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Reciprocal theorem for calculating the flow rate-pressure drop relation for complex fluids in narrow geometries

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We study the mechanically driven flows of non-Newtonian fluids in narrow and confined configurations. Using the Lorentz reciprocal theorem, we derive a closed-form expression for the flow rate–pressure drop relation of complex fluids in such geometries, which holds for a wide class of non-Newtonian constitutive models. For the weakly non-Newtonian limit, our theory provides the first-order non-Newtonian correction for the flow rate–pressure drop relation solely using the corresponding Newtonian solution, eliminating the need to solve the non-Newtonian flow problem. In particular, for the flow-rate-controlled situation, we find that the first-order non-Newtonian pressure drop correction may increase, decrease, or not change the total pressure drop for a viscoelastic second-order fluid, depending on the geometry, but always decreases it for a shear-thinning Carreau fluid.

I. INTRODUCTION

Pressure-driven flows of non-Newtonian fluids in narrow and confined geometries are ubiquitous in natural processes and technological applications. Examples include blood flow in microvessels, such as arterioles and venules [1, 2], polymeric flows in industrial processes [3], and non-Newtonian flows in microfluidic devices, such as a microviscometer [4] and viscoelastic fluidic rectifier [5–7]. For such confined flows, which are usually created either by imposing the flow rate q or pressure drop Δp, one of the main interests is to understand the relationship between the flow rate and pressure drop for a given geometry. Beyond the aforementioned examples, understanding the $q-\Delta p$ relation is also important in medical applications, for instance, for precise estimation of the injection force of subcutaneous drug administration, which may exhibit non-Newtonian rheology [8, 9]. Conventionally, obtaining the flow rate−pressure drop relation requires first solving the governing equations for the detailed distribution of the velocity and pressure fields, which may involve cumbersome calculations for non-Newtonian flows even in simple geometries. However, as we show, these detailed calculations of the non-Newtonian flow problem can be bypassed, at least in some cases, by applying the Lorentz reciprocal theorem [10]. While we hereafter consider stable flows to analyze the $q-\Delta p$ relation, it should be noted that the flow of non-Newtonian fluids within nonuniform geometries may become unstable above a certain flow rate even at low Reynolds numbers due to the fluid's complex rheology [11–13].

The reciprocal theorem for low-Reynolds-number hydrodynamics has been applied widely to facilitate some calculations by eliminating the need for calculating the detailed flow and pressure fields (e.g., [14]). The reciprocal theorem is not limited to the flow of Newtonian fluids and has been extended to describe the motion of particles in non-Newtonian flows [15–19], in particular, to the determination of the non-Newtonian contribution to the forces and torques on particles [20–23], and to calculate the speed [24–30] and force moments [31, 32] of self-propelled particles in a complex fluid. All of the above studies considered the weakly non-Newtonian limit, thus enabling the use of the Newtonian solution for the flow field to find the first-order non-Newtonian effects without solving the detailed flow field in the non-Newtonian problem.

The integral form of the reciprocal theorem is particularly convenient for calculating integrated hydrodynamic quantities such as force, torque, and flow rate [14]. Thus, given this convenience, one would expect to find the application of the reciprocal theorem to determine all of these quantities. However, to date, its use has been primarily limited to obtaining the force and torque acting on particles in unbounded and semi-bounded flows of Newtonian and non-Newtonian fluids, and only a few studies have utilized the reciprocal theorem to obtain the flow rate of Newtonian fluids in channel flows [33–35]. Moreover, to the best of our knowledge, no application of the reciprocal theorem has been presented to date for obtaining the flow rate−pressure drop relation for the flow of non-Newtonian fluids.

