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# Quasi-static magnetoconvection with a tilted magnetic field

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A numerical study of convection with stress-free boundary conditions in the presence of an imposed magnetic field that is tilted with respect to the direction of gravity is carried out in the limit of small magnetic Reynolds number. The dynamics are investigated over a range of Rayleigh number Ra and Chandrasekhar numbers up to  $Q = 2 \times 10^6$ , with the tilt angle of the imposed magnetic field vector fixed at  $45^{\circ}$  relative to vertical. For a fixed value of Q and increasing Ra, the convection dynamics can be broadly characterized by three primary flow regimes: (1) quasi-two-dimensional convection rolls near the onset of convection; (2) isolated convection columns aligned with the imposed magnetic field; and (3) unconstrained convection reminiscent of non-magnetic convection. The influence of varying Q and Ra on the various fields is analyzed. Heat and momentum transport, as characterized by the Nusselt and Reynolds numbers, are quantified and compared with the vertical field case. Ohmic dissipation dominates over viscous dissipation in all cases investigated. Various mean fields are investigated and their scaling behavior is analyzed. Provided Ra is sufficiently large, all investigated values of Q exhibit an inverse kinetic energy cascade that yields strong 'zonal' flows with an amplitude that scales as  $Q^{1/3}$ . Relaxation oscillations, as characterized by a quasi-periodic shift in the predominance of either the zonal or non-zonal component of the mean flow, occur when Ra and Q are sufficiently large.

## I. INTRODUCTION

Convection in the presence of externally imposed magnetic fields, or magnetoconvection (MC), is important in stars and planetary interiors [1]. Magnetic fields can lead to novel flow regimes relative to non-conducting fluids. Notably, imposed magnetic fields induce flow anisotropy. The resulting change in flow structure, however, is dependent on the direction and magnitude of the imposed magnetic field, implying that a rich variety of dynamics can be realized in MC. In the context of planets and stars, the magnetic field tends to be self-generated through dynamo action and is therefore spatially (and temporally) complex. It is therefore of interest to understand how field direction, in addition to field magnitude, influences the underlying convective motions.

The periodic plane layer geometry provides a particularly simple system in which to study MC. The linear theory of MC for the plane layer is well established and provides a useful starting point for understanding the resulting nonlinear dynamics [2]. The constant gravitational field is denoted by  $\mathbf{g} = -g\hat{\mathbf{z}}$  (where  $\hat{\mathbf{z}}$  is the unit vector pointing normal to the planar boundaries) and the uniform imposed magnetic field is  $\mathbf{B}_0$ . In the present work we focus solely on the limit in which the induced magnetic field is weak relative to the imposed field – known as the quasi-static limit [e.g. 3] – we therefore limit our present discussion to this case. When  $\mathbf{g}$  and  $\mathbf{B}_0$  are aligned, which we refer to as vertical MC (VMC), convection is stabilized and the horizontal scale of the most unstable eigenmodes decreases with increasing field strength. When  $\mathbf{B}_0$  is horizontal (HMC), the preferred mode consists of two-dimensional (2D) convection rolls with their axes aligned with  $\mathbf{B}_0$ . The general case of a tilted magnetic field (TMC) is essentially a mixture of these two previous cases in which the most unstable eigenmodes consist of two-dimensional rolls aligned with the horizontal component of the imposed magnetic field, but with a horizontal length scale that decreases with increasing magnetic field strength.

The nonlinear evolution of MC has been investigated both experimentally and numerically for a variety of magnetic field configurations. VMC dynamics has been studied for both the quasi-static limit, and for the case of arbitrarily large induced magnetic fields using numerical simulations [e.g. 4]. In the discussion here we focus on those results pertaining to the quasi-static limit. Given the preference for fluid to move parallel to the imposed magnetic field direction, the vertical field geometry tends to limit horizontal mixing when the field strength is sufficiently large [5]. In confined geometries, such as cylinders, distinct flows such as convective wall modes are also present [6–8]. Heat and momentum transport, while always weaker than non-magnetic Rayleigh-Bénard convection (RBC), both increase at a rate that is faster than RBC [5, 8]. For sufficiently strong buoyancy forcing, VMC data appears to approach the corresponding RBC data [5, 8, 9], which suggests an expected weakening dynamical role of the imposed magnetic field.

Studies of HMC show that 2D rolls persist over a significant range of parameter space and can yield heat transport that is more efficient than RBC [10–13]. The flow eventually transitions to anisotropic 3D convection that can exhibit rich time dependent motions [14–16]. In confined geometries the lateral walls can have a significant influence on the dynamics due to the formation of Hartmann boundary layers [11]. The stabilizing role played by these boundary layers leads to an increase in the critical temperature gradient required to initiate convection.

In comparison to VMC and HMC, less is presently known about the nonlinear behavior of TMC. Previous 2D studies of TMC have found mean flows and traveling wave solutions that are generated by the broken symmetry



FIG. 1. Geometry used in the present study. The fluid layer has depth H and horizontal extent L.

associated with the tilted magnetic field [17]. This previous work suggests that the mean flows tend to travel in the direction of the tilt, though the dependence of the amplitude of this mean flow on the input parameters, namely the imposed field strength, has not been investigated in detail. In addition, to our knowledge the efficiency of heat and momentum transport has been unexplored for TMC.

Rotating convection (RC) has been studied in great detail due to its relevance for planetary and stellar applications [18]. Like MC, RC with a vertical rotation vector stabilizes the convection and yields anisotropic structures [2]. The inverse kinetic energy cascade is prevalent in RC provided that the influence of rotation is strong; the resulting flows are characterized by vortices that span the horizontal length of the system and are approximately invariant in the direction of the rotation axis [19–22]. When the two horizontal dimensions are unequal, the inverse cascade is instead manifested by the presence of large scale horizontally-directed jets [23]. Recent studies of RC with a tilted rotation axis find that both jets and vortices are present, depending on the tilt of the rotation axis (and likely also the relative importance of rotation and inertia) [24, 25].

In the present work we carry out a systematic investigation of three-dimensional TMC using direct numerical simulations. Flow regimes are delineated, and heat and momentum transport are quantified over a range of imposed field strengths and buoyancy forcing. When possible, a comparison is made with recent VMC simulations; we find that the primary difference between TMC and VMC is the presence of magnetically constrained turbulent states in the former. We find that TMC, like rotating convection, yields an inverse cascade of kinetic energy, characterized by energetic (relative to the convection) mean flows that tend to be dominated by a meandering, alternating jet structure. In some cases we also find that these jets can become unstable and give rise to relaxation oscillations.

# II. METHODS

We employ a periodic plane layer geometry, as shown in Fig. 1. The Cartesian coordinate system is denoted by  $\mathbf{x} = (x, y, z)$ . The fluid layer has depth H and horizontal extent L. The spatially uniform imposed magnetic field vector is given by

$$\mathbf{B}_0 = B_0 \widehat{\boldsymbol{\eta}},\tag{1}$$

where we define the vector

$$\widehat{\boldsymbol{\eta}} = \eta_1 \, \widehat{\mathbf{x}} + \eta_3 \, \widehat{\mathbf{z}},\tag{2}$$

with  $\eta_1 = \sin \theta$  and  $\eta_3 = \cos \theta$ . The angle  $\theta$  is measured relative to the vertical and fixed in the present study to  $\theta = 45^{\circ}$ .

