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Unifying Lengthscale-Based Rheology of Dense Suspensions

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We present a new lengthscale-based rheology for dense sheared particle suspensions as they transition from inertial- to viscous-dominated. We derive a lengthscale ratio using straightforward physics-based considerations for a particle subjected to pressure and drag forces. In doing so, we demonstrate that an appropriately chosen length-scale ratio intrinsically provides a consistent relationship between normal stress and system proximity to its "jammed" or solid-like state, even as a system transitions between inertial and viscous states, captured by a variable Stokes number.

I. INTRODUCTION

Particle flows and particle-fluid flows are ubiquitous in natural phenomena and industrial processes, such as landslides, debris flows, rock falls, and concrete [1–3]. Complex environmental conditions make it difficult to obtain a unified constitutive law for their flow characteristics, particularly for dense flows, where short-range interactions are enduring and often generate long-range correlations [4].

In last two decades, significant progress has been made in modeling wet and dry granular flows, focusing on dimensional analysis and time scales. For dry granular flows, the local normal stress P_p (which is associated with interparticle interactions only), particle density ρ_s , particle size d , and shear rate $\dot{\gamma}$ are combined into a single dimensionless ratio of two timescales, microscopic ($\sqrt{\rho_s d^2 / P_p}$) and macroscopic ($1/\dot{\gamma}$): $I = \dot{\gamma} d / \sqrt{P_p / \rho_s}$ [5–7]. Various articles have shown that dynamic parameters such as the apparent friction coefficient $\mu = \tau / P_p$ (here, τ is a local shear stress) and the solid fraction ϕ can be represented using functions of I in both steady-state [5] and transient (e.g., column collapse) systems [8]. Cassar *et al.* [9] adapted this framework to submerged particle systems by replacing the microscopic (inertial) timescale with a viscous timescale (η_f / P_p , where η_f is the fluid viscosity), and the appropriate dimensionless control parameter is $J = \dot{\gamma} \eta_f / P_p$. Boyer *et al.* [10] further validated the saturated framework and generalized the mathematical form to include much sparser suspensions.

In the last decade, work on dense flows has broadened to include systems in which both hydrodynamic forces and particle inertial effects contribute to the rheology. Toward this, Trulsson *et al.* [11] proposed an effective shear stress in the form of a linear superposition of inertial and viscous stresses ($\tau_{\text{eff}} = \lambda \times \rho_s d^2 \dot{\gamma}^2 + \eta_f \dot{\gamma}$ normalized by P_p that produces a new dimensionless number: $K = \lambda I^2 + J$, where λ is a single-valued fitting parameter). Based on their experimental data, Tapia *et al.* [12] argued that $\lambda = 1/St_{\text{tr}}$, where St_{tr} is a transitional Stokes number ($St = I^2/J = \rho_s \dot{\gamma} d^2 / \eta_f$) close to 1. They observed that ϕ and μ exhibit distinct scaling behaviors, thus requiring *two* fitted parameters to collapse their μ and ϕ data ($St_{\text{tr},\mu} = 114$ and $St_{\text{tr},\phi} = 10$, respectively).

These efforts have revolutionized the representation of dense suspension flows. However, significant knowledge gaps remain, associated with issues of material properties [12, 13] and representations of transitions from inertial-dominated to viscous-dominated dynamics [14]. Recent rheologies based on combinations of I and J only include static fitted variables to represent the transition from I - to J -dominated behaviors [11, 12, 15], but the lack of physical meaning of these parameters, and the multiple values they can take for the different variable involved (μ and ϕ), greatly limits the physical insight that can be obtained from them.

In this work, we address these points by deriving a new lengthscale-based rheology for dense sheared flows based on the ratio of average relative ("macroscopic") displacements (associated with streamwise travel in the shearing direction) to deviations from those displacements. The latter ("microscopic" displacements), are increasingly restricted

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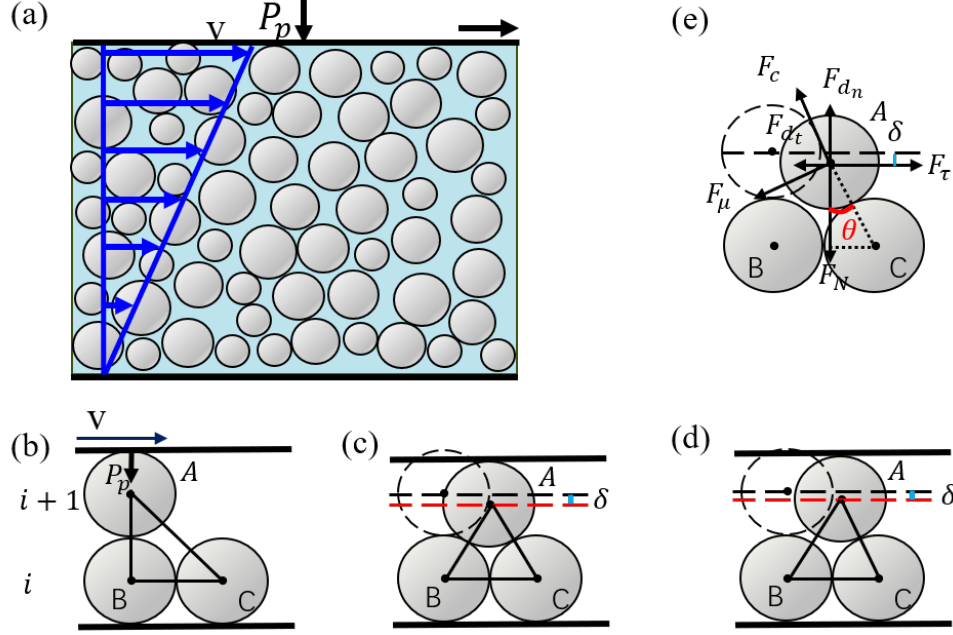