In this Letter, we show that the reciprocal theorem allows one to obtain the flow rate−pressure drop relation for flows of non-Newtonian fluids in channels of arbitrary shape, bypassing the detailed calculations of the non-Newtonian flow problem. We first employ the shallowness of the geometry and derive a general expression for flow rate−pressure

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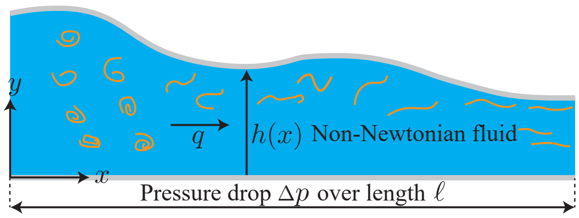

FIG. 1: Schematic illustration of the geometry consisting of a two-dimensional spatially-varying shallow channel of height $h(x)$ and length ℓ. The channel contains a non-Newtonian fluid steadily driven by an imposed flow rate q resulting in the pressure drop Δp.

drop relation in nonuniform lubrication flows, which holds for a wide class of shear-thinning and viscoelastic non-Newtonian constitutive models. Considering the weakly non-Newtonian limit, we then show that calculation of the first-order non-Newtonian correction for the $q-\Delta p$ relation involves only the use of the corresponding Newtonian solution in the same geometry without the need to solve for non-Newtonian flow at this order. We illustrate the use of our approach for the weakly viscoelastic second-order fluid model and the weakly shear-thinning Carreau fluid and show that the first-order non-Newtonian pressure drop correction can increase, decrease or not change the total pressure drop, depending on the flow geometry.

II. GOVERNING EQUATIONS AND LUBRICATION SCALING

Consider incompressible steady flow of a non-Newtonian fluid in a two-dimensional nonuniform channel of height $h(x)$ and length ℓ, where $h \ll \ell$, as shown in Fig. 1. We assume that the fluid motion with velocity $\boldsymbol{u}=(u,v)$ and pressure distribution p is induced by the imposed flow rate q (per unit depth), and we are interested in determining the resulting pressure drop Δp for a given q. There is no loss of generality in considering the two-dimensional case, as contrasted with an axisymmetric configuration or a three-dimensional case provided the Newtonian flow solution is known.

We consider low-Reynolds-number flows so that the fluid motion is governed by the continuity equation and Cauchy momentum equations in the absence of inertia,

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=\boldsymbol{0}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, which is assumed to take the form [16, 26]

$$\boldsymbol{\sigma}=-p\boldsymbol{I}+\boldsymbol{\tau}=-p\boldsymbol{I}+2\eta_0\boldsymbol{E}+\boldsymbol{A} \quad \text{with} \quad \boldsymbol{E}=\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}). \tag{2}$$

Here, $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{\tau}$ is the deviatoric stress tensor, η_0 is a constant viscosity, $\boldsymbol{E}$ is the rate-of-strain tensor, and $\boldsymbol{A}$ is a symmetric tensor, which may represent a deformable microstructure, but see below for an application to the Carreau model, which is an example of a generalized Newtonian fluid. The term $2\eta_0\boldsymbol{E}$ represents the Newtonian contribution to the deviatoric stress, while $\boldsymbol{A}$ represents the contribution that gives rise to the non-Newtonian effects. The governing equations (1)−(2) are supplemented by the no-slip and no-penetration boundary conditions along the channel walls, $\boldsymbol{u}=\boldsymbol{0}$ at $y=0$, $h(x)$, and the integral constraint for the flow rate, $\int_0^{h(x)} u(x,y)\mathrm{d}y=q$.

We introduce nondimensional variables based on lubrication theory,

$$X=\frac{x}{\ell}, \quad Y=\frac{y}{h_0}, \quad U=\frac{u}{q/h_0}, \quad V=\frac{v}{\epsilon q/h_0}, \quad P=\frac{p-p_{\mathrm{ref}}}{\eta_0 q/(\epsilon^2 h_0\ell)}, \quad H=\frac{h}{h_0}, \quad \Delta P=\frac{\Delta p}{\eta_0 q/(\epsilon^2 h_0 \ell)}, \tag{3a}$$

$$\mathcal{A}_{xx}=\frac{A_{xx}}{\eta_0 q/(\epsilon^2 h_0\ell)}, \qquad \mathcal{A}_{xy}=\frac{A_{xy}}{\eta_0 q/(\epsilon h_0\ell)}, \qquad \mathcal{A}_{yy}=\frac{A_{yy}}{\eta_0 q/(h_0\ell)}, \tag{3b}$$