The fluid is Oberbeck-Boussinesq with density  $\rho$ , kinematic viscosity  $\nu$ , thermal diffusivity  $\kappa$ , thermal expansion coefficient  $\alpha$ , magnetic diffusivity  $\lambda$ , and vacuum permeability  $\mu$ . The dimensional temperature difference between the bottom and top boundaries is denoted by  $\Delta T > 0$ . The governing equations are non-dimensionalized with length H, viscous diffusion timescale  $(H^2/\nu)$ , flow speed  $(\nu/H)$ , magnetic field  $B_0$ , pressure  $\rho(\nu/H)^2$ , and temperature  $\Delta T$ , to give

$$D_t \mathbf{u} = -\nabla P + \frac{Ra}{Pr} T \,\widehat{\mathbf{z}} + Q \,\widehat{\boldsymbol{\eta}} \cdot \nabla \mathbf{b} + \nabla^2 \mathbf{u},\tag{3}$$

$$0 = \widehat{\boldsymbol{\eta}} \cdot \nabla \mathbf{u} + \nabla^2 \mathbf{b},\tag{4}$$

$$D_t T = \frac{1}{Pr} \nabla^2 T,\tag{5}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{6}$$

$$\nabla \cdot \mathbf{b} = 0, \tag{7}$$

where t is the non-dimensional time, the material derivative is denoted by  $D_t(\cdot) = \partial_t(\cdot) + \mathbf{u} \cdot \nabla(\cdot)$ ,  $\mathbf{u} = (u, v, w)$  is the velocity field, **b** is the induced magnetic field, T is the temperature, and P is the reduced pressure.

The non-dimensional parameters appearing in the above equations are the Rayleigh number, the thermal Prandtl number, and the Chandrasekhar number defined by, respectively

$$Ra = \frac{\alpha g \Delta \mathcal{T} H^3}{\kappa \nu}, \quad Pr = \frac{\nu}{\kappa}, \quad Q = \frac{B_0^2 H^2}{\rho \nu \mu \lambda}.$$
(8)

In all cases presented here the thermal Prandtl number is fixed at Pr = 1, whereas Ra and Q are both varied.

The quasi-static magnetohydrodynamic equations (3)-(4) represent the physical scenario in which the induced magnetic field is asymptotically smaller than the imposed magnetic field. Denoting the magnetic Prandtl number as  $Pm = \nu/\lambda$ , the induced magnetic field is of size O(Pm), and the limit  $Pm \to 0$  is taken in the governing equations. A consequence of this approximation is that the Lorentz force becomes linear and the induced magnetic field adjusts instantaneously to the velocity field, as indicated by the lack of a time derivative in equation (4). The quasi-static approximation is well-satisfied in laboratory experiments that use liquid metals in which  $Pm = O(10^{-6})$ [e.g. 11, 15, 26–28]. For further details on this approximation we refer the reader to Refs. [3, 29, 30].

The mechanical boundary conditions are impenetrable and stress-free, constant temperature and electrically insulating, which can be written as

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$
 at  $z = 0, 1.$  (9)

The thermal boundary conditions are constant temperature

$$T = 1$$
 at  $z = 0$ ,  $T = 0$  at  $z = 1$ . (10)

Electrically insulating electromagnetic boundary conditions are used such that the current density  $\mathbf{j} = \nabla \times \mathbf{b}$  is zero within the insulator. With these boundary conditions the magnetic field at z = 0 and z = 1 must be matched to a potential field; the numerical details of this matching procedure can be found in Ref. [31].

All flow variables are assumed to be periodic in the horizontal dimensions. Both the velocity and magnetic fields are represented in terms of poloidal and toroidal scalars such that the solenoidal conditions are satisfied exactly. The resulting equations are solved numerically with a pseudo-spectral algorithm using Fourier series in the horizontal directions and Chebyshev polynomials in the vertical direction. A third-order implicit-explicit Runge-Kutta time stepping method is used. Further details of the code are given in Ref. [32].

An important parameter for the simulations is the aspect ratio of the computational domain, defined by

$$\Gamma \equiv \frac{L}{H}.$$
(11)

For the simulations we scale the horizontal length in integer multiples (n) of the (dimensionless) critical horizontal wavelength,  $\lambda_c$ , such that  $L = n\lambda_c H$ . The aspect ratio then becomes

$$\Gamma = n\lambda_c. \tag{12}$$

$$Ra_m = \frac{\left(\pi^2 + k_y^2\right)}{k_y^2} \left[ \left(\pi^2 + k_y^2\right)^2 + \pi^2 \eta_3^2 Q \right].$$
 (13)

Minimizing the above expression gives

field (i.e.  $k_x = 0$ ) with marginal Rayleigh number [2]

$$2k_c^{\ 6} + 3\pi^2 k_c^{\ 4} - \pi^4 \eta_3^2 Q - \pi^6 = 0, \tag{14}$$

where  $k_c = 2\pi/\lambda_c$  is the critical horizontal wavenumber. For the field strengths used in the present study,  $Q = (2 \times 10^3, 2 \times 10^5, 2 \times 10^6)$ , we find  $k_c \approx (5.6842, 12.8343, 18.9823)$  with corresponding critical Rayleigh numbers  $Ra_c \approx (1.5207 \times 10^4, 1.0784 \times 10^6, 1.0281 \times 10^7)$ . We found that n = 10 is sufficient for convergence of bulk quantities such as the heat transport. Thus, this value is used unless stated otherwise. For the imposed field strengths used in the present study, the aspect ratios corresponding to n = 10 are given by  $\Gamma \approx (11.1, 4.9, 3.3)$ . In section III.4.1 we report on the effects of varying the aspect ratio.

For future reference it is also helpful to investigate the asymptotic limit  $Q \to \infty$ . In this limit the critical wavenumber and the critical Rayleigh number are given by, respectively,

$$k_c^{(a)} \to \left(\frac{1}{2}\pi^4 \eta_3^2 Q\right)^{1/6}, \qquad Ra_c^{(a)} \to \pi^2 \eta_3^2 Q. \qquad (Q \to \infty)$$
 (15)

#### II.1. Definitions

Various forms of averaging are used to present the results. For some generic scalar quantity,  $f(\mathbf{x}, t)$ , horizontal averages are defined by

$$\overline{f}(z,t) = \frac{1}{\Gamma^2} \int_{(x,y)} f(\mathbf{x},t) \, dx \, dy.$$
(16)

Other averages will be denoted with an overline and a superscript to denote the dimension in which the quantity is averaged. For instance, the zonal average is defined by

$$\overline{f}^{y}(x,z,t) = \frac{1}{\Gamma} \int f(\mathbf{x},t) \, dy.$$
(17)

Averages in time and in the direction of the magnetic field are similarly defined. Volume and time averaged quantities are denoted with angled brackets,

$$\langle f \rangle = \frac{1}{t' \Gamma^2} \int_{(t', \mathbf{x})} f(\mathbf{x}, t) \, d\mathbf{x} \, dt, \tag{18}$$

where t' is the time interval.

The Reynolds number is defined by  $Re = UH/\nu$ , where U is a characteristic speed. We find it useful to compute Reynolds numbers that utilize both the three-dimensional velocity field (denoted by Re) and only the vertical component of the velocity (denoted by  $Re_z$ ). In the non-dimensionalization used here these Reynolds numbers become

$$Re = \sqrt{\langle u^2 + v^2 + w^2 \rangle}, \qquad Re_z = \sqrt{\langle w^2 \rangle}.$$
 (19)

The Nusselt number quantifies the non-dimensional global heat transport across the layer and is defined as

$$Nu = 1 + Pr\langle wT' \rangle, \tag{20}$$

where  $T'(x, y, z, t) = T(x, y, z, t) - \overline{T}(z, t)$  is the fluctuating temperature.