FIG. 1. (a) Shear behavior of dense suspensions: The top plate applies a pressure P_p on the particles, inducing shear within the granular assembly as the top plates move at a velocity v , while the bottom plate remains fixed. (b) Schematic of a 2-D conceptual model of a particle in layer $i + 1$ subjected to shear and normal stresses over particles in layer i . (c) Relative motion for low $\dot{\gamma}$ (and/or high P_p). (d) Relative motion for higher $\dot{\gamma}$ (and/or lower P_p). (e) Force analysis of the particle A during shear.

relative to the streamwise displacements as the system ϕ approaches its maximum value ϕ_c . We start by using a conceptualized model to show how near ϕ_c dynamics are partially represented in increasingly limited streamwise-normal movements (compared to streamwise movements) as the system deviates from ϕ_c . Through the momentum balance equation at particle scale, we demonstrate that μ depends on three factors: inter-particle friction μ_p , solid structure (which can be represented by the lengthscale ratio), and fluid-structure interaction (which is a function of rescaled confining pressure $I/J = \sqrt{\rho_s P_p} d / \eta_f$, which is introduced in the work of Trulsson *et al.* [11]). Then, we theoretically derive the appropriate lengthscale ratio between average displacements and average variants from those displacements to model how the ratio varies with solid and fluid properties and forcing conditions. This gives rise to a structure based rheological parameter that requires *no* additional fitting parameters and includes a function of the Stokes number St_{tr} that varies through the inertial-to-viscous transition, intrinsically capturing the transition at varying ϕ/ϕ_c proximities to 1 [16]. Moreover, we present a unified expression for μ/μ_c based on a function involving the structural parameter (lengthscale ratio), and the fluid-structure interaction parameter I/J . We demonstrate that the new framework contextualizes previously proposed timescale based frameworks, and also fits published experimental and computational data [10, 12, 15, 17, 18].

II. THEORETICAL DERIVATION OF GENERALIZED LENGTHSCALE AND TIMESCALE RATIOS

We begin with a conceptual model of one particle among a conglomeration of particles sheared under uniform shear and normal stresses τ and P_p (Fig.1(a)) in a fluid with a viscosity of η_f . For a shear rate of $\dot{\gamma}$, the average streamwise velocity between particles in layer $i + 1$ and i is $\approx d\dot{\gamma}$. During the *macroscopic* timescale $1/\dot{\gamma}$ our particle in the layer $i + 1$ (A in Fig.1(b)) moves streamwise by $\approx d$. It also moves down and up in the y -direction, a distance “ δ ”, e.g., Fig.1(b) to 1(d). When $\dot{\gamma}$ is relatively low and P_p high, a particle in layer $i + 1$ remains in contact with particles in layer i and it travels a maximum vertical distance (2δ , Fig.1(c)), generating a dense packing $\phi_c = V_p/V_s$, where V_p is the volume of the three particles (due to the volume of particles is constant, we assume $V_p = 1$), $V_s = \sqrt{3}d^2/4$ is the volume occupied by the three particles, represented as the equilateral triangle ($\triangle ABC$). In contrast, when P_p is low and $\dot{\gamma}$ is high, we expect our particle A to break contact with particle B in layer i before connecting with C , reducing the value for 2δ , e.g., Fig.1(b) to 1(d). Hence, a general expression of the solid fraction is expressed as $V_s = d(d - \delta)/2$, and, the solid fraction is given by:

$$\frac{\phi}{\phi_c} = \frac{\sqrt{3}d^2/4}{d(d-\delta)/2} = \frac{\sqrt{3}}{2(1-\frac{\delta}{d})} = f(d/\delta). \quad (1)$$

This shows that ϕ/ϕ_c is dependent on the internal structure identified by δ/d . This suggests the importance of displacement lengthscales in capturing the evolution of the solid fraction.

As shown in Fig.1(e), we analyze the forces acting on particles. In the quasi-steady state, momentum transport can be described as follows: $F_N = \pi d^2 P_p / 4$ in the vertical direction and $F_\tau = \pi d^2 \tau / 4$ in the shearing direction for different layers. In a densely packed suspension where $\phi \rightarrow \phi_c$, there is a small space for a particle to fluctuate. The fluid flow inside the dense suspension is a result of fluid-granular interaction, distinct from fluid flow through a porous medium, where the velocity can be considerably higher than that of the solid phase. In this condition, the relative velocity between the fluid and solid phases is small, resulting in a low Reynolds number. Consequently, we assume that the hydrodynamic force of the particle inside a dense suspension is the Stokes force. We aspire to achieve a comprehensive description of the hydrodynamic forces under various conditions. However, it's a significant challenge that requires a substantial breakthrough in fluid mechanics. While the Stokes force may not encompass all aspects of the dense suspension system's behavior, it can still enhance our understanding of the interplay between fluid dynamics and inertial effects in this complex system.