where h_0 is the height at $x=0$, p_{ref} is an appropriate reference pressure, and $\epsilon=h_0/\ell$ is the aspect ratio of the configuration, which is assumed to be small, $\epsilon \ll 1$. The nondimensional shape of the channel is denoted $H(X)$ and will be an important parameter in our main results below. With this nondimensionalization, the governing equations (1)−(2) take the following form for two-dimensional flows:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \qquad \frac{\partial P}{\partial X}=\epsilon^2\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}+\frac{\partial \mathcal{A}_{xx}}{\partial X}+\frac{\partial \mathcal{A}_{xy}}{\partial Y}, \qquad \frac{\partial P}{\partial Y}=\epsilon^2\left[\epsilon^2\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial \mathcal{A}_{xy}}{\partial X}+\frac{\partial \mathcal{A}_{yy}}{\partial Y}\right]. \tag{4}$$

From (4), it follows that $P = P(X) + O(\epsilon^2)$, i.e., the pressure is independent of Y up to $O(\epsilon^2)$, consistent with the classical lubrication approximation.

III. RECIPROCAL THEOREM FOR NON-NEWTONIAN FLOWS IN NARROW GEOMETRIES

Let $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\sigma}}$ denote, respectively, the velocity and stress fields corresponding to the solution of the Newtonian problem in the same domain with the same viscosity η_0. The corresponding governing equations are

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}} = 0, \qquad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}} = \boldsymbol{0} \quad \text{with} \quad \hat{\boldsymbol{\sigma}} = -\hat{p}\boldsymbol{I} + 2\eta_0\hat{\boldsymbol{E}}. \tag{5}$$

From (1) and (5), it follows that $(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\hat{\boldsymbol{u}} = 0$ and $(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}})\cdot\boldsymbol{u} = 0$, and thus $\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\hat{\boldsymbol{u}} = 0$ and $\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}) - \hat{\boldsymbol{\sigma}}:\boldsymbol{\nabla}\boldsymbol{u} = 0$. Using the latter result, incompressibility, and the symmetry of the stress tensor, we obtain $\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}) - \boldsymbol{\sigma}:\hat{\boldsymbol{E}} = 0$ and $\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}) - \hat{\boldsymbol{\sigma}}:\boldsymbol{E} = 0$. Subtracting these identities and using (2) and (5) yields $\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}) = \boldsymbol{A}:\hat{\boldsymbol{E}}$. This equation can be integrated over the entire fluid volume $\mathcal{V}$ bounded by the surface of the top and bottom walls S_w, and the surfaces at the inlet and outlet S_0 and S_ℓ at $x=0$ and $x=\ell$, respectively. Then, applying the divergence theorem over the fluid domain $\mathcal{V}$ leads to the reciprocal theorem in the form

$$\int_{S_0}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}\,\mathrm{d}S + \int_{S_\ell}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}\,\mathrm{d}S - \int_{S_0}\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}\,\mathrm{d}S - \int_{S_\ell}\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}\,\mathrm{d}S = \int_{\mathcal{V}}\boldsymbol{A}:\hat{\boldsymbol{E}}\,\mathrm{d}\mathcal{V}, \tag{6}$$

where $\boldsymbol{n}$ is the unit outward normal to $S_{0,\ell}$, and we have used the fact that the integrals over the walls S_w vanish since there $\boldsymbol{u} = \hat{\boldsymbol{u}} = \boldsymbol{0}$.

Previously, the reciprocal theorem in a form equivalent to (6) has been applied to analyze the motion of active and passive particles [15–19, 24–29, 31, 32] and calculate the non-Newtonian contribution to the forces and torque on particles [20–23] in unbounded/semi-bounded non-Newtonian flows. Instead, here, we use the reciprocal theorem to calculate the flow rate−pressure drop relation of complex fluids in confined and narrow geometries. Our derivation applies to a wide class of non-Newtonian constitutive models and assumes only negligible fluid inertia, a shallow geometry, $\epsilon \ll 1$, and two-dimensional flow, where the latter assumption is for simplicity and clarity of presentation.