FIG. 2. Parametric overview of the simulations, as characterized by the ratio  $Ha/Re_z$ , where  $Ha = \sqrt{Q}$  is the Hartmann number and  $Re_z$  is the Reynolds number based on the rms of the vertical component of the velocity field. The critical Rayleigh number is denoted by  $Ra_c$ . The ratio  $Ha/Re_z$  represents the relative size of the Lorentz force and inertia: cases with  $Ha/Re_z \gtrsim O(1)$ are considered magnetically constrained.

# III. RESULTS

# III.1. Parametric overview and flow regimes

We investigate imposed magnetic field strengths of  $Q = (2 \times 10^3, 2 \times 10^5, 2 \times 10^6)$ , which were chosen so that the critical Rayleigh number,  $Ra_c$ , for each field strength corresponds with those used in recent VMC simulations [5]. We explore Rayleigh numbers up to  $Ra \approx (132Ra_c, 28Ra_c, 7Ra_c)$  for each of the three values of Q. Data for the simulations is summarized in Table I. While spatial resolution requirements are significant for many of the simulated flows, we find that the primary limiting factor for the computations done at larger values of Q and Ra is the presence of slowly evolving relaxation oscillations that require extremely long computation times to obtain converged statistics.

A useful measure for characterizing the relative importance of the imposed magnetic field is the ratio,  $Ha/Re_z$ , where the Hartmann number is defined as  $Ha = \sqrt{Q}$ . The ratio  $Ha/Re_z$  represents the relative influence of the Lorentz force to inertial forces in the quasi-static limit [e.g. 3, 29]. By restricting the Reynolds number to only include the vertical component of the velocity field, we are attempting to better characterize the relative influence of the magnetic field on the small scale convection. As discussed later, mean flows develop that have magnitudes significantly larger than the vertical component of the velocity field. Fig. 2 shows this ratio for all of the simulations. Cases in which  $Ha/Re_z \gg 1$  are considered magnetically constrained in the sense that the Lorentz force enters the leading order force balance [cf. 5]. For values of  $Ha/Re_z \leq 1$ , inertia plays a leading order dynamical role and the resulting motions are only weakly influenced by the imposed magnetic field. In the present study we only find cases with  $Ha/Re_z \leq 1$  for the smallest field strength of  $Q = 2 \times 10^3$ ; for these cases we find that the convective structures are no longer aligned with the (tilted) magnetic field.

To illustrate the structure of the flow field as both Q and Ra are varied (and therefore also  $Ha/Re_z$ ) we show volumetric renderings of the fluctuating temperature in Fig. 3 for each of the three values of Q and three particular values of Ra. As predicted by linear theory, for  $Ra \approx Ra_c$ , we observe anisotropic convective rolls that are predominantly aligned with the x-component of the imposed magnetic field – these structures are evident in Fig. 3(a) and to a lesser degree in panels (d) and (g). As the Rayleigh number is increased we find that the rolls develop a large scale,  $k_x = 1$ , modulation; with our aspect ratio this corresponds to a wavelength of  $\lambda = 10\lambda_c$ . This modulation interacts nonlinearly with the convective rolls and leads to the formation of a large scale mean flow that is discussed in more detail below. Further increases in Ra lead to the development of convective columns that are aligned with the imposed magnetic field. These structures are particularly evident in the  $Q = 2 \times 10^5$  and  $Q = 2 \times 10^6$  cases, e.g. Figures 3(e,f,h,i). These cases also show that the convective structures are elongated in the x-direction. For  $Q = 2 \times 10^3$  the tilt of the convective structures is less noticeable; in Fig. 3(b) some tilt is observable, though for sufficiently large Rayleigh number the tilt is no longer obvious (Fig. 3(c)). Despite the lack of constraint in simulations with  $Ha/Re_z \lesssim 1$ , we find that the magnetic field still plays an important dissipative role, as discussed in the next section.

