $x_\pm = \omega/U \pm 2\rho$, $\zeta(x)$ is the zeta function and $f_{\text{saw}}(x)$ is the sawtooth function, which has period 2 and unit jumps at every odd integer: $f_{\text{saw}}(x) = (-x + 2n)/2$ for $2n - 1 < x <$

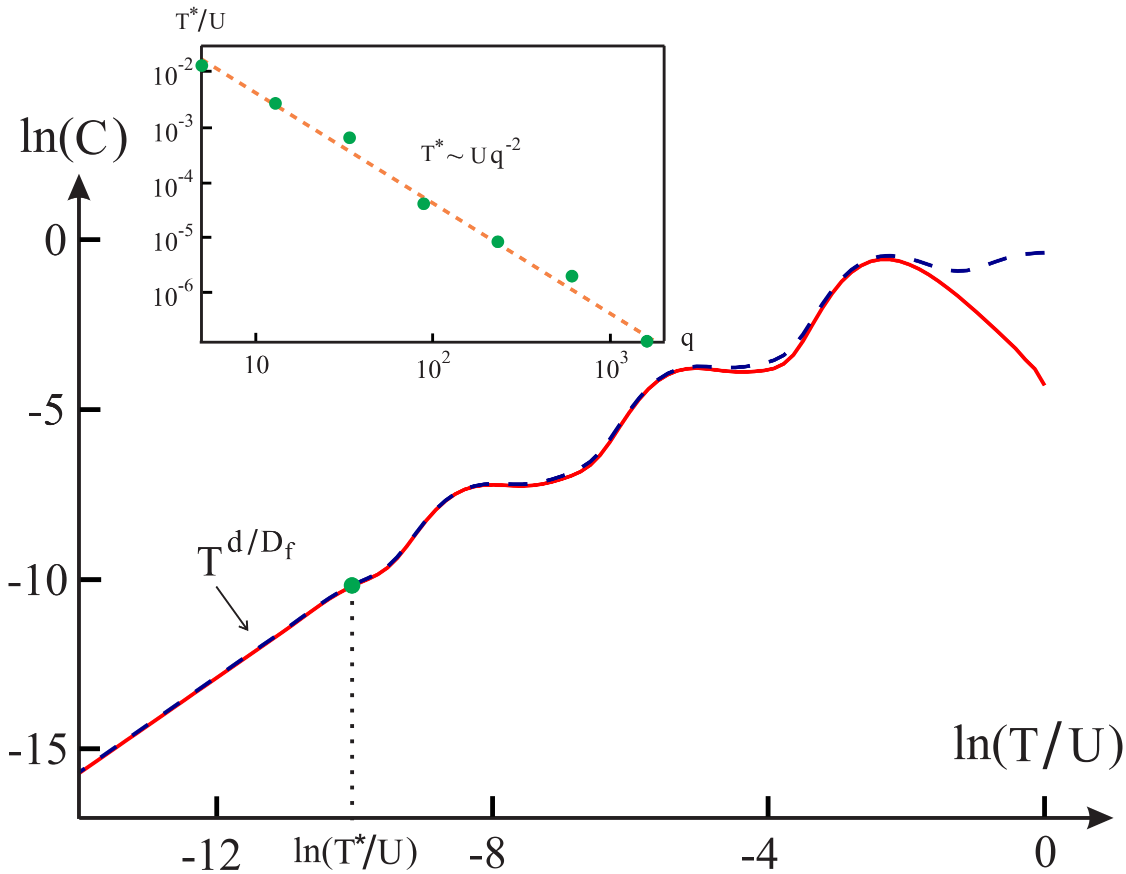

Figure 3: Specific heat C as function of temperature T for the rational $\rho = 55/89$. The dashed blue line is the exact result and the solid red line is the fermionic approximation. The onset of power-law behavior is marked by T^*(dotted line). The inset shows the log-log variation of T^* with respect to the denominator q for a series of rationals with comparable values, $\rho = \left\{\frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{55}{89}, \frac{144}{233}, \frac{377}{610}, \frac{987}{1597}\right\}$.

$2n+1$. These jumps give rise to discontinuities in $D(\omega)$ at frequencies:

$$\omega_j = 2U \left|\rho - \frac{p_j}{q_j}\right|, \tag{8}$$

where p_j is an odd integer and q_j is even. At $\omega \ll U$, the fractions p_j/q_j that satisfy Eq. (8) are the ones that best approximate ρ, i.e. the Diophantine approximants, which are given by the convergents of the continued fraction expansion of ρ [21]. Physically, the jumps are a consequence of the existence of a set of energetically degenerate (i.e. "resonating") clusters with sizes that are odd multiples of $q_j/2$, $s = (n+1/2)\,q_j$ (see Fig. 2). Summing over all these clusters in Eq. (7), we find that each jump in $D(\omega)$ is given by $\Delta_j = q_j^{-\tau}\zeta(\tau)(2^\tau - 1)$.