Using the scaling analysis and (2), (3), and (5), the terms $\boldsymbol{A}:\hat{\boldsymbol{E}}$, $\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}$, and $\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}$, appearing in (6), are approximately:

$$\boldsymbol{A}:\hat{\boldsymbol{E}} = \frac{\eta_0 q^2}{h_0^4}\left[\mathcal{A}_{xx}\frac{\partial\hat{U}}{\partial X} + \mathcal{A}_{xy}\frac{\partial\hat{U}}{\partial Y} + O(\epsilon^2)\right], \tag{7a}$$

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}|_{X=0,\,1} = \mp\frac{\eta_0 q^2\ell}{h_0^4}\Big[[-P + \mathcal{A}_{xx}]\,\hat{U} + O(\epsilon^2)\Big]_{X=0,\,1}, \qquad \boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u}|_{X=0,\,1} = \mp\frac{\eta_0 q^2\ell}{h_0^4}\Big[-\hat{P}U + O(\epsilon^2)\Big]_{X=0,\,1}, \tag{7b}$$

where the minus sign in (7b) corresponds to S_0 and the plus sign corresponds to S_ℓ.

Substituting (7) into (6), we obtain

$$\int_0^{H(0)}\Big[[P - \mathcal{A}_{xx}]\,\hat{U}\Big]_{X=0}\mathrm{d}Y - \int_0^{H(1)}\Big[[P - \mathcal{A}_{xx}]\,\hat{U}\Big]_{X=1}\mathrm{d}Y - \int_0^{H(0)}\Big[\hat{P}U\Big]_{X=0}\mathrm{d}Y + \int_{-H(1)}^{H(1)}\Big[\hat{P}U\Big]_{X=1}\mathrm{d}Y = \int_0^1\int_0^{H(X)}\left[\mathcal{A}_{xx}\frac{\partial\hat{U}}{\partial X} + \mathcal{A}_{xy}\frac{\partial\hat{U}}{\partial Y}\right]\mathrm{d}Y\,\mathrm{d}X + O(\epsilon^2), \tag{8}$$

where $H(X)$ is the nondimensional shape of the channel. Noting that $P = P(X) + O(\epsilon^2)$, $\hat{P} = \hat{P}(X) + O(\epsilon^2)$ and $\int_0^{H(X)}U\mathrm{d}Y = \int_0^{H(X)}\hat{U}\mathrm{d}Y = 1$, and defining $\Delta P = P(0) - P(1)$ and $\Delta\hat{P} = \hat{P}(0) - \hat{P}(1)$, (8) yields the reciprocal theorem for two-dimensional flows of non-Newtonian fluids in narrow geometries:

$$\Delta P = \Delta\hat{P} + \int_0^{H(0)}\Big[\mathcal{A}_{xx}\hat{U}\Big]_{X=0}\mathrm{d}Y - \int_0^{H(1)}\Big[\mathcal{A}_{xx}\hat{U}\Big]_{X=1}\mathrm{d}Y + \int_0^1\int_0^{H(X)}\left[\mathcal{A}_{xx}\frac{\partial\hat{U}}{\partial X} + \mathcal{A}_{xy}\frac{\partial\hat{U}}{\partial Y}\right]\mathrm{d}Y\,\mathrm{d}X + O(\epsilon^2). \tag{9}$$

The solution of the corresponding Newtonian lubrication problem is (see e.g., [36]),

$$\Delta\hat{P} = 12\int_0^1\frac{\mathrm{d}X}{H(X)^3}, \qquad \hat{U} = \frac{6}{H(X)^3}Y\left(H(X) - Y\right), \qquad \hat{V} = \frac{6}{H(X)^4}\frac{\mathrm{d}H(X)}{\mathrm{d}X}Y^2\left(H(X) - Y\right). \tag{10}$$

Equation (9) is the main result of this Letter, clearly indicating that the pressure drop of non-Newtonian flow in a narrow channel consists of four contributions. The first term on the right-hand side of (9) represents the Newtonian contribution to the pressure drop. The second and third terms represent the contribution of the non-Newtonian normal stress of the complex fluid at the inlet and outlet of the channel. Finally, the last term represents the non-Newtonian contribution due to elongational ($\mathcal{A}_{xx}\partial\hat{U}/\partial X$) and shearing ($\mathcal{A}_{xy}\partial\hat{U}/\partial Y$) effects within the fluid domain $\mathcal{V}$.