Q	Ra	Nu	Re	$Re_z$	$\Delta t$	$N_x \times N_y \times N_z$
$2\ \times 10^3$	$1.53~{\times}10^4$	$1.012\pm0.000$	$0.822\pm0.000$	$0.719\pm0.000$	$10^{-3}$	$96\times96\times48$
$2\ \times 10^3$	$2 \times 10^4$	$1.30\pm0.03$	$5.05\pm0.19$	$4.16\pm0.18$	$10^{-3}$	$96\times144\times48$
$2\ \times 10^3$	$2.5 \times 10^4$	$1.54\pm0.04$	$8.29\pm0.24$	$6.61\pm0.21$	$10^{-3}$	$144\times192\times48$
$2\ \times 10^3$	$4 \times 10^4$	$2.19\pm0.04$	$17.86\pm0.24$	$11.34\pm0.30$	$5\times 10^{-4}$	$144\times192\times48$
$2\ \times 10^3$	$6 \times 10^{4}$	$2.83\pm0.05$	$30.98\pm0.35$	$16.47\pm0.36$	$2\times 10^{-4}$	$192\times288\times72$
$2\ \times 10^3$	$1 \times 10^5$	$3.75\pm0.05$	$54.80\pm0.37$	$25.12\pm0.43$	$1\times 10^{-4}$	$288\times288\times144$
$2 \times 10^3$	$2 \times 10^5$	$5.22\pm0.07$	$103.43 \pm 0.71$	$40.81\pm0.66$	$4 \times 10^{-5}$	$288\times 384\times 144$
$2 \times 10^3$	$4 \times 10^{5}$	$7.06\pm0.09$	$179.71 \pm 0.79$	$63.67\pm1.03$	$4 \times 10^{-5}$	$384\times576\times144$
$2 \times 10^3$	$6 \times 10^{5}$	$8.38\pm0.09$	$229.58 \pm 1.90$	$81.39\pm1.14$	$2 \times 10^{-5}$	$576\times576\times144$
$2 \times 10^3$	$1 \times 10^{6}$	$10.35\pm0.11$	$285.17 \pm 2.17$	$110.31\pm1.25$	$1 \times 10^{-5}$	$768\times768\times192$
$2 \times 10^{3}$	$2 \times 10^{6}$	$13.42\pm0.12$	$214.17 \pm 2.16$	$154.75\pm1.39$	$5 \times 10^{-6}$	$768\times768\times288$
$2 \times 10^5$	$1.1 \times 10^{6}$	$1.024\pm0.006$	$2.11\pm0.29$	$2.01\pm0.28$	$10^{-5}$	$96\times144\times96$
$2 \times 10^5$	$1.3 \times 10^{6}$	$1.17\pm0.02$	$8.13\pm0.52$	$7.16\pm0.55$	$10^{-5}$	$96\times144\times96$
$2 \times 10^{5}$	$1.5 \times 10^{6}$	$1.31\pm0.02$	$13.84 \pm 0.54$	$10.73\pm0.67$	$10^{-5}$	$144\times192\times96$
$2 \times 10^{5}$	$1.7 \times 10^{6}$	$1.46\pm0.03$	$20.69 \pm 0.62$	$14.33\pm0.79$	$10^{-5}$	$144\times192\times96$
$2 \times 10^{5}$	$2 \times 10^{6}$	$1.72\pm0.03$	$32.44 \pm 0.63$	$19.53\pm0.92$	$10^{-5}$	$192\times192\times96$
$2 \times 10^5$	$2.2 \times 10^{6}$	$1.87\pm0.04$	$41.26 \pm 0.64$	$22.65\pm0.96$	$10^{-5}$	$192\times192\times96$
$2 \times 10^5$	$2.5 \times 10^{6}$	$2.10\pm0.04$	$55.79 \pm 0.70$	$27.05 \pm 1.02$	$10^{-5}$	$192\times192\times96$
$2 \times 10^5$	$3 \times 10^{6}$	$2.45\pm0.04$	$81.53 \pm 0.76$	$34.23\pm1.11$	$10^{-5}$	$192\times288\times96$
$2 \times 10^5$	$4 \times 10^{6}$	$3.11\pm0.07$	$134.82 \pm 1.32$	$47.60 \pm 1.70$	$10^{-5}$	$288\times 384\times 96$
$2 \times 10^5$	$6 \times 10^{6}$	$4.24\pm0.13$	$240.66 \pm 1.88$	$70.43\pm3.00$	$10^{-5}$	$288\times 384\times 144$
$2 \times 10^5$	$8 \times 10^{6}$	$5.18\pm0.16$	$344.60 \pm 2.37$	$90.74 \pm 3.77$	$5 \times 10^{-6}$	$576 \times 576 \times 192$
$2 \times 10^5$	$1 \times 10^{7}$	$6.55\pm0.27$	$323.43 \pm 72.48$	$112.83\pm3.82$	$5 \times 10^{-6}$	$576 \times 576 \times 192$
$2 \times 10^5$	$1.5 \times 10^{7}$	$8.30\pm0.38$	$548.14\pm114.34$	$154.27\pm6.46$	$2 \times 10^{-6}$	$576 \times 576 \times 288$
$2 \times 10^{5}$	$3 \times 10^{7}$	$12.67\pm0.40$	$921.73\pm195.51$	$251.40\pm6.44$	$2 \times 10^{-6}$	$576\times768\times288$
$2 \times 10^{6}$	$1.04 \times 10^{7}$	$1.015\pm0.000$	$2.651 \pm 0.000$	$2.575\pm0.000$	$10^{-6}$	$96\times144\times144$
$2 \times 10^{6}$	$1.1 \times 10^{7}$	$1.07\pm0.01$	$6.75\pm0.37$	$6.54\pm0.35$	$10^{-6}$	$96\times144\times144$
$2 \times 10^{6}$	$1.3 \times 10^{7}$	$1.21\pm0.01$	$16.17\pm0.75$	$13.33\pm0.76$	$10^{-6}$	$144\times192\times144$
$2 \times 10^{6}$	$1.5 \times 10^{7}$	$1.37\pm0.02$	$29.24 \pm 0.91$	$20.16 \pm 1.20$	$10^{-6}$	$144\times192\times144$
$2 \times 10^{6}$	$2 \times 10^{7}$	$1.81\pm0.03$	$73.14 \pm 1.03$	$36.13 \pm 1.59$	$10^{-6}$	$144\times288\times144$
$2 \times 10^6$	$2.5 \times 10^{7}$	$2.25\pm0.04$	$125.88 \pm 1.29$	$51.13 \pm 2.28$	$10^{-6}$	$192\times 288\times 144$
$2 \times 10^{6}$	$3 \times 10^{7}$	$2.64\pm.06$	$184.21 \pm 1.91$	$63.98\pm3.06$	$10^{-6}$	$288\times 288\times 144$
$2 \times 10^{6}$	$4 \times 10^{7}$	$3.41\pm.07$	$305.61 \pm 2.88$	$89.59 \pm 4.62$	$10^{-6}$	$384\times 384\times 192$
$2 \times 10^{6}$	$5 \times 10^{7}$	$4.14\pm.10$	$422.72 \pm 5.20$	$113.64\pm5.24$	$10^{-6}$	$384\times576\times192$
$2 \times 10^{6}$	$7 \times 10^{7}$	$5.46\pm.14$	$663.70 \pm 3.09$	$156.39\pm8.06$	$10^{-6}$	$576 \times 576 \times 288$

TABLE I. Details of the simulations. Q is the Chandraskehar number, Ra is the Rayleigh number, Nu is the Nusselt number, Re is the Reynolds number,  $Re_z$  is the Reynolds number based only on the vertical component of the velocity,  $\Delta t$  is the timestep size and  $N_x \times N_y \times N_z$  denotes the physical space resolution. The thermal Prandtl number is fixed at Pr = 1 for all simulations.

#### III.2. Heat and momentum transport

Fig. 4 shows the Nusselt number, where both the new TMC data and the VMC data from Ref. [5] are shown for comparison. The slope of the Nu-Ra data increases with Q, similar to VMC. However, in VMC there is a steeper increase in Nu near the onset of convection and a subsequent reduction in the growth rate of Nu with increasing Ra. This difference between VMC and TMC is exhibited in the corresponding compensated value,  $Nu/(Ra/Ra_c)$ , as shown in Fig. 4(b). The  $Nu \sim Ra/Ra_c$  scaling behavior is independent of viscous dissipation, which might be expected in the limit  $Q \to \infty$  [e.g. 33]. However, we find that viscous dissipation becomes important in all cases just after the onset of convection, which suggests a possible explanation for why such a scaling is not observed. For sufficiently large values of  $Ra/Ra_c$ , both VMC and TMC show similar scaling behavior in Nu, at least for  $Q = 2 \times 10^3$  and  $Q = 2 \times 10^5$ .

Vertical profiles of the rms temperature fluctuation are shown for all values of Q in Fig. 5. For a fixed value of Q we find that the temperature fluctuation within the interior begins to decrease for sufficiently large  $Ra/Ra_c$  once robust thermal boundary layers form; the growth of T' with increasing  $Ra/Ra_c$  is then achieved within the vicinity of the thermal boundary layer. Comparing the profiles for different values of Q reveals that the rms temperature



(a)

(b)

(c)

(f)



(e)

(d)



FIG. 3. Volumetric renderings of the fluctuating temperature for various cases. Top row  $(Q = 2 \times 10^3)$ : (a)  $Ra/Ra_c = 1.3$ ; (b)  $Ra/Ra_c = 13.2$ ; (c)  $Ra/Ra_c = 132$ . Middle row  $(Q = 2 \times 10^5)$ : (d)  $Ra/Ra_c = 1.4$ ; (e)  $Ra/Ra_c = 7.4$ ; (f)  $Ra/Ra_c = 27.8$ . Bottom row  $(Q = 2 \times 10^6)$ : (g)  $Ra/Ra_c = 1.5$ ; (h)  $Ra/Ra_c = 3.9$ ; (i)  $Ra/Ra_c = 6.8$ . The orientation for all visualizations is shown in panel (a).

fluctuation is generally a decreasing function of Q, as predicted by asymptotic theory [34–36]. Although not shown, we find that for all values of Q the horizontally averaged temperature shows a trend toward a nearly isothermal bulk with well developed thermal boundary layers as Ra is increased. These results indicate that, like RBC and VMC [5], the heat transfer is ultimately limited by the thermal boundary layers in TMC. This data suggests that the initial rapid growth in Nu for small  $Ra/Ra_c$  is due to the formation of thermal boundary layers.