Particle A is subjected to Stokes drag in both vertical ($F_{d_n} = 3\pi\eta_f v_n^* d$) and horizontal directions ($F_{d_t} = 3\pi\eta_f v_t^* d$), where v_n^* and v_t^* denote the magnitude of relative velocities between the fluid and the particle in the vertical and horizontal directions, respectively. The frictional force between particles A and C is $F_\mu = \mu_p F_c$, where F_c is the contact force in the normal direction and μ_p is the particle friction coefficient. Hence, we can obtain

$$F_c \cos \theta - \mu_p F_c \sin \theta + 3\pi\eta_f v_n^* d = F_N = \pi d^2 P_p / 4, \quad (2a)$$

$$F_c \sin \theta + \mu_p F_c \cos \theta + 3\pi\eta_f v_t^* d = F_\tau = \pi d^2 \tau / 4. \quad (2b)$$

The apparent frictional coefficient $\mu = \tau / P_p$ can be expressed as

$$\mu = \frac{12\eta_f v_t^*}{P_p d} + \frac{(1 - \frac{12\eta_f v_n^*}{P_p d})(\tan \theta + \mu_p)}{1 - \mu_p \tan \theta}. \quad (3)$$

We define the characteristic velocity v_n^* as the final velocity of a particle settling in fluid over a distance of a particle diameter d under P_p , and t_μ is the traveling time. We can obtain $v_n^* = \frac{P_p d}{12\eta_f} (1 - h(I/J))$, where $h(I/J) = e^{-216J^2/I^2 - 1 - W(-e^{-216J^2/I^2 - 1})}$, where $W(\cdot)$ is the Lambert-W function. We can calculate the magnitude of horizontal velocity $v_t^* = \tan \theta_v v_n^*$, where θ_v is the angle generated by the relative velocity to the vertical direction. Refer to Appendices A and B for detailed calculations. Due to

$$\tan \theta = \frac{\sqrt{d^2 - (d - \delta)^2}}{d - \delta} = \sqrt{\left(\frac{1}{1 - \delta/d}\right)^2 - 1}, \quad (4)$$

we can express the apparent frictional coefficient as

$$\mu = (1 - e^{-h(I/J)}) \tan \theta_v + \frac{e^{-h(I/J)}(\tan \theta + \mu_p)}{1 - \mu_p \tan \theta} = g(\mu_p, d/\delta, I/J, \tan \theta_v). \quad (5)$$

We assume that the stokes drag force is parallel to the contact direction, implying $\tan \theta_v = \tan \theta$. Consequently, the alteration of μ is contingent on three key components: (1) Particle scale friction μ_p , which directly impacts momentum exchange between distinct layers; (2) The solid skeleton structure represented by the ratio d/δ (or ϕ/ϕ_c , due to $d/\delta = f^{-1}(\phi/\phi_c)$, as shown in Eqns.3, and 4.). As depicted in Fig.1(b), the angle $\theta(d/\delta)$ generated by the solid structure plays a crucial role in the momentum exchanges; (3) Fluid-structure interaction denoted by I/J in various systems. Given a specific particle distribution and particle friction, i.e., keeping d/δ and μ_p constant, variations in fluid and solid properties result in different momentum exchanges. As I/J increases from 0 to infinity, signifying a decrease in the role of fluid effect from maximum to minimum, Eqn.5 undergoes the following variation:

$$\lim_{I/J \rightarrow 0} \mu = \frac{\tan \theta + \mu_p}{1 - \mu_p \tan \theta}, \quad (6a)$$

$$\lim_{I/J \rightarrow +\infty} \mu = (1 - e^{-1}) \tan \theta_v + \frac{e^{-1}(\tan \theta + \mu_p)}{1 - \mu_p \tan \theta}. \quad (6b)$$

This implies that when fluid-structure interaction reaches its maximum or becomes negligible, μ becomes independent of I/J . This observation aids us in establishing a unifying expression for the rheological model, a discussion of which follows. Although the dynamic process of granular material during shearing is much more complex, this simplified conceptual model still provides physical insights into the behavior of granular materials and allows us to use a length-scale ratio to build a rheological model.

We next derive a quantitative expression for what we might call a “microscopic” lengthscale l_μ in the context of representative forces on a particle. This is analogous to δ , as conceptualized above, though the direction of l_μ is arbitrary. As previously noted by Cassar *et al.* [9], when inertia, drag force, and P_p are all significant, we may write:

$$(\pi/6)\rho_s d^3 \frac{dv}{dt} = (\pi/4)P_p d^2 - F_d, \quad (7)$$

for the response of a particle to a contact force on one side (scaling with P_p), in a viscous fluid, with no particle contact on the other side (e.g., due to a “hole” in the contact network). As discussed earlier, we assume that a particle within the dense suspension is subjected to a Stokes drag, expressed as $F_d = 3\pi\eta_f v d$. We integrate Eqn.7 to find the speed $v(t)$ and the distance traveled $l(t)$ for a particle released from rest:

$$v(t) = \int_{t'=0}^t [dv(t')/dt'] dt' = v_f (1 - e^{-t/t_0}), \quad (8a)$$

$$l(t) = \int_{t'=0}^t v(t') dt' = v_f t - v_f t_0 (1 - e^{-t/t_0}). \quad (8b)$$

$v_f = P_p d / (12\eta_f)$ is analogous to a settling velocity, and $t_0 = (\rho_s d^2) / 18\eta_f$, to a settling timescale. We use Eqn.8b to derive either a “microscopic timescale” t_μ for a particle to travel distance $l = d$ or a “microscopic lengthscale” l_μ a particle can travel in time $\mathcal{T} = 1/\dot{\gamma}$:

$$t_\mu = \frac{12\eta_f}{P_p} + \frac{\rho_s d^2}{18\eta_f} \left[1 + W \left(-e^{-216\eta_f^2 / (\rho_s d^2 P_p) - 1} \right) \right] \quad (9a)$$

$$l_\mu = \frac{P_p d}{12\eta_f \dot{\gamma}} - \frac{P_p d^3 \rho_s}{216\eta_f^2} (1 - e^{-18\eta_f / (\rho_s d^2 \dot{\gamma})}) \quad (9b)$$

where $W(\cdot)$ is the Lambert-W function. See Appendix.A for calculation details.