The expression (9) is not restricted to a particular choice of the constitutive rheological equation and can be used with various constitutive models, provided the deviatoric stress can be written as $\boldsymbol{\tau} = 2\eta_0\boldsymbol{E} + \boldsymbol{A}$. For example, we can use (9) with viscoelastic models such as the Oldroyd-B, second-order fluid, FENE-CR, FENE-P, Phan-Thien-Tanner, and Giesekus models, and shear-thinning models such as the Carreau model [37, 38]. We note that although the FENE-P and Phan-Thien-Tanner constitutive models do not exactly take the form of (2), they agree with it for the weakly non-Newtonian limit, and thus our approach allows assessing the non-Newtonian correction to pressure drop for these models as well.

IV. THE WEAKLY NON-NEWTONIAN LIMIT

Equation (9) clearly shows that the pressure drop depends on the $\mathcal{A}_{xx}$ and $\mathcal{A}_{xy}$ components of the non-Newtonian contribution to the deviatoric stress, and thus, generally, requires the solution of the nonlinear non-Newtonian problem. However, in the weakly non-Newtonian limit, where the non-Newtonian deviatoric stress $\boldsymbol{\tau} = 2\eta_0\boldsymbol{E} + \boldsymbol{A}$ is slightly perturbed from a Newtonian stress $\boldsymbol{\tau} = 2\eta_0\boldsymbol{E}$, we can apply the reciprocal theorem (9) to obtain the non-Newtonian correction to pressure drop only with the knowledge of the solution of the Newtonian problem.

To this end, we expand the velocity and pressure as $\{U, V, P\} = \{U_0, V_0, P_0\} + \alpha\{U_1, V_1, P_1\} + O(\alpha^2)$ and $\boldsymbol{\mathcal{A}}$ as $\{\mathcal{A}_{xx}, \mathcal{A}_{xy}, \mathcal{A}_{yy}\} = \alpha\{\mathcal{A}_{xx,0}, \mathcal{A}_{xy,0}, \mathcal{A}_{yy,0}\} + O(\alpha^2)$, where $\alpha \ll 1$ is a small dimensionless parameter indicating the deviation from Newtonian behavior, for example, small Carreau number or viscosity ratio for shear-thinning fluids [27] or small Deborah (or Weissenberg) number for weakly viscoelastic fluids [16, 26]. Substituting these expansions into (9), at the leading order we obtain the Newtonian contribution to the pressure, $\Delta P_0 = \Delta\hat{P}$, while the first-order terms yield the non-Newtonian correction,

$$\Delta P_1 = \int_0^{H(0)} \left[\mathcal{A}_{xx,0}\hat{U}\right]_{X=0} \mathrm{d}Y - \int_0^{H(1)} \left[\mathcal{A}_{xx,0}\hat{U}\right]_{X=1} \mathrm{d}Y + \int_0^1 \int_0^{H(X)} \left[\mathcal{A}_{xx,0}\frac{\partial\hat{U}}{\partial X} + \mathcal{A}_{xy,0}\frac{\partial\hat{U}}{\partial Y}\right] \mathrm{d}Y\,\mathrm{d}X, \tag{11}$$

where $\mathcal{A}_{xx,0}$ and $\mathcal{A}_{xy,0}$ depend on the corresponding Newtonian flow field $(U_0, V_0) = (\hat{U}, \hat{V})$.

To illustrate the use of these results, in subsequent sections we calculate the first-order non-Newtonian correction to the pressure drop for a weakly viscoelastic second-order fluid and a weakly shear-thinning Carreau fluid.