The Nusselt number and the energy dissipation are related via

$$(Nu-1)\frac{Ra}{Pr^2} = \varepsilon_u + \varepsilon_b,\tag{21}$$

where the viscous and ohmic dissipation are given by  $\varepsilon_u = \langle \boldsymbol{\zeta}^2 \rangle$  and  $\varepsilon_b = Q \langle \mathbf{j}^2 \rangle$ , respectively. The vorticity vector is denoted by  $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ .



FIG. 4. Nusselt number data: (a) Nu versus Ra; (b) compensated Nusselt number,  $Nu(Ra_c/Ra)$ , versus  $Ra/Ra_c$ . Tilted magnetoconvection cases are denoted by TMC, and the vertical field cases from Ref. [5] are denoted by VMC.

It is helpful to define the ratios of viscous and ohmic dissipation according to

$$\phi_u = \frac{\varepsilon_u}{\varepsilon_b + \varepsilon_u}, \qquad \phi_b = \frac{\varepsilon_b}{\varepsilon_b + \varepsilon_u},\tag{22}$$

where we note that  $\phi_u + \phi_b = 1$ . The dissipation ratios are shown in Fig. 6 for all values of Q. For all cases, near the onset of convection we observe a rapid decrease (increase) in the ohmic (viscous) dissipation ratio as  $Ra/Ra_c$  is increased. For  $Q = 2 \times 10^3$ , there is an approximate saturation in both dissipation ratios in the range  $25 \leq Ra/Ra_c \leq$ 75. For the largest value of  $Ra/Ra_c \approx 132$  we find a slight increase (decrease) in the ohmic (viscous) dissipation ratio. We note that within the saturated regime we find significant mean flows, though the mean flow is negligibly small at the largest value of  $Ra/Ra_c \approx 132$ . This increase (decrease) in the ohmic (viscous) dissipation ratio may be due to the strongly 3D flow that occurs, thus leading to significant induced magnetic field and associated current. The data suggests that viscous dissipation is important for all Q, though it is always smaller than ohmic dissipation.

To examine the depth-dependence of the dissipation we compute horizontally averaged profiles of the squared vorticity and current density, i.e.  $\overline{\zeta}^2$  and  $Q\overline{j}^2$ . Fig. 7 shows these profiles, normalized by the total dissipation, for  $Q = 2 \times 10^3$  and three different values of Ra. Due to the stress-free mechanical boundary conditions, we find that viscous dissipation is dominant within the interior of the domain, and momentum boundary layers are evident. In contrast, we find that ohmic dissipation is dominant near the boundaries though we do not observe obvious boundary layer regions even for the largest values of  $Ra/Ra_c$ . As a result, it is not relevant to separate the ohmic dissipation into interior and boundary layer contributions.

The convective flow speeds, as characterized by the Reynolds number based on the vertical component of the velocity,  $Re_z$ , are shown in Fig. 8(a). For the VMC data we show the total Reynolds number, as derived from Ref. [5], though the vertical component of the velocity is larger than the corresponding horizontal components so long as the Lorentz force remains dominant [e.g. 34]. We find that the scaling of the convective flow speeds are qualitatively similar in both TMC and VMC where there is a rapid rise in amplitude near the onset of convection, and a slower growth for larger  $Ra/Ra_c$ . There is a general trend of increasing  $Re_z$  with increasing Q for both data sets – we find a good collapse of the data by rescaling the convective flow speeds according to  $Re_z Q^{-1/4}$ , as shown in Fig. 8(b). This  $Q^{1/4}$  scaling was used in the asymptotic models of [35, 36], though only single mode (i.e. single wavenumber) solutions were analyzed. Asymptotic behavior is only expected in the magnetically constrained regime in which  $Ha/Re_z \gg 1$ ; as previously mentioned, for  $Q = 2 \times 10^3$  the cases with  $Ra/Ra_c \gtrsim 10$  are characterized by  $Ha/Re_z = O(1)$ .

The anisotropy in the velocity field can be characterized by computing the ratio of the volumetric rms of each velocity component to the total volumetric rms velocity (Re); the resulting data is shown in Fig. 9. Near the onset of convection, in which the flow consists of two-dimensional rolls, we find that the vertical component of the velocity dominates. However, we find a rapid decrease in the relative size of  $w_{rms}$  as the large scale flow forms and the y-component of the velocity field,  $v_{rms}$ , dominates. All cases show a region in parameter space in which a saturation of  $v_{rms}/Re$  occurs within the magnetically constrained regime. Moreover, this saturation is observed to occur at smaller

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FIG. 5. Vertical profiles of the time-averaged temperature for all cases. Top row (a)-(c): horizontally averaged temperature. Bottom row (d)-(f): rms temperature fluctuation. (a,d)  $Q = 2 \times 10^3$ ; (b,e)  $Q = 2 \times 10^5$ ; (c,f)  $Q = 2 \times 10^6$ . Darker greyscale lines correspond to larger Rayleigh numbers.



FIG. 6. Dissipation ratios: (a)  $Q = 2 \times 10^3$ ; (b)  $Q = 2 \times 10^5$ ; (c)  $Q = 2 \times 10^6$ . Note the difference in scale on the horizontal axis.

values of  $Ra/Ra_c$  as Q increases. The relative size of  $u_{rms}/Re$  exhibits a slow but steady increase with  $Ra/Ra_c$  and we find that the standard deviation also increases. For  $Q = 2 \times 10^5$  we find relaxation oscillations (discussed more below in section III.4.2) for supercriticalities  $Ra/Ra_c \gtrsim 7.5$  – in this regime both  $u_{rms}$  and  $v_{rms}$  become of comparable magnitude and exhibit large amplitude variations with time.

# III.3. Horizontally averaged mean fields

As previously noted, mean flows form in TMC for all values of Q investigated here. These mean flows take various forms and are associated with corresponding mean magnetic fields. In the present section we analyze mean fields, both velocity and magnetic, that are defined by averages over the entire horizontal plane. In the next subsection we



FIG. 7. Normalized dissipation profiles averaged in time for select cases from  $Q = 2 \times 10^3$ . (a) normalized viscous dissipation; (b) normalized ohmic dissipation.



FIG. 8. Reynolds number data for both TMC and VMC.  $Re_z$  is shown for TMC and Re is shown for VMC. (a) Reynolds number  $Re_z$  versus  $Ra/Ra_c$ ; (b) Rescaled Reynolds number,  $Re_z Q^{-1/4}$ , versus  $Ra/Ra_c$ .

examine the dynamics of what we refer to as zonal flows, which are y-directed flows averaged only in the y-direction.

We restrict the present analysis to the x-component of the mean velocity field,  $\overline{u}$ , and the associated magnetic field  $\overline{b}_x$ , since the y-components are observed to be significantly smaller in magnitude and approach zero for sufficiently long time averages. The mean momentum equation in the x-direction is given by

$$\partial_t \overline{u} + \partial_z \overline{(u'w')} = Q\eta_3 \partial_z \overline{b}_x + \partial_z^2 \overline{u},\tag{23}$$

where  $u' = u - \overline{u}$ , etc. The corresponding mean magnetic field is governed by

$$0 = \eta_3 \partial_z \overline{u} + \partial_z^2 \overline{b}_x,\tag{24}$$

so that vertical shear in the mean flow is associated with a mean magnetic field.