With these, we find a dimensionless timescale ratio by dividing t_μ by the (“macroscopic”) timescale $\mathcal{T} = 1/\dot{\gamma}$:

$$\Gamma = \frac{t_\mu}{\mathcal{T}} = 12J + \frac{\text{St}}{18} \left[1 + W \left(-e^{-216J/\text{St} - 1} \right) \right], \quad (10)$$

and a dimensionless lengthscale ratio by dividing a (“macroscopic”) lengthscale $\mathcal{L} = d$ by l_μ :

$$G = \frac{\mathcal{L}}{l_\mu} = \frac{216J^2}{18J - I^2(1 - e^{-18/\text{St}})} = 12(J + \lambda_{\text{St}} I^2); \quad (11a)$$

$$\lambda_{\text{St}} = \frac{(1 - e^{-18/\text{St}})}{18 - \text{St} \times (1 - e^{-18/\text{St}})}. \quad (11b)$$

III. RELATIONSHIPS AMONG RHEOLOGICAL VARIABLES: G AND Γ VS. I, J, K

The relationship between G and K is apparent. The expressions are nearly identical, except for the need for a fitting parameter λ . In the case of K , $\lambda = \lambda_o$, is a constant that needs to be determined based on the system e.g. ,

material properties and boundary conditions. In the case of G , $\lambda = \lambda(St)$, a variable that adjusts naturally to the system as it transitions between J and I and thus has one fewer fitting parameter.

When the inertial force is dominant over the hydrodynamic force ($St \gg 1$), we find

$$\lim_{St/J \rightarrow +\infty} \Gamma = \frac{2I}{\sqrt{3}} \left[6\sqrt{3}\sqrt{\frac{J}{St}} + \frac{\sqrt{3}}{36}\sqrt{\frac{St}{J}} + \frac{\sqrt{3}}{36}\sqrt{\frac{St}{J}}W\left(-e^{-\frac{216J}{St}-1}\right) \right] \quad (12)$$

$$= \frac{2I}{\sqrt{3}} \times \frac{\sqrt{3}}{36}\sqrt{\frac{St}{J}} \left[1 + W\left(-e^{-\frac{216J}{St}-1}\right) \right]. \quad (13)$$

Due to $\lim_{St/J \rightarrow +\infty} W(-e^{-216J/St-1}) = W(-1/e) = -1$, we apply the Taylor expansion to $W(x)$ as $x \rightarrow -1/e$ so that $W(x) = -1 + \sqrt{2(1+ex)} - \mathcal{O}(1+ex)$ [cf. 19]. Thus,

$$\lim_{St/J \rightarrow +\infty} \Gamma = \frac{2I}{\sqrt{3}} \frac{\sqrt{3}}{36}\sqrt{\frac{St}{J}} \left[\sqrt{2(1-e^{-\frac{216J}{St}})} - \mathcal{O}\left(1-e^{-\frac{216J}{St}}\right) \right] \quad (14)$$

$$= \frac{2I}{\sqrt{3}} \frac{\sqrt{3}}{36}\sqrt{\frac{St}{J}} \left[\sqrt{\frac{2 \times 216J}{St}} - \mathcal{O}\left(\frac{216J}{St}\right) \right] \quad (15)$$

$$= \frac{2I}{\sqrt{3}} \left[1 - \mathcal{O}\left(6\sqrt{3}\sqrt{\frac{J}{St}}\right) \right] \approx 2I/\sqrt{3}. \quad (16)$$

For the dimensionless length scale

$$\lim_{St \rightarrow +\infty} G = \frac{216J^2}{18J - I^2(1 - e^{-18J/I^2})} = \frac{216J}{18 - St(1 - e^{-18/St})} \quad (17)$$

$$= \frac{216J}{18 - St\left(\frac{18}{St} - \frac{9 \times 18}{St^2} + \frac{3 \times 18^2}{St^3} - \mathcal{O}\left(\frac{1}{St^4}\right)\right)} = \frac{9 \times 18}{St} - \frac{3 \times 18^2}{St^2} - \mathcal{O}\left(\frac{1}{St^3}\right) \quad (18)$$

$$= \frac{4J \times St}{3 - \frac{18}{St} - \mathcal{O}\left(\frac{1}{St^2}\right)} \approx 4I^2/3. \quad (19)$$