Back to Eq. 7, we find that the regular part of the sawtooth function cancels out the linear in ω term. Thus, the frequency dependence of $D(\omega)$ is governed by the successive jumps Δ_j

at ω_j of Eq. (8) and, consequently, by the sequence of convergents of the continued fraction expansion of ρ with even denominator q_j. Although the determination of this sequence for an arbitrary irrational ρ is an outstanding problem in number theory, it is simplified in the case of quadratic irrationals, which have periodic continued fraction expansions [21]. Then, one finds that the sequence of even q_j is also periodic.

In fact, for quadratic irrationals with a single period $a \in \mathbb{Z}$, which are solutions of the algebraic equation $y^2 - ay - 1 = 0$, we find that if q_j is even, so is q_{j+N}, with $N = 2 + \mathrm{mod}\,(a, 2)$. Consequently, the distance between consecutive jumps is a constant in log-scale, $\ln(\omega_j/\omega_{j+1}) = 2N \ln y_+$, as well as the ratio between their amplitudes, $\ln(\Delta_j/\Delta_{j+1}) = \tau N \ln y_+$, where y_+ is the positive solution of the algebraic equation. Using these properties, we can show that $D(\omega)$ is a fractal function with fractal dimension $d_\omega = \tau/2$, characterized by a power-law decay in ω and periodic oscillations in $\ln \omega$. Since $C(T) = \int d\omega\, \nu(\omega)\, C_1(\omega, T)$, where $\nu = dD/d\omega$ is the density of states, we obtain:

$$C(T) = T^{d/z_{ir}} A(\ln T), \tag{9}$$

where $z_{ir} = 2D_f d/(D_f + d)$, i.e. $z_{ir} > z_r$, and $A(t)$ is a periodic function of period $z_{ir} \ln b_0 \equiv 2N \ln y_+$. For the compressibility, we find $\kappa(T) = T^{(d/z_{ir})-1} B(\ln T)$, where $B(t)$ has the same period as $A(t)$. Thus, the system has complex critical exponents as in Eq. (1). In Fig. 4, we show numerical results for $\rho = \sqrt{2}$; we also verified numerically that the scaling form in Eq. 9 holds for quadratic irrationals ρ with more complicated continued-fraction periods. For non-quadratic irrationals, our numerical calculations indicate that Eq. 9 still describes the critical behavior, but now $A(t)$ oscillates irregularly without a well-defined period.

The breakdown of continuous scale invariance for irrational ρ can be attributed to the resonating clusters with arbitrarily low excitation energies, as they cause jumps in the entire spectrum, prohibiting to replace $\sum_s \to \int ds$ in Eq. 7. For rational ρ, such a replacement is allowed, leading to full scaling $s \to s/b^{D_f}$ and to Eq. 4. Yet, when ρ is a quadratic irrational, the dynamically broken scale invariance is partially restored as discrete scale invariance. In this case, the periodic structure of the continued-fraction expansion of ρ gives rise to log-periodic relations between sizes $s = (n + 1/2)\, q_j$ and energies $\omega_j = 2U\,|\rho - p_j/q_j|$ of different sets of resonating clusters, establishing an invariant scale b_0 and complex critical exponents $(d/z_{irr}) + in\pi/(N \ln y_+)$ for $C(T)$.

Going back to the contribution of the spin-waves, their density of states is $\nu_{sw} \propto k^{d_s - 1}$ at