A. Viscoelasticity: second-order fluid

Viscoelastic fluids exhibit both viscous and elastic responses to applied shear and extensional rates. For low-Deborah-number flows, $\mathrm{De} \ll 1$, where the relevant De is defined below, viscoelasticity may be described using the second-order fluid model, obtained via the retarded motion expansion of the deviatoric stress tensor as a polynomial in the rate-of-strain tensors up to second order in the expansion [37, 38],

$$\boldsymbol{\tau} = 2\eta_0\boldsymbol{E} - \Psi_1\overset{\triangledown}{\boldsymbol{E}} + 4\Psi_2\boldsymbol{E}\cdot\boldsymbol{E} \quad \text{with} \quad \overset{\triangledown}{\boldsymbol{E}} = \partial\boldsymbol{E}/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{E} - (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot(\boldsymbol{\nabla}\boldsymbol{u}). \tag{12}$$

Here, η_0 is the constant viscosity of the viscoelastic solution, the triangle is the upper-convected derivative of $\boldsymbol{E}$, and Ψ_1 and Ψ_2 are the first and second normal stress-difference coefficients, respectively. The first and second normal stress-difference coefficients have units of $\mathrm{Pa\,s^2}$, and $\Psi_1 > 0$ and $\Psi_2 \leq 0$ for polymer solutions. In this work, we consider steady flows and thus drop the time derivative $\partial\boldsymbol{E}/\partial t$ in $\overset{\triangledown}{\boldsymbol{E}}$.

From (2) and (12), it follows that $\boldsymbol{A} = -\Psi_1\left[\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{E} - (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\right] + 4\Psi_2\boldsymbol{E}\cdot\boldsymbol{E}$. Using (3) and performing order-of-magnitude analysis, we obtain the dimensionless expressions for $\mathcal{A}_{xx}$ and $\mathcal{A}_{xx}$,

$$\mathcal{A}_{xx} = \mathrm{De}\left[(2+\mathcal{B})\left(\frac{\partial U}{\partial Y}\right)^2 + O(\epsilon^2)\right], \qquad \mathcal{A}_{xy} = -\mathrm{De}\left[U\frac{\partial^2 U}{\partial X\partial Y} + V\frac{\partial^2 U}{\partial Y^2} - 2\frac{\partial U}{\partial Y}\frac{\partial V}{\partial Y} + O(\epsilon^2)\right], \tag{13}$$

where $\mathcal{B} = -2\Psi_2/\Psi_1 \geq 0$ is the ratio of the second to the first normal stress-difference coefficients and $\mathrm{De} = \Psi_1 q/2\eta_0\ell h_0$ is the Deborah number, which is the product of the relaxation time scale of the fluid, $\Psi_1/2\eta_0$, and the characteristic extension rate of the flow, $q/\ell h_0$. Alternatively, De can be interpreted as the ratio of the relaxation time to the residence time of the fluid in the channel, $\ell/(q/h_0)$. The Deborah number De is related to the Weissenberg number $\mathrm{Wi} = \Psi_1 q/2\eta_0 h_0^2$ through $\mathrm{De} = \epsilon \mathrm{Wi}$, where Wi is the product of the relaxation time scale of the fluid and the characteristic shear rate of the flow, q/h_0^2. Since we assume $\epsilon \ll 1$, De can be small while keeping $\mathrm{Wi} = O(1)$ [39].

Considering the weakly viscoelastic limit, corresponding to $\mathrm{De} \ll 1$, where now $\alpha = \mathrm{De}$, we expand the velocity, pressure, and $\boldsymbol{\mathcal{A}}$ in powers of De, similarly to the previous section. Noting that $(U_0, V_0) = (\hat{U}, \hat{V})$ and using (10) and (13), we obtain

$$\mathcal{A}_{xx,0} = \frac{36(2+\mathcal{B})\left(H(X)-2Y\right)^2}{H(X)^6}, \qquad \mathcal{A}_{xy,0} = \frac{72Y\left(2Y-H(X)\right)\left(4Y-3H(X)\right)}{H(X)^7}\frac{\mathrm{d}H(X)}{\mathrm{d}X}. \tag{14}$$