The computed time-averaged x-components of the means fields,  $\overline{u}$  and  $\overline{b}_x$ , are shown in Fig. 10 for a range of Ra and all three values of Q. In a subset of the  $Q = 2 \times 10^6$  simulations we observed a depth invariant (i.e. constant) component of  $\overline{u}$ ; this component of the flow was subtracted off from all of the data presented since it has no dynamical significance. We find that the magnitude of the mean fields increases with  $Ra/Ra_c$ , with the singular exception of



FIG. 9. Time-averaged velocity ratios: (a)  $Q = 2 \times 10^3$ ; (b)  $Q = 2 \times 10^5$ ; (c)  $Q = 2 \times 10^6$ . All quantities are volumetric rms values and the error bars show the standard deviation. Note that the scale of the horizontal axis is different for each figure.



FIG. 10. Time and horizontally averaged x-component of the velocity field and magnetic field: (a,d)  $Q = 2 \times 10^3$ ; (b,e)  $Q = 2 \times 10^5$ ; (c,f)  $Q = 2 \times 10^6$ . Darker lines correspond to higher Rayleigh numbers.

the case  $(Q = 2 \times 10^3, Ra/Ra_c = 132)$ , which shows a sudden decrease; profiles for this case are indicated by the darkest shade in panels (a) and (d). This latter case has reached sufficiently large Rayleigh number such that the magnetic field is no longer constraining the motion. We find that the *x*-components of the mean fields are robust in the sense that they maintain a similar structure with varying Q and Ra, and evolve on a timescale much longer than the underlying convection.

The mean velocity is observed to point in the positive (negative) x-direction for z > 0.5 (z < 0.5) for nearly all cases investigated. The direction of the mean flow is controlled by the Reynolds stress term, u'w', appearing in equation

(23). This component of the Reynolds stress represents the vertical flux of x-directed momentum. Because of the propensity for the convection to align with the direction of the imposed magnetic field, one expects  $\overline{u'w'} > 0$  in both upwelling and downwelling regions and this behavior was confirmed in the simulation data. Thus, a net vertical flux of x-directed momentum implies the mean flow should be positive in the upper half of the fluid layer (z > 0.5), and the corresponding deficit of x-directed momentum in the bottom half of the layer (z > 0.5) yields a negative mean flow in this region.

# III.3.1. Scaling analysis

Estimates for the sizes of the various terms in the above equations can be made, and help to explain some of the numerical findings shown in Fig. 10. We scale vertical derivatives of mean quantities as  $\partial_z \to \bar{\ell}_z^{-1}$ , where  $\bar{\ell}_z$  is some characteristic length scale in the vertical dimension, and the arrow is used to denote the scaling form for a particular operation or term in the governing equations. In what follows we assume  $\eta_3 = O(1)$ , and the symbol '~' is used to indicate a balance of terms in the governing equations. Equation (24) then gives

$$\overline{u} \sim \frac{b_x}{\overline{\ell}_z}.$$
(25)

If, for the purpose of the present scale analysis, we interpret the overline as also including a time average, then we have three terms in the mean momentum equations to consider, which we scale as

$$\partial_z \overline{(u'w')} \to \frac{\overline{u'w'}}{\overline{\ell}_z}, \qquad Q\eta_3 \partial_z \overline{b}_x \to \frac{Q\overline{b}_x}{\overline{\ell}_z}, \qquad \partial_z^2 \overline{u} \to \frac{\overline{u}}{\overline{\ell}_z^2}.$$
 (26)

Using relationship (25) in the above Lorentz force term we then have

$$\frac{Qb_x}{\bar{\ell}_z} \to Q\overline{u}.$$
 (27)

Balancing the mean Lorentz force with the mean viscous force would imply  $\overline{\ell}_z \sim Q^{-1/2}$ , which is the well-known Hartmann boundary layer scaling. We assume that such a strong dependence on Q is only relevant within the Hartmann layer, and not in the bulk of the domain. We do observe a Hartmann boundary layer in the horizontally averaged fields, though its effects appear to be small. Outside of the Hartmann layer it is unclear that there should be any Q-dependence on the vertical length scale with regards to mean quantities. Therefore, as a first approximation we neglect any Q-dependence on this length scale and the mean Lorentz force can then only be balanced by the divergence of the Reynolds stress in the bulk; the resulting balance gives

$$\overline{b}_x \sim \frac{1}{Q} \overline{u'w'}.$$
(28)

This result shows that Reynolds stresses are directly responsible for the generation of a mean magnetic field. Using the mean induction equation then gives a relationship between the mean flow and the Reynolds stresses,

$$\partial_z \overline{u} \sim -\frac{1}{Q\eta_3^2} \partial_z^2 \overline{(u'w')}.$$
(29)

Assuming  $\overline{\ell}_z = O(1)$ , the above balance leads to

$$\overline{u} \sim \frac{\overline{u'w'}}{Q},\tag{30}$$

so that both the mean velocity and mean magnetic field scale similarly with Q. These relationships suggest that we require knowledge on the asymptotic size of the fluctuating velocity components in order to estimate the asymptotic size of both  $\bar{u}$  and  $\bar{b}_x$ . Fig. 8 suggests that  $w' = O(Q^{1/4})$  and, though not shown, we also find that the  $u' = O(Q^{1/4})$ , which suggests that

$$\overline{u} = O\left(Q^{-1/2}\right), \qquad \overline{b}_x = O\left(Q^{-1/2}\right). \tag{31}$$

Figs. 11(a) and (c) show rms values of the mean velocity and magnetic field, respectively. The corresponding rescaled components are given in Figs. 11(b) and (d). The collapse of the rescaled quantities suggests that the scaling arguments given above lead to good estimates for the asymptotic dependence of these mean fields. These results suggest that the mean flow (and associated mean magnetic field) becomes less significant dynamically in the limit  $Q \to \infty$ .



FIG. 11. Scalings of the rms values for the horizontally averaged quantities: (a)  $\overline{u}_{rms}$ ; (b)  $\overline{u}_{rms} Q^{1/2}$ ; (c)  $\overline{b}_{x,rms}$ ; (d)  $\overline{b}_{x,rms} Q^{1/2}$ .

#### III.4. Zonal flow dynamics

The horizontally averaged fields analyzed in the previous section represent one particular component of the mean flows that are observed in TMC. However, the strongest mean flows observed in the simulations are dominated by horizontal wavenumber  $k_x = 1$ , and such motions are eliminated by a horizontal average over the entire horizontal plane. These mean flows are a combination of 'zonal' (y-directed) flows and large scale vortices that result in a meandering jet structure. For a fixed value of Q, we find that the y-component of the flow dominates, though for sufficiently strong forcing for  $Q = 2 \times 10^5$  the non-zonal (x-directed) flow becomes comparable to the zonal component and relaxation oscillations occur. Given the directional dependence of the flow it is helpful to average in only a single horizontal direction, either x or y. Here we focus only on the zonal, or y-directed, mean flows since they are observed for all values of Q and tend to dominate over much of the parameter space investigated here. A brief description of the relaxation oscillations is given in the next section.

As Q is increased we find that the zonal flow quickly becomes the dominant component of the velocity. To illustrate this behavior, Fig. 12 shows instantaneous volumetric renderings of the y-component of the velocity, v, for  $Q = 2 \times 10^6$ and three different values of Ra, increasing from left to right. Just beyond the onset of convection we find a mode with  $k_x = 1$  appears, as shown in panel (a), which suggests the formation of the zonal flow. As Ra increases in panels (b) and (c), this zonal flow becomes stronger and eventually becomes energetically dominant relative to the small scale convection by which it is driven.