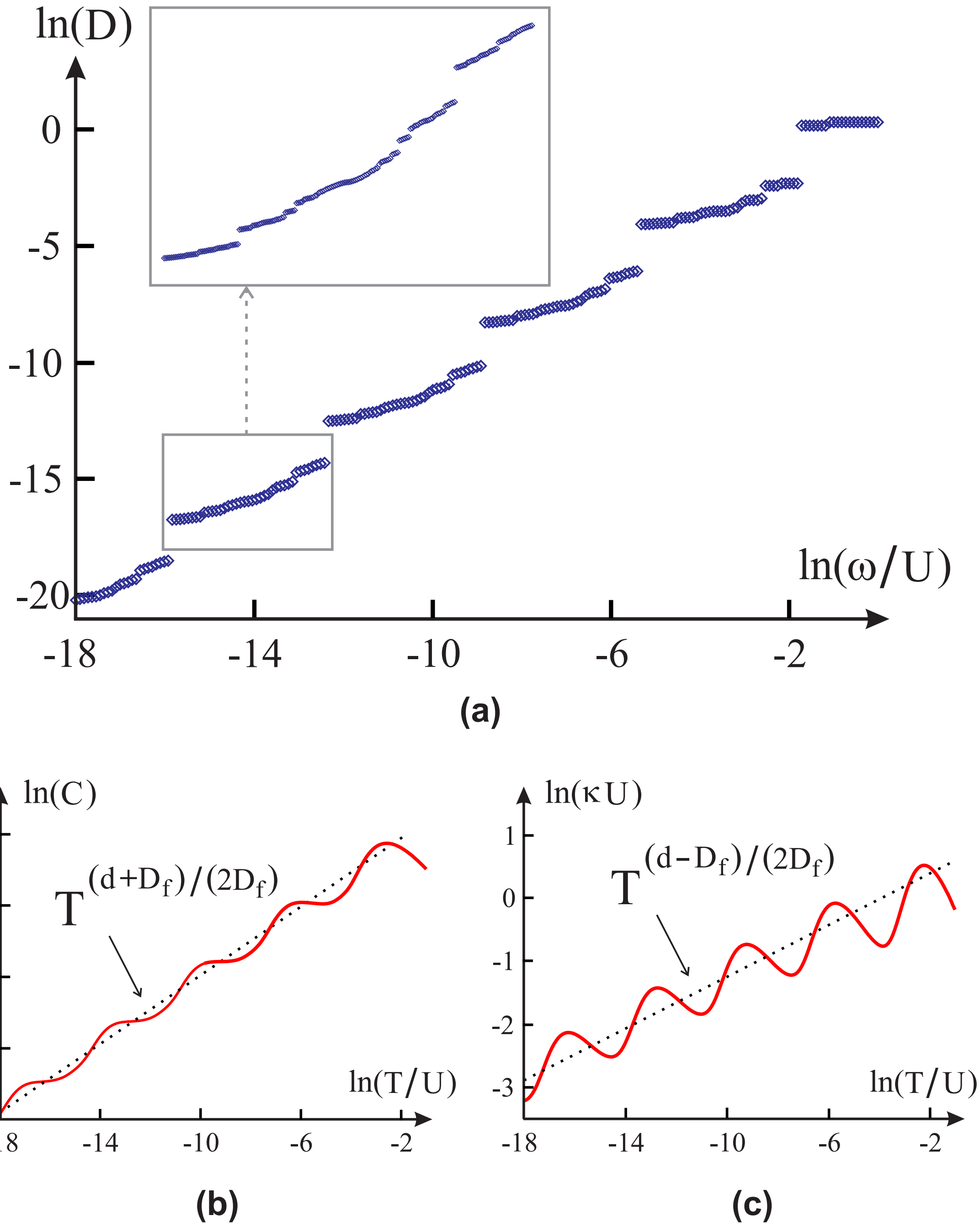

Figure 4: (**a**) Frequency dependent integrated density of states $D(\omega)$, as well as the temperature dependent (**b**) specific heat $C(T)$ and (**c**) compressibility $\kappa(T)$ for the system with $\rho = \sqrt{2}$. Dashed lines show the underlying power-law behavior superimposed to the log-periodic oscillations. Inset is a zoom of $D(\omega)$.

P_c, where the fracton dimension d_s characterizes the spectrum of the eigenvalues k^2 of the Laplacian on the percolating cluster [13, 16]. From the spin-waves dispersion $\Omega_{s,k}^2 = \varepsilon_s^2 + c^2k^2$ and the scaling properties of the fermionic density of states, we find $\mathcal{O}_{sw} \propto \mathcal{O}T^\phi$ for both $\mathcal{O} = C, \kappa$, with $\phi = d_s - 1$ ($\phi = d_s - 1/2$) for rational (irrational) ρ. As $d_s > 1$ [13], it follows that $\phi > 0$. Since the internal modes are sub-leading compared to the coherent modes, the spectrum of a single cluster depends solely on its size and not on its shape, in accordance to Eq. 3.

In summary, we solved the XY quantum-rotor problem at low T and close to the percolation threshold, which describes diluted systems as diverse as JJ-arrays with d.c.-bias voltage, canted QAF in a perpendicular magnetic field, and interacting bosons coupled to a particle reservoir. Their topological Berry phase $2\pi\rho$ dramatically alters the percolation QCP, since the low-T behavior is governed by emergent spinless fermions with fractal spectrum, giving rise to generally irregular $\log T$-oscillations of thermodynamic variables. While for irrational ρ they persist to $T \to 0$, for rational $\rho = p/q$ they occur in the temperature range $q^{-2} \lesssim T/U \lesssim 1$, which can be broad for $q \gg 1$. Remarkably, for a quadratic irrational ρ, they become regular, leading to complex critical exponents. Our results demonstrate that the quantum criticality in disordered systems governed by a topological Berry phase is beyond the GLW paradigm of critical systems.

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