Substituting (14) into (11) provides an analytical expression for the first-order pressure drop correction of the second-order fluid:

$$\Delta P_1 = \frac{18(5+3\mathcal{B})}{5}\left[\frac{1}{H(0)^4} - \frac{1}{H(1)^4}\right] = \frac{18(5+3\mathcal{B})}{5}\frac{\left(H(1)^4 - H(0)^4\right)}{H(0)^4 H(1)^4}, \tag{15}$$

indicating that ΔP_1 may increase, decrease, or not change the total pressure drop of the viscoelastic second-order fluid, depending on the geometry. Specifically, (15) shows that the pressure drop at the first order solely depends on the height of the channel at the inlet and outlet; for $H(1) > H(0)$ the first-order correction leads to an increase in the pressure drop, for $H(1) < H(0)$ to a decrease in the pressure drop, and for $H(1) = H(0)$ there is no first-order in De contribution to the pressure drop. We note that such an increase (decrease) in the pressure drop for $H(1) > H(0)$ ($H(1) < H(0)$) is in qualitative agreement with two-dimensional numerical simulations using the Oldroyd-B model for abruptly expanding (contracting) channels [40]. We expect the higher-order corrections to depend on the channel curvature and have a significant contribution to the pressure drop, similar to the recent studies of squirmers at low Deborah (or Weissenberg) number that reported large changes to the results when higher-order corrections were taken into account [28, 30, 41].

B. Shear thinning: Carreau fluid

Shear-thinning fluids exhibit a decrease in viscosity with applied shear rate. The Carreau model is an example of a generalized Newtonian model with $\boldsymbol{\tau} = 2\tilde{\eta}(\dot{\gamma})\boldsymbol{E}$, which reproduces the realistic rheological behavior of shear-thinning fluids over the entire range of shear rates. The constitutive equation for the Carreau model is $\boldsymbol{\tau} = 2\tilde{\eta}(\dot{\gamma})\boldsymbol{E} = 2\left[\eta_\infty + (\eta_0 - \eta_\infty)\left(1+(\lambda\dot{\gamma})^2\right)^{(n-1)/2}\right]\boldsymbol{E}$ [37], where $\dot{\gamma} = \sqrt{2\boldsymbol{E}:\boldsymbol{E}}$ is the shear rate, and η_0 and η_∞ are the zero- and infinite-shear-rate viscosities, respectively. The power-law index n characterizes the degree of shear thinning ($0 < n < 1$) and λ is the inverse of a characteristic shear rate at which shear thinning becomes apparent. The Carreau model can be written in the form of (2) as

$$\boldsymbol{\tau} = 2\eta_0\boldsymbol{E} + \boldsymbol{A} \quad \text{with} \quad \boldsymbol{A} = -2(\eta_0 - \eta_\infty)\left[1 - \left(1+(\lambda\dot{\gamma})^2\right)^{(n-1)/2}\right]\boldsymbol{E}. \tag{16}$$

Using (3) and performing order-of-magnitude analysis, we obtain the form of the shear rate, $\dot{\gamma}^2/(q^2/h_0^4) = (\partial U/\partial Y)^2 + O(\epsilon^2)$, and $\mathcal{A}_{xx}$ scales as $O(\epsilon^2)$. This implies that the non-Newtonian contribution to the pressure drop of the Carreau fluid arises solely from shearing effects ($\mathcal{A}_{xy}\partial\hat{U}/\partial Y$), as represented by the last term in (11), and $\mathcal{A}_{xx}$ does not contribute to the pressure drop up to $O(\epsilon^2)$. We therefore calculate only the $\mathcal{A}_{xy}$ component of the deviatoric stress and find

$$\mathcal{A}_{xy} = -(1-\beta)\left[1 - \left(1 + \mathrm{Cu}^2\left(\frac{\partial U}{\partial Y}\right)^2\right)^{(n-1)/2}\right]\frac{\partial U}{\partial Y} + O(\epsilon^2), \tag{17}$$

where $\beta = \eta_\infty/\eta_0$ and $\mathrm{Cu} = \lambda q/h_0^2$ is the Carreau number, which is the ratio of the characteristic shear rate in the flow, q/h_0^2, to the cross-over shear rate in the fluid, $1/\lambda$.