In other words, in the limit of large St , Γ recover to the inertial number (time ratio in inertial regime), while, G transform into a square of the inertial number. Albeit subtle, this difference in the exponent of I is critical as will be shown later on. When the hydrodynamic force is dominant, one obtains

$$\lim_{St/J \rightarrow 0} \Gamma = 12J \left[1 + \frac{St}{216J} + \frac{St}{216J}W\left(-e^{-\frac{216J}{St}-1}\right) \right] = 12J, \quad (20)$$

where $W(0) = 0$. The dimensionless length scale is

$$\lim_{St \rightarrow 0} G = \frac{216J^2}{18 - St(1 - e^{-18/St})} \approx \frac{216J^2}{18J} = 12J. \quad (21)$$

In the limit of small St , both Γ and G have the same limiting value: $\approx 12J$. As granular materials change from a free fall regime to a viscous regime, the generalized time scale ratio Γ naturally transform from the inertial number I (time scale ratio in inertial regime defined in work of [5]) to the viscous number J (time scale ratio in inertial regime defined in the work of [9]), when ignoring the constant parameter. However, G naturally recover relations from the inertial number based $(\phi_c/\phi - 1) \propto I \propto \sqrt{G}$ in the inertial regime to the viscous number based $(\phi_c/\phi - 1) \propto J^{0.5} \propto \sqrt{G}$ in the viscous regime. This is another argument to justify the choice of G as the key dimensionless parameter.

We summarize some details in Table I including their relationships with I , J , and K . The first column details conditions for comparisons (i.e., high and low St and in between). The second column summarizes previously published relationships for I , J , and K . In the third and fourth columns, we see that in the high- St limits, Γ and $\sqrt{G}$ scale identically with I . However, at the lowest values of St , Γ scales with J , while $\sqrt{G}$ scales with $\sqrt{J}$ suggesting the need for an additional fit parameter may be needed for Γ to fully reconcile with the data. In fact, the form of G is similar to K with an important difference which will be highlighted later on.

TABLE I. Near- ϕ_c dynamics and rheological parameters for viscous to inertial behaviors ($St \approx 0 \rightarrow \infty$)

Flow regimes	Published $\Delta\phi(P_p, \dot{\gamma})$	$\Gamma = t_f/\mathcal{T}$	$G = \mathcal{L}/l$
Viscous ($St \rightarrow 0$)	$\frac{\phi_c}{\phi} - 1 \propto J^{0.5}$ [9, 10]	$\lim_{St/J \rightarrow 0} \Gamma = 12J$	$\lim_{St \rightarrow 0} G = 12J$
Inertial ($St \rightarrow \infty$)	$\frac{\phi_c}{\phi} - 1 \propto I$ [5–7]	$\lim_{St/J \rightarrow \infty} \Gamma = \frac{2}{\sqrt{3}}I$	$\lim_{St \rightarrow \infty} G = \frac{4}{3}I^2$
Visco-inertial ($St \approx 0 \rightarrow \infty$)	$\phi_c - \phi \propto \sqrt{K}$ [11, 12] $\frac{\phi_c}{\phi} - 1 \propto \sqrt{K}$ [15]	$\frac{\phi_c}{\phi} - 1 \propto \Gamma^\alpha$ $\alpha = 0.5 \rightarrow 1$	$\frac{\phi_c}{\phi} - 1 \propto G^{0.5}$

^a $K = \lambda_i I^2 + J$; where λ_i is a fit parameter that can have two values, one each for $i = \phi$ or μ [12];

^b Γ and G are in Eqns.10-11a. α is a fit parameter for Γ . No fit parameters are needed for G .

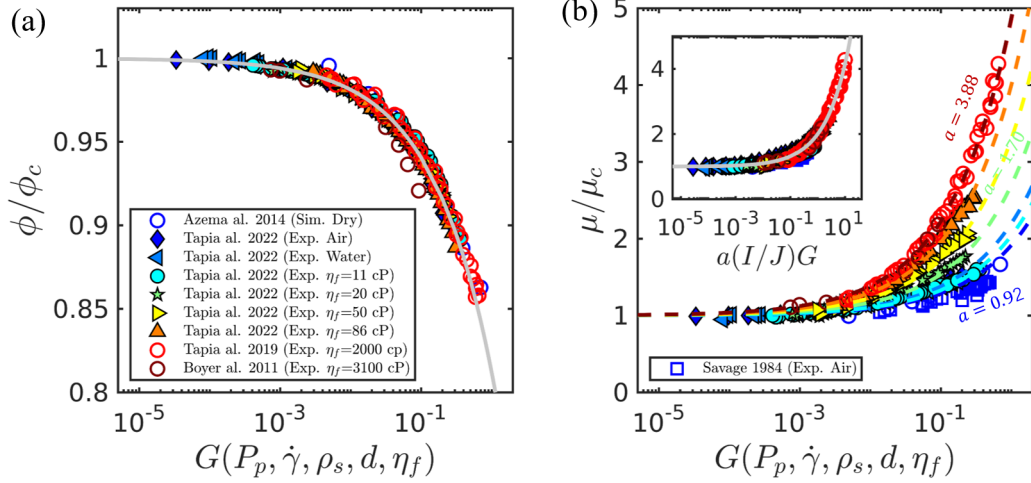


FIG. 2. Normalized solid fraction (a) and apparent frictional coefficient (b) plotted vs. data from G for data from Refs.[10, 12, 17, 18, 20]. The solid line in (a) are from Eqn.22b, using $b = 0.188$, and the solid line inset of (b) are from Eqns.22a and 23 with $a_i = 1.0$, $a_u = 4.0$, and $c = 41$.

