# Mathematical framework for analysis of internal energy dynamics and spectral distribution in compressible turbulent flows

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Phys. Rev. Fluids **4**, 042601 — Published 5 April 2019

DOI: 10.1103/PhysRevFluids.4.042601

# A mathematical framework for analysis of internal energy dynamics and spectral distribution in compressible turbulent flows

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The objective of this work is to develop the framework and governing equations needed to analyze internal energy interactions and its spectral distribution in a manner similar to that of kinetic energy analysis. To emulate the role of velocity in kinetic energy analysis, a new variable ($\phi \sim \sqrt{p}$ where $p$ is pressure) is introduced to enable the examination of internal energy dynamics. Evolution equations for the mean and fluctuating components of $\phi$ are derived. These equations enable precise examination of mean–turbulent flow internal energy interactions, internal–kinetic energy exchange and spectral distribution of internal energy.

Kinetic energy dynamics such as inter-scale transfer and spectral distribution are key features of turbulence and have been the subject of several investigations over many decades [1]. In incompressible flows, wherein density is constant and uniform, the velocity ($u_i$) evolution equation, Navier-Stokes or the momentum conservation equation, forms the basis of kinetic energy analysis. For example, scale-to-scale energy transfer characteristics are dictated by the triadic interactions incumbent in the advective term of the spectral Navier-Stokes equation [2, 3]. The kinetic energy spectrum is computed from Fourier-transforming the auto-covariance function of the velocity field [4].

The advent of compressibility renders the kinetic energy dynamics more complicated due to two factors: (i) spatio-temporal variations in density; and (ii) interactions with internal energy. Some of the complexities in kinetic energy dynamics due to density variations can be adequately addressed by considering density-weighted velocity field – $\rho u_i$ leading to Favre-averaging [5]. But investigation of scale-to-scale transfer and spectral distribution require additional considerations. A new variable $\sqrt{\rho} u_i$, first proposed by Kida and Orszag [6], has been utilized in many works for computing kinetic energy spectrum in compressible turbulence [6–9].

The focus of this paper is on the investigation of internal energy dynamics and its interaction with kinetic energy. It is desirable to investigate and characterize internal energy interactions in turbulence in a manner similar to that of kinetic energy analysis. Many studies perform insightful investigations of turbulent fluctuations of pressure [10–13]. However, such studies do not address internal energy spectral distribution or transfer in a manner similar to that of kinetic energy. For example, the scale-to-scale transfer of internal energy cannot be computed from the advection term in the spectral pressure equation. Further, the turbulent internal energy spectrum is not the same as the turbulent pressure spectrum. To overcome these limitations, Miura and Kida [14] propose a new internal energy variable ($\phi \sim \sqrt{p}$) corresponding to velocity for investigating the internal energy spectrum.

The objective of this paper is to extend the proposal of Miura and Kida [14] and develop the mathematical framework and evolution equations required for performing a comprehensive and rigorous analysis of turbulent internal energy dynamics based on $\phi$. Similar to kinetic energy analysis, we aim at partitioning the internal energy into two parts corresponding to contribution of the mean field and the fluctuating field. The equations based on $\phi$ will enable rigorous investigations of (i) internal–kinetic energy interactions at each scale; (ii) mean–turbulent internal energy exchange; and (iii) spectral distribution of internal and total energies in a turbulent flow field. The proposed framework is very important for a comprehensive understanding of total energy dynamics and spectral energy distribution in high-speed, compressible transition and turbulent flows.

The compressible Navier-Stokes equations for a calorically perfect ideal gas form the basis of this analysis:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \tag{1a}$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{1b}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) + \frac{\partial}{\partial x_j}\left(\frac{p u_j}{\gamma-1}\right) = \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) - p\frac{\partial u_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j}, \tag{1c}$$

$$p = \rho R T, \tag{1d}$$

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where $\rho$ is the fluid density, $u_i$ is velocity component, $\tau_{ij}$ is the viscous stress tensor, $p$ is the gas pressure, $\gamma$ is ratio of specific heats at constant pressure and volume, $\kappa$ is the coefficient of thermal conductivity, $T$ is the temperature, $x_i$ is the spacial coordinate and $t$ is time. The viscous stress tensor is given by:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij}, \tag{2}$$

where $\mu$ is the coefficient of dynamic viscosity, $\lambda$ is the coefficient of second viscosity and $\delta_{ij}$ is the Kronecker delta.