Figs. 13(a)-(c) show instantaneous views of  $\overline{v}^y(x, z, t)$  in the x-z plane for all three values of Q with values of Ra chosen such that the zonal flow is energetically dominant (relative to the convection). For all values of Q we find that the zonal flow is aligned with the imposed magnetic field, and dominated by a  $k_x = 1$  structure at all depths.



FIG. 12. Volumetric renderings of the y-component of the velocity, v, illustrating the development of the zonal flow for  $Q = 2 \times 10^6$  and increasing Ra: (a)  $Ra = 1.04 \times 10^7$ ; (b)  $Ra = 1.1 \times 10^7$ ; (c)  $Ra = 1.3 \times 10^7$ . The orientation for all visualizations is shown in panel (a).

Corresponding top-down views of v at depth  $z \approx 0.5$  are shown in panels (d)-(f), where the meandering structure of this mean flow can be seen. Comparing the *y*-averaged flows shown in (a)-(c) with the full field in (d)-(f) shows how the action of *y*-averaging removes much of the small scale features present in (d)-(f).

## III.4.1. Scaling analysis

The scaling behavior of the zonal velocity is shown in Fig. 14(a) where rms values are plotted for all values of Q. We observe a trend of increasing magnitude with both increasing  $Ra/Ra_c$  and increasing Q. As shown in Fig. 14(b), the data can be collapsed by scaling the zonal velocity as  $\overline{v}^y = O(Q^{1/3})$ . As done previously for the horizontally averaged mean flows, the cause of this zonal flow scaling can be found by analyzing dominant balances in the zonal momentum equation.

The alignment of the zonal flow with the imposed magnetic field suggests that the use of a non-orthogonal (i.e. skewed) coordinate system is helpful. Such a coordinate system was used, for example, in Ref. [37] for studying the asymptotic behavior of rapidly rotating convection with a tilted rotation axis. We follow Ref. [37] and define the non-orthogonal coordinate system with unit vectors  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\boldsymbol{\eta}}$  and coordinates ( $\tilde{x}, \tilde{y}, \eta$ ). The relevant transformations between the orthogonal and non-orthogonal variables are given by

$$\widetilde{x} = x - \frac{\eta_1}{\eta_3} z, \qquad \widetilde{y} = y, \qquad \eta = \frac{1}{\eta_3} z$$
(32)

$$\widetilde{u} = u - \frac{\eta_1}{\eta_3}w, \qquad \widetilde{v} = v, \qquad \widetilde{w} = \frac{1}{\eta_3}w.$$
(33)

Similarly we have  $\widetilde{b}_y = b_y$ . The y-component of the momentum equation can now be written as

$$\frac{\partial v}{\partial t} + \partial_{\widetilde{x}} \left( \widetilde{u}v \right) + \partial_{y} \left( v^{2} \right) + \partial_{\eta} \left( v\widetilde{w} \right) = -\partial_{y}P + \left( \frac{1}{\eta_{3}^{2}} \partial_{\widetilde{x}}^{2} + \partial_{y}^{2} + \frac{1}{\eta_{3}^{2}} \partial_{\eta}^{2} - 2\frac{\eta_{1}}{\eta_{3}} \partial_{\widetilde{x}\eta}^{2} \right) v + Q \partial_{\eta} b_{y}.$$
(34)

Averaging in the y-direction,  $\eta$ -direction, and in time yields

$$\overline{\widetilde{u}^{y\eta}}\partial_{\widetilde{x}}\overline{v}^{y\eta}{}^{t} + \partial_{\widetilde{x}}\overline{(\widetilde{u}'v')}^{y\eta t} = \left(\frac{1}{\eta_3} - 2\right)\partial_{\eta}\overline{v}^{yt}\Big|_{0}^{1/\eta_3} + \frac{1}{\eta_3^2}\partial_{\widetilde{x}}^2\overline{v}^{y\eta t}.$$
(35)

Here the fluctuating terms are given by  $v' = v - \overline{v}^{y\eta}$ , etc. We note the absence of the Lorentz force in the above equation, indicating that it plays an indirect role in the zonal dynamics. The first term on the left side represents advection by the mean whereas the first term on the right side contains boundary terms associated with averaging the viscous force along  $\eta$ ; the numerical simulations show that both of these terms are small in comparison to the two other terms present in the above equation. Advection by the mean is small because  $\tilde{u}^y$  is not generally aligned with



FIG. 13. Structure of the zonal flow from representative cases. (a,d)  $Q = 2 \times 10^3$ ,  $Ra = 1 \times 10^5$ ; (b,e)  $Q = 2 \times 10^5$ ,  $Ra = 8 \times 10^6$ ; (c,f)  $Q = 2 \times 10^6$ ,  $Ra = 7 \times 10^7$ . The top row shows instantaneous views of the zonal velocity,  $\overline{v}^y$ , and the bottom row shows instantaneous views of the *y*-component of the velocity, *v*, in the horizontal plane at depth  $z \approx 0.5$ .

 $\hat{\eta}$ . In addition, because  $\overline{v}^y$  is aligned with  $\hat{\eta}$ , the  $\eta$ -averaged value of  $\partial_{\eta}\overline{v}^y$  is expected to be small. Therefore, the largest terms are the Reynolds stress term and the viscous force,

$$\partial_{\widetilde{x}}\overline{(\widetilde{u}'v')}^{y\eta t} \sim \frac{1}{\eta_3^2} \partial_{\widetilde{x}}^2 \overline{v}^{y\eta t}.$$
(36)

Furthermore, if we let  $\bar{\ell}^{y\eta}$  denote a length scale associated with y and  $\eta$ -averaged quantities, we find

$$\overline{v}^{y\eta t} \sim \overline{\ell}^{y\eta} \,\overline{(\widetilde{u}'v')}^{y\eta t},\tag{37}$$

where we have dropped the factor of  $\eta_3^2$  since it is of order unity. In all simulations in which a zonal flow is observed we find that the zonal flow grows to fill the domain and is dominated by a  $k_x = 1$  structure. As mentioned previously, as Q is varied we fix the total number of horizontal critical wavelengths in the domain to be n = 10. Since the critical wavelength changes with Q, i.e.  $\lambda_c \sim Q^{-1/6}$  as Q becomes large, this implies that the horizontal (and vertical) scale of the zonal flow is also changing as we vary Q in our simulations. Therefore, we can scale the characteristic zonal length scale as

$$\bar{\ell}^{y\eta} \sim n\lambda_c \qquad \Rightarrow \qquad \bar{\ell}^{y\eta} \sim nQ^{-1/6}. \tag{38}$$

Combining this scaling with relationship (37) then yields

$$\overline{v}^{y\eta t} = O\left(Q^{1/3}\right),\tag{39}$$

where we have again used the numerical data that indicates  $(\tilde{u}', v') = O(Q^{1/4})$ .

The zonal length scaling relation (38) implies that for a fixed value of Q the zonal flow magnitude increases linearly with the horizontal dimension (as quantified by n) of the simulation domain. Fig. 15 shows various measurements



FIG. 14. Scaling behavior of the rms value for the zonal flow: (a) raw data; (b) rescaled data.