IV. RESULTS

Based on the theoretical analysis in Sec.III, we plot ϕ/ϕ_c and μ/μ_c vs. G using previously published experimental and computational data [10, 12, 17, 18, 20] in Figs.2(a) and (b). And the rheological parameters can be expressed by previously proposed relationships [12]:

$$\mu = \mu_c(1 + aG^{0.5}), \text{ and} \quad (22a)$$

$$\phi = \phi_c(1 - bG^{0.5}). \quad (22b)$$

We determined that $b \approx 0.188$ remains a constant across all systems, as illustrated in Fig.3(a), aligning with our analytical findings from the conceptual model. $\phi/\phi_c = \Phi(G)$ is consistently represented by a single function of a structure-based parameter, the lengthscale ratio G . However, μ_c , ϕ_c , a are system-dependent parameters. Both ϕ_c and μ_c represent critical values in the jamming transitions, and recent work has shown that μ_c account for the effect of the particle-scale friction μ_p [21]. Parameter a varies, starting from a constant value in dry conditions ($a_i \approx 1.0$) and increasing to a larger constant value in highly viscous conditions ($a_v \approx 4.0$), as depicted in Fig.2(b). The fitting parameter ϕ_c , μ_c , a , and b for different systems are presented in Table.II (refer to Appendix.C for details).

As discussed in Sec.II, the introduction of the rescaled confining pressure I/J is necessary to account for the effect of fluid-solid interaction on μ . Consequently, we plot the last fitting parameter a in Eqn.22a against I/J and fit their relationship with

$$a = a_v - \frac{a_v - a_i}{1 + c/(I/J)}, \quad (23)$$

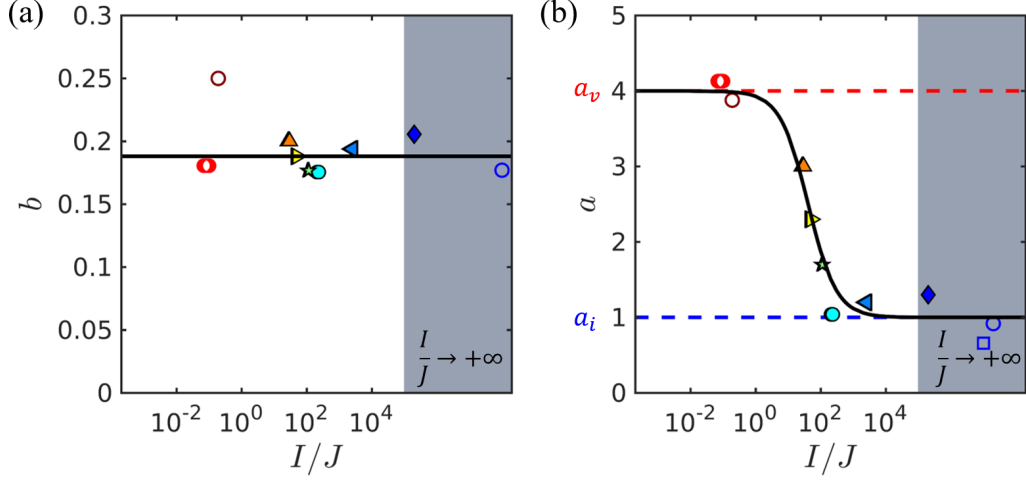


FIG. 3. Evolution of fitting parameter (a) b and (b) a vs. I/J for data from Refs.[10, 12, 17, 18]. The solid black line in (b) is from Eqn.23 with $a_i = 1$, $a_v = 4$, and $c = 41$.

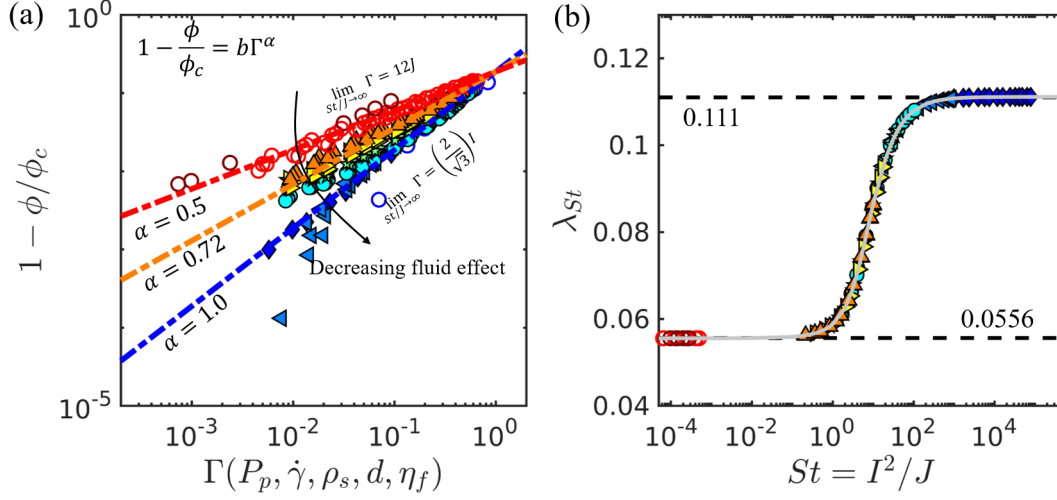


FIG. 4. (a) $1 - \phi/\phi_c$ vs. generalized time scale ratio Γ , where Γ transforms from an inertial number $(2/\sqrt{3})I$ to a viscous number $12J$ and $\alpha = 0.5 \rightarrow 1$ as the fluid-granular system varies from a dry condition to a very viscous condition, the dash line is plotted by $1 - \phi/\phi_c = 0.188\Gamma^\alpha$ with $\alpha = 0.5, 0.72$, and 1 . (b) shows λ vs. St , the solid line is plotted by Eqn.11b.