We will first present the kinetic ($K$), internal ($e$) and total ($E$) energy equations to identify the key interactions. The energy equations can be derived using eqs. (1b) and (1c):

$$\frac{\partial K}{\partial t} + \frac{\partial (K u_k)}{\partial x_k} + \frac{\partial}{\partial x_k}\Big[p u_k - \tau_{ik} u_i\Big] = p\frac{\partial u_k}{\partial x_k} - \tau_{ij}\frac{\partial u_i}{\partial x_j}, \tag{3a}$$

$$\frac{\partial e}{\partial t} + \frac{\partial (e u_k)}{\partial x_k} + \frac{\partial q_k}{\partial x_k} = -p\frac{\partial u_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j}, \tag{3b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial (E u_k)}{\partial x_k} + \frac{\partial}{\partial x_k}\Big[p u_k - \tau_{ik} u_i + q_k\Big] = 0, \tag{3c}$$

where,

$$K = \frac{1}{2}\rho u_i u_i, \qquad e = \frac{p}{\gamma - 1}, \qquad q_k = -\kappa\frac{\partial T}{\partial x_k} \qquad \text{and} \qquad E = K + e. \tag{4}$$

Equations (3a) and (3b) indicate interactions between kinetic and internal energy via pressure-dilatation and viscous action. Pressure-dilatation permits a two-way exchange while viscous action can only lead to a one-way transfer from kinetic to internal energy.

The velocity field is decomposed into a Favre average and a corresponding fluctuation field [5]:

$$u_i = \tilde{U}_i + u_i'', \tag{5}$$

while other variables ($\psi$) are decomposed using Reynolds averaging:

$$\psi = \overline{\psi} + \psi'. \tag{6}$$

For some interactions Reynolds averaging of the velocity field is also considered. Throughout the paper $(\tilde{\cdot})$ corresponds to Favre averaging and $(\bar{\cdot})$ to Reynolds averaging. The evolution of the mean kinetic ($\overline{K}$) and internal ($\overline{e}$) energies is given by,

$$\frac{\partial \overline{K}}{\partial t} + \frac{\partial (\tilde{U}_k \overline{K})}{\partial x_k} + \frac{\partial}{\partial x_k}\Big[\overline{K u_k''} - \overline{\tau_{ik} u_i} + \overline{p u_k}\Big] = \overline{p}\frac{\partial \overline{U}_k}{\partial x_k} + \overline{p'\frac{\partial u_k'}{\partial x_k}} - \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} - \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}, \tag{7a}$$

$$\frac{\partial \overline{e}}{\partial t} + \frac{\partial (\tilde{U}_k \overline{e})}{\partial x_k} + \frac{\partial}{\partial x_k}\Big[\overline{e u_k''} + \overline{q_k}\Big] = -\overline{p}\frac{\partial \overline{U}_k}{\partial x_k} - \overline{p'\frac{\partial u_k'}{\partial x_k}} + \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}. \tag{7b}$$

It is important to recognize that the mean kinetic energy of the flow has two contributions: the kinetic energy associated with the mean flow field and that associated with the fluctuations.

$$\overline{K} = K_m + k \qquad \text{where,} \qquad K_m = \frac{1}{2}\overline{\rho}\tilde{U}_i\tilde{U}_i \qquad \text{and} \qquad k = \frac{1}{2}\overline{\rho u_i'' u_i''}. \tag{8}$$

The mean field ($\tilde{U}_i$) kinetic energy is $K_m$ and turbulent field ($u_i''$) kinetic energy is $k$ in eq. (8).

Governing equation for $K_m$ can be derived using the Favre averaged Navier-Stokes equation [15]:

$$\overline{\rho}\frac{\partial \tilde{U}_i}{\partial t} + \overline{\rho}\tilde{U}_j\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \overline{\rho u_i'' u_j''}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j} \tag{9}$$

The equation for the mean-flow kinetic energy ($K_m$) can be obtained by multiplying eq. (9) with $\tilde{U}_i$:

$$\frac{\partial K_m}{\partial t} + \frac{\partial (K_m \tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j}\Big[\overline{\rho u_i'' u_j''}\tilde{U}_i + \overline{p}\overline{U}_j - \overline{\tau_{ij}}\ \overline{U}_i\Big] = \overline{\rho u_i'' u_j''}\frac{\partial \tilde{U}_i}{\partial x_j} + \overline{p}\frac{\partial \overline{U}_k}{\partial x_k} - \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_k''}\frac{\partial \overline{p}}{\partial x_k} - \overline{u_k''}\frac{\partial \overline{\tau_{kj}}}{\partial x_j}. \tag{10}$$