Considering the weakly shear-thinning limit, corresponding to $\mathrm{Cu}^2 \ll 1$, where now $\alpha = \mathrm{Cu}^2$, we expand the velocity, pressure, and $\mathcal{A}_{xy}$ in powers of Cu^2, similarly to the previous sections, and using (10) and (17) obtain

$$\mathcal{A}_{xy,0} = -\frac{1}{2}(1-\beta)(1-n)\left(\frac{\partial U_0}{\partial Y}\right)^3 = 108(1-\beta)(1-n)\frac{\left(2Y-H(X)\right)^3}{H(X)^9}. \tag{18}$$

Substituting (18) into (11) and recalling that the terms involving $\mathcal{A}_{xx}$ scale as $O(\epsilon^2)$, and thus are negligible at this order, provides an analytical expression for the first-order pressure drop correction of the Carreau fluid:

$$\Delta P_1 = \int_0^1 \int_0^{H(X)} \mathcal{A}_{xy,0} \frac{\partial \hat{U}}{\partial Y} \mathrm{d}Y \mathrm{d}X = -\frac{648}{5}(1-\beta)(1-n) \int_0^1 \frac{\mathrm{d}X}{H(X)^7}. \tag{19}$$

Since in a shear-thinning fluid, we have $n < 1$ and $\beta < 1$ and the integrand is always positive, (19) indicates that ΔP_1 is always negative and thus reduces the total pressure drop of shear-thinning fluids at low Carreau number. The decrease in the pressure drop is anticipated for shear-thinning fluids and is associated with the reduction in viscosity $\tilde{\eta}(\dot{\gamma})$ with the applied shear rate $\dot{\gamma} \sim q/h_0^2$. Such decrease can be obtained from scaling analysis $\Delta p \sim \tilde{\eta}(q/h_0^2) q\ell/h_0^3$, clearly showing that the pressure drop always decreases. This is in contrast to viscoelastic fluids, where the first-order correction may enhance, reduce, or not affect the total pressure drop depending on the geometry. We note that our findings are in qualitative agreement with the results of Akbar and Nadeem [42], who determined the $q - \Delta p$ relation for a Carreau fluid in a nonuniform cylinder, considering the low-Carreau-number limit and calculating the detailed first-order flow field. Furthermore, for a straight channel, our results agree with the $q - \Delta p$ expression recently derived by Boyko and Stone [43] in the small-Cu limit. Note, however, that in that study the Carreau number was based on the Δp rather than q, and the pressure and flow rate were normalized differently.

V. CONCLUDING REMARKS

In this Letter, we presented a general framework that employs the reciprocal theorem to obtain the flow rate$-$pressure drop relation for complex fluids in narrow channels of arbitrary shape. Our approach applies to a wide class of shear-thinning and viscoelastic constitutive models in the weakly non-Newtonian limit and allows finding the first-order $q - \Delta p$ correction, bypassing the detailed calculations of the non-Newtonian flow problem and relying only on the corresponding Newtonian solution. Our approach is not limited to the case of two-dimensional channels and can be applied to calculate the first-order pressure drop correction in narrow axisymmetric and three-dimensional geometries provided the Newtonian flow solution is known. In fact, our results directly apply to narrow and shallow three-dimensional channels in which $h \ll w \ll \ell$, to the leading order in $\epsilon = h/\ell \ll 1$ and $\delta = h/w \ll 1$, where w is the width of the channel. Furthermore, our method is not restricted to calculating the first-order pressure drop correction and, in the weakly non-Newtonian limit, can be utilized to determine the pressure drop at higher orders only with the knowledge of the velocity and stress fields at the previous orders.

The dependence of the pressure drop on the flow rate of non-Newtonian fluids in nonuniform geometries, such as contracting and expanding channels, is widely studied experimentally and numerically in the fluid mechanics and rheology communities [5–7, 40, 44–55]. Therefore, it would be interesting, as a future direction, to use our theoretical method and calculate the higher-order terms of pressure drop for a particular rheological model and compare it with the available experimental and numerical data. Given the inability of numerical simulations using the elastic dumbbell models such as the Oldroyd-B and FENE-CR models to predict the experimental $q - \Delta p$ behavior of viscoelastic fluids in some cases (see e.g., discussions in [48, 50]), such comparison is of fundamental importance as it may provide insight into the cause of this disagreement.

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