where $a_v \approx 4.0$ is a constant upper limit at extreme viscous regime [derived from the data of 10, 20], $a_i \approx 1.0$ is a lower limit value in the dry condition [derived from the data of 12, 17, 18]. The parameter $c = 41$ is a fitted constant parameter. Eqn.23 demonstrates that for a fixed G (in relation to a specific ϕ/ϕ_c , assuming the similar internal solid structure) within a P-imposed system with constant particle friction μ_p , increasing the fluid viscosity from 0 to a high value (resulting in a decrease of I/J from infinity to 0) leads to a progression in the influence of the fluid, transitioning from negligible to a peak. This aligns with the theoretical analysis presented in our conceptual model.

As depicted in Fig.4(a), we observe scaling relations between ϕ/ϕ_c and the general form of the time scale ratio Γ^α as the system transforms from an inertial to viscous regime, the exponent α undergoes a transition from 1.0 to 0.5. We find Γ impractical and do not consider it further herein. In contrast, G does not require such a compromise. This study confirms recent discoveries showcasing distinct scaling laws for ϕ and μ [cf. 12]. We attribute this discrepancy to the variations in fluid-structure interaction as a function of I/J , as elucidated through theoretical analysis using our conceptual model (see Eqn.5). Furthermore, we propose a universal constitutive law among different systems. The Root Mean Square Error for predicting μ using the data from [12] decreases from $2.2\text{e-}3$ to $4.3\text{e-}4$ when utilizing Eqn.22a in comparison to their established relations.

Before we conclude, we briefly revisit the question of appropriate system scales. As we recall, I and J were derived

using ratios of macro- to micro- timescales for inertial and viscous systems, respectively, while the intentionally-designed cross-rheology parameter K was initially proposed based on a linear superposition of stress scales [see 11] (similar to $I_m = \sqrt{\alpha_1 I^2 + \alpha_2 J}$ [cf. 15]). Starting from these superpositions ends with forms that require additional fit parameters (λ_o or λ_μ and λ_ϕ for K or α_1/α_2 for I_m). Our new timescale ratio Γ provides a generalized form of J and I from the viscous regime to the inertial regime. However, it still requires a fit parameter α , that varies from 1/2 to 1 as the system transitions from viscous to inertial (Table I). While undesirable, this is still intuitive: it forces a transition between a $\dot{\gamma}$ - rheology in the viscous limit to a $\dot{\gamma}^2$ - rheology in the inertial limit, as shown previously to hold for these systems [10, 12]. Our new lengthscale ratio G does not require a fitting parameter and, at the same time, contains the functional form of $\lambda(St)$ necessary to transform G into K (Eqn.11b). The lengthscale can effectively capture the evolution of ϕ/ϕ_c of the granular assembly. Furthermore, the lengthscale ratio combined with I/J can better capture the evolution of μ/μ_c . The importance of lengthscales is highlighted by the proposed conceptual model. We elaborate on additional relationships between the lengthscale ratio and macroscopic properties ϕ and μ through this conceptual model along with force analysis and geometric operation. The proposed constitutive relations (Eqns.22a, 22b, and 23) provide a precise reflection and correlation with the physical influencing factors for each rheological parameters. For example, $\phi/\phi_c = f(d/\delta) = \Phi(G)$, and $\mu = g(\mu_p, d/\delta, I/J) = \mathcal{G}(\mu_c, G, I/J)$, where G is analogous to d/δ and μ_c represents the effect of μ_p [21].

In addition to the rheological studies reviewed here, studies of near-jammed (near ϕ_c) systems have illustrated that spatial structures play important roles during particle transport [e.g. 16, 22, 23]. For example, the dynamical processes of granular flows are strongly correlated to larger-scale correlated movements within a granular assembly. While the relationships between the structures and flows remain open questions, we have provided significant justifications to give renewed importance to the proposed lengthscale ratios over the existing alternatives.

V. CONCLUSIONS

To summarize, in this work, we considered the inertial response of a sphere subjected to contact pressure P_p and a linear drag force, and derived a particle-response micro-timescale $t_\mu = t(l = d)$ and equivalent micro-lengthscale $l_\mu = l(t = 1/\dot{\gamma})$, i.e., the time for a particle to travel by d and the distance for a particle to travel in time $1/\dot{\gamma}$, respectively, in response to this forcing. From these, we derived a timescale ratio and a lengthscale ratio: $\Gamma = t_\mu/(1/\dot{\gamma})$ and $G = d/l_\mu$. When considering these in the context of previously published data, we found that the lengthscale ratio (rather than a timescale ratio) is intrinsically related to the ϕ/ϕ_c and requires no additional fitting parameters in comparison with existing alternatives. The apparent frictional coefficient μ is influenced by three key components: particle-particle friction μ_p , solid structure, and the fluid-structure interaction. Previous research [21] has shown that μ_c is influenced by μ_p , and our study demonstrates that the solid structure is accurately characterized by the proposed G . Additionally, the effect of the fluid-structure interaction is universally captured by a function of I/J .