Here we have used the relation $\overline{U}_k = \tilde{U}_k + \overline{u''_k}$ to simplify some terms. The evolution of turbulent kinetic energy ($k$) in compressible flows follows from [16, 17]:

$$\frac{\partial k}{\partial t} + \frac{\partial (k\tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\overline{\rho u''_i u''_i u''_j} + \overline{p' u''_j} - \overline{\tau'_{ij} u''_i}\right] = -\overline{\rho u''_i u''_j}\frac{\partial \tilde{U}_i}{\partial x_j} + \overline{p'\frac{\partial u''_k}{\partial x_k}} - \overline{\tau'_{ij}\frac{\partial u''_i}{\partial x_j}} - \overline{u''_k}\frac{\partial \overline{p}}{\partial x_k} + \overline{u''_k}\frac{\partial \overline{\tau_{kj}}}{\partial x_j}. \tag{11}$$

Utilizing $u''_k = \overline{U}_k - \tilde{U}_k + u'_k$ and $\overline{\psi'} = 0$, we can establish that:

$$\overline{\psi'\frac{\partial u''_k}{\partial x_k}} = \overline{\psi'\frac{\partial u'_k}{\partial x_k}}. \tag{12}$$

Therefore, equation for turbulent kinetic energy ($k$) can be re-written as,

$$\frac{\partial k}{\partial t} + \frac{\partial (k\tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\overline{\rho u''_i u''_i u''_j} + \overline{p' u'_j} - \overline{\tau'_{ij} u'_i}\right] = -\overline{\rho u''_i u''_j}\frac{\partial \tilde{U}_i}{\partial x_j} + \overline{p'\frac{\partial u'_k}{\partial x_k}} - \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}} - \overline{u''_k}\frac{\partial \overline{p}}{\partial x_k} + \overline{u''_k}\frac{\partial \overline{\tau_{kj}}}{\partial x_j}. \tag{13}$$

The various terms in these equations and the incumbent physics will be discussed further below. In incompressible flows, the spectral transfer and distribution of kinetic energy across different scales of motion can be examined by performing Fourier transform of $u'_i$. As proposed by Kida and Orszag [6], similar spectral energy analysis of compressible flows requires a new variable,

$$w_i = \sqrt{\rho} u''_i, \tag{14}$$

to account for spatio-temporal variations in density. The Fourier transform of the auto-correlation function of $w_i$ yields the kinetic energy spectrum. The quartic interactions for turbulent kinetic energy cascade are also reduced to triadic interactions as:

$$\frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho u''_i u''_i u''_j\right] = \frac{\partial}{\partial x_j}\left[\frac{1}{2} w_i w_i u''_j\right]. \tag{15}$$

Unlike kinetic energy, it is more complicated to clearly identify the internal energy associated with the mean (or background field) and the turbulent internal energy. In the linear regime, when the density fluctuations are small, Sarkar *et al.* [10] identified that the potential energy incumbent in pressure fluctuations can be approximated as $\overline{p'p'}/(2\gamma\overline{p})$. However, this representation is inadequate when the density fluctuations are large. Miura and Kida [14] propose a more general approach which is valid for strongly compressible flows. They propose a new variable,

$$\phi \equiv \sqrt{e} = \sqrt{\frac{p}{\gamma - 1}}, \tag{16}$$

for examining the internal energy spectrum in simple isotropic flows. The justification is that $\phi^2$ corresponds to internal energy.

In this study we develop the framework of equations based on $\phi$ to investigate flows with strong mean fields and more importantly to examine internal–kinetic energy interactions. First the choice of $\phi$ as the internal energy equivalent of velocity is further justified on the following physics-based argument. Using the relation $\gamma p = \rho a^2$, where $a$ is the speed of sound, $\phi$ can be written as:

$$\phi = \frac{1}{\sqrt{\gamma(\gamma - 1)}}\sqrt{\rho} a. \tag{17}$$

Eq. (17) shows that $\phi$ includes density-weighted acoustic speed information which relates to the internal energy of the system. The state variable $\phi$ for internal energy is thus analogous to $w_i$ for turbulent kinetic energy. $\phi$ can now be decomposed using Reynolds averaging:

$$\phi = \overline{\phi} + \phi' \quad \text{where,} \quad \overline{\phi} = \overline{\sqrt{\frac{p}{\gamma - 1}}} \quad \text{and} \quad \phi' = \sqrt{\frac{p}{\gamma - 1}} - \overline{\sqrt{\frac{p}{\gamma - 1}}}. \tag{18}$$