(a)

FIG. 15. Behavior of various measurements of flow speed with increasing horizontal dimension of the simulation domain for  $Q = 2 \times 10^5$  and  $Ra = 6 \times 10^6$ . The horizontal length of the simulation domain is represented by the number of unstable horizontal wavelengths, n; the aspect ratio for the cases shown is  $\Gamma = (3.31, 4.30, 4.97, 5.63, 6.62)$ . Dotted lines show the corresponding least squares fits to the data.

of flow speed as a function of n for  $Q = 2 \times 10^5$  and  $Ra = 6 \times 10^6$ . This particular combination of parameters is used since it shows significant zonal flows. The linear dependence between Re and n, as well as  $\overline{v}_{rms}^y$  and n, is clearly observed, as is the independence of  $Re_z$  on n. This linear dependence between the large scale flow and the horizontal domain size is in agreement with a previous investigation of the inverse kinetic energy cascade in rotating convection where the same scaling was observed [38].

# III.4.2. Relaxation Oscillations

Relaxation oscillations were observed in simulations with  $Q = 2 \times 10^5$  and  $10Ra_c \leq Ra \leq 30Ra_c$ , where  $30Ra_c$  was the highest supercritical Rayleigh number achieved for this value of Q. Relaxation oscillations were also found for a case with  $Q = 2 \times 10^6$  but no statistics were collected due to the long time integration required. The relaxation oscillations are characterized by a zonal flow magnitude that exhibits large oscillations in time. Figs. 16(a) and (b) show the Reynolds number and Nusselt number, respectively, as a function of time for  $Q = 2 \times 10^5$  and  $Ra = 1.5 \times 10^7$ . During



FIG. 16. Times series data illustrating the temporal behavior of relaxation oscillations for  $Q = 2 \times 10^5$  and  $Ra = 1.5 \times 10^7$ : (a) Reynolds number, Re(t); (b) Nusselt number, Nu(t).

times when the mean flow is strong (weak), the Nusselt number is generally smaller (larger) than the corresponding time-averaged value. Fig. 17 shows how the horizontal components of velocity change when the Reynolds number goes from increasing to decreasing. The y-component of the velocity decays rapidly and the x-component of velocity grows to be the dominant velocity component.

# IV. CONCLUSION

The direct numerical simulations reported here have shown that magnetoconvection with a tilted magnetic field (TMC) has both dynamic similarities, and key differences with magnetoconvection with a vertical field (VMC). For a meaningful comparison between these two cases we have chosen imposed field strengths that yield critical parameters that are identical to those used in an analogous VMC study [5]; i.e. the strength of the vertical component of the imposed magnetic fields are identical for our chosen values of Q. The simulations span dynamical regimes that are characterized as magnetically constrained, in which  $Ha/Re_z \gg 1$ , and those that might be considered magnetically influenced when  $Ha/Re_z = O(1)$ . Like VMC, we find that when  $Ha/Re_z \gg 1$  and for sufficiently strong buoyancy forcing, the convective structures consist of anisotropic 'columns' that align with the direction of the magnetic field. This alignment is no longer obvious when  $Ha/Re_z \lesssim O(1)$ .

The heat transport for TMC is qualitatively similar to VMC in which there is a rapid increase in the Nusselt number near the onset of convection that is associated with the formation of the thermal boundary layers, and the growth of Nu with increasing  $Ra/Ra_c$  slows beyond this regime. We do not observe a clear power law scaling of the Nusselt number for TMC over our investigated range of parameter space. Ohmic dissipation dominates viscous dissipation in all simulations, including those in which  $Ha/Re_z = O(1)$ . However, viscous dissipation remains important and represents approximately 40% of the total dissipation as the Rayleigh number is increased.

Convective flow speeds, as characterized by the Reynolds number based on the vertical component of the velocity,  $Re_z$ , show behavior that is similar to VMC. However, it is unclear whether  $Re_z$  exhibits power law behavior due to the limited range of parameter space accessible in the present simulations. We find that an asymptotic scaling of  $Re_z \sim Q^{1/4}$  describes the Q-dependence of the convective flow speeds. This scaling is the same that was used in the asymptotic models of Refs. [35, 36]. Further investigation, particularly simulations with larger values of Q, is necessary to confirm the robustness of this scaling.

Mean flows, and associated mean magnetic fields, form for all values of Q investigated. We find and study two distinct forms of mean flows, as characterized by the horizontal wavevector  $\mathbf{k}$ . Those mean flows in which  $\mathbf{k} = (0,0)$ (i.e. averaged over the horizontal plane) are dominated by the x-components,  $\overline{u}$  and  $\overline{b}_x$ . The propensity for convective structures to align with the magnetic field yields a vertical flux of x-directed momentum that generates a positive mean flow above the mid-plane (i.e. z > 0), and a negative mean flow below the midplane (z < 0). The simulations show that the rms value of  $\overline{u}$  is a decreasing function of Q; a balance analysis suggests that  $\overline{u} = O(Q^{-1/2})$ , which is in agreement with the numerical findings. An identical scaling holds for  $\overline{b}_x$ , suggesting that this particular component of the mean fields becomes less significant dynamically as Q is increased.



FIG. 17. Instantaneous plots of the horizontal velocity components illustrating relaxation oscillations for  $Q = 2 \times 10^5$  and  $Ra = 1.5 \times 10^7$ . The two plots on the left (a,c) are taken at time t = 5.200, just prior to the peak in Reynolds number. The two plots on the right (b,d) are taken at time t = 5.228, just after the Reynolds number reaches a peak. (a,b) y-velocity component; (c,d) x-velocity component.

The energetically dominant mean flows are characterized by a meandering jet structure that tends to be aligned with the tilt of the imposed magnetic field. These mean flows are dominated by either  $\mathbf{k} = (1,0)$  at lower values of  $Ra/Ra_c$ , or exhibit relaxation oscillations that shift between the  $\mathbf{k} = (0,1)$  and  $\mathbf{k} = (1,0)$  modes for sufficiently large  $Ra/Ra_c$  and  $Q > 2 \times 10^3$ . We find that the 'zonal' flows scale strongly with Q, i.e.  $\overline{v}^y \sim Q^{1/3}$ ; this scaling is the result of the Q-dependent Reynolds stresses and the fact that the zonal flow magnitude scales linearly with the horizontal dimensions of the simulation domain. The zonal flows become energetically negligible for sufficiently large  $Ra/Ra_c$ , since then  $Ha/Re_z \leq 1$  and the convection is no longer aligned with the imposed magnetic field. The relaxation oscillations evolve on the timescale of a large scale viscous diffusion unit and therefore require substantial computational resources to study.

It is interesting to note that several phenomena observed here for convection in a tilted magnetic field are also observed for rotating convection [24, 25]. In convection with a tilted rotation vector, the fluid structures tend to align along the tilt axis of the rotation, much like how the fluid structures in this investigation were aligned along the magnetic field. In addition, a strong shear flow arises in convection with a tilted rotation vector that closely resembles the mean flow found for convection in a tilted magnetic field. We anticipate that scalings for the strength of the mean flow and convective flow speeds can be found in tilted rotating convection, as was done in the present investigation for magnetoconvection.

The structure of the observed mean flows in the present work is a manifestation of the plane periodic geometry that was used. Confined geometries will yield mean flows of different structure. However, the present work has shown that a robust inverse kinetic energy cascade is nevertheless present in TMC, and there is no a priori reason why such a cascade would not also be present in confined geometries such as cylinders or cubes. We might expect that the inverse cascade would manifest itself in the form of only large scale vortices in confined geometries, but future work

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