Based on these results along with recent work connecting particle properties with rheology parameters [21], we propose general constitutive relationships suitable for a wide range of dense, sheared granular flows across the viscous-inertial transition expressed using dimensionless scales G and I/J where the effects of viscous drag and inertia change under different confining pressures, fluid viscosities, and macroscopic deformations. These results provide insight toward expanding our understanding of the influence of different lengthscales in dense particle-fluid flows, providing a greater promise for formulating constitutive models for larger-scale physics-based flow than the ones we explored herein.

In our current study, the derivation of the lengthscale ratio relies on the assumption of Stokes' law. However, in cases where Stokes' law may not be applicable, a more precise expression for hydrodynamic effects or an additional force field is necessary to derive the lengthscale accurately. Nevertheless, we have presented a physically grounded conceptual model that can be extended to comprehend the dynamics of complex dense granular systems.

Appendix A: Derivation details for microscopic time t_μ

As defined by Cassar *et al.* [9], the microscopic characteristic time scale t_μ is the time it takes for a particle to travel for a distance of d . From Eqn.8b,

$$d = v_f t_\mu - v_f t_0 (1 - e^{-t_\mu/t_0}), \quad (A1)$$

where $v_f = P_p d / (12\eta_f)$, $t_0 = (\rho_s d^2) / 18\eta_f$. With some algebra we can obtain

$$\frac{1}{t_0} \left(t_\mu - t_0 - \frac{d}{v_f} \right) \exp \left[\frac{1}{t_0} \left(t_\mu - t_0 - \frac{d}{v_f} \right) \right] = -\exp \left[- \left(1 - \frac{d}{t_0 v_f} \right) \right]. \quad (A2)$$

Equation.A2 has the form of the Lambert W-function, and X can be calculated using the Lambert W-function W as $X = W(Y)$ [24]. Hence, the solution for t_μ is expressed as

$$t_\mu = \frac{d}{v_f} + t_0 + t_0 W \left(-e^{\frac{-d}{t_0 v_f} - 1} \right) \quad (\text{A3})$$

$$= \frac{12\eta_f}{P_p} + \frac{\rho_s d^2}{18\eta_f} \left[1 + W \left(-e^{-216\eta_f^2/(\rho_s d^2 P_p) - 1} \right) \right] \quad (\text{A4})$$

$$= \frac{12\eta_f}{P_p} + \frac{\rho_s d^2}{18\eta_f} \left[1 + W \left(-e^{-216J^2/I^2 - 1} \right) \right]. \quad (\text{A5})$$

Appendix B: Derivation details for characteristic velocity v_n^*

We can calculate the characteristic velocity $v_n^* = v_t^* / \tan \theta_v = v(t_\mu)$ in vertical direction with Eqn.8a,

$$v_n^* = \frac{P_p d}{12\eta_f} \left(1 - e^{-t_\mu \frac{18\eta_f}{\rho_s d^2}} \right) = \frac{P_p d}{12\eta_f} \left(1 - e^{-(216\eta_f^2/\rho_s P_p d^2 - 1 - W(-e^{-216J^2/I^2 - 1}))} \right) \quad (\text{B1})$$

$$= \frac{P_p d}{12\eta_f} \left(1 - e^{-216J^2/I^2 - 1 - W(-e^{-216J^2/I^2 - 1})} \right) \quad (\text{B2})$$

$$= \frac{P_p d}{12\eta_f} (1 - h(I/J)) \quad (\text{B3})$$

where

$$h(I/J) = e^{-216J^2/I^2 - 1 - W(-e^{-216J^2/I^2 - 1})}. \quad (\text{B4})$$

Hence, Eqn.5 is given by

$$\mu = \frac{12\eta_f v_t^*}{P_p d} + \frac{(1 - \frac{12\eta_f v_n^*}{P_p d})(\tan \theta + \mu_p)}{1 - \mu_p \tan \theta} \quad (\text{B5})$$

$$= \tan \theta_v (1 - h(I/J)) + \frac{h(I/J)(\tan \theta + \mu_p)}{1 - \mu_p \tan \theta}. \quad (\text{B6})$$

Due to $\tan \theta$ is a function of d/δ ,

$$\mu = g(\mu_p, d/\delta, I/J, \tan \theta_v) \quad (\text{B7})$$

Appendix C: Critical values

The critical solid fraction, apparent friction, and the best fit parameter a and b in Eqns.22a and 22b in different works are listed in Table.II.

TABLE II. Critical apparent friction and solid fraction.

	$\eta_f(\text{cp})$	ϕ_c	b	μ_c	a
Savage [17]	Air	-	-	0.414	0.66
Azéma and Radjaï [18]	Dry	0.578	0.178	0.382	0.92
Tapia <i>et al.</i> [12]	0.0183 (Air)	0.596	0.206	0.394	1.30
Tapia <i>et al.</i> [12]	1	0.586	0.194	0.411	1.20
Tapia <i>et al.</i> [12]	11	0.615	0.176	0.315	1.04
Tapia <i>et al.</i> [12]	20	0.617	0.177	0.315	1.70
Tapia <i>et al.</i> [12]	50	0.615	0.188	0.312	2.30
Tapia <i>et al.</i> [12]	86	0.615	0.200	0.311	3.02
Tapia <i>et al.</i> [20]	2000	0.584	0.181	0.372	4.13
Boyer <i>et al.</i> [10]	3000	0.584	0.250	0.30	3.88

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