With this formulation, it is possible to clearly separate the mean internal energy of the flow into a mean field and a fluctuating field contribution,

$$\overline{e} = \overline{\phi^2} = \overline{\phi}\,\overline{\phi} + \overline{\phi'\phi'} \qquad \text{therefore,} \qquad e_m = \overline{\phi}\,\overline{\phi} \qquad \text{and} \qquad e_t = \overline{\phi'\phi'}. \tag{19}$$

The strong similarity of internal energy decomposition in eq. (19) and kinetic energy partitioning in eq. (8) is clearly evident. The spectral transfer and spectral distribution of turbulent internal energy can be characterized from the behavior $\phi'$ much in the same manner as the turbulent kinetic energy transfer/distribution can be analyzed using $w_i = \sqrt{\rho} u_i''$. The Fourier transform of the auto-correlation of $\phi'$ yields the turbulent internal energy spectrum. Thus, $\phi' - w_i$ interactions hold the key to examining internal–kinetic energy exchange.

In the linear limit, as indicated by small turbulent Mach number - $M_t \equiv u'/a \ll 1$ [10], the pressure fluctuations can be considered small relative to mean pressure. Therefore, higher order moments are negligible $\left(\frac{\overline{p'p'}}{\overline{p}^2} \approx 0,\ \frac{\overline{p'p'p'}}{\overline{p}^3} \approx 0,\ ...\right)$ and can be ignored. $\overline{\phi}$ and $\phi'$ can now be approximated as:

$$\overline{\phi} = \overline{\sqrt{\frac{\overline{p}}{\gamma - 1}}\left[1 + \frac{p'}{\overline{p}}\right]^{1/2}} \approx \sqrt{\frac{\overline{p}}{\gamma - 1}} \quad \text{and} \quad \phi' \approx \frac{p'}{2\sqrt{\overline{p}(\gamma - 1)}} + O\left(\frac{p'p'}{\overline{p}^{3/2}}\right). \tag{20}$$

Then the turbulent internal energy in the linear limit can be shown to be:

$$e_t = \overline{\phi'\phi'} \approx \frac{1}{4(\gamma - 1)}\left[\frac{\overline{p'p'}}{\overline{p}}\right]. \tag{21}$$

The above expression is similar to the linear limit potential energy of pressure fluctuations as identified in literature [10, 18].

The mean and the fluctuation pressure field can be written in terms of the internal energy velocity variable $\phi$ as:

$$\overline{p} = (\gamma - 1)\left[\overline{\phi}\,\overline{\phi} + \overline{\phi'\phi'}\right] \quad \text{and} \quad p' = (\gamma - 1)\left[\phi'\phi' + 2\overline{\phi}\phi' - \overline{\phi'\phi'}\right]. \tag{22}$$

We can clearly see that mean and fluctuating pressure have a linear and nonlinear component:

$$\overline{p} = \overline{p}_m + \overline{p}_t \quad \text{and} \quad p' = p'_m + p'_t \tag{23}$$

where,

$$\begin{aligned} \overline{p}_m &= (\gamma - 1)\overline{\phi}\,\overline{\phi} \quad \text{and} \quad \overline{p}_t = (\gamma - 1)\overline{\phi'\phi'}, \\ p'_m &= 2(\gamma - 1)\overline{\phi}\phi' \quad \text{and} \quad p'_t = (\gamma - 1)\left[\phi'\phi' - \overline{\phi'\phi'}\right]. \end{aligned} \tag{24}$$

Here, $\overline{p}_m$ and $\overline{p}_t$ are the components of mean pressure associated with the mean field and the perturbation field respectively. Similarly, $p'_m$ and $p'_t$ are the components of fluctuating pressure associated with the mean field and the perturbation field. $\overline{p}_m$ and $p'_m$ can also be understood as linear (or rapid) components while $\overline{p}_t$ and $p'_t$ as nonlinear (or slow) components.

We can now derive the evolution equations for $e_m = \overline{\phi}\,\overline{\phi}$ and $e_t = \overline{\phi'\phi'}$. We start with the internal energy equation, eq. (3b) and use definitions $e = \phi^2$ and $p = (\gamma - 1)\phi^2$ to obtain an equation for $\phi$:

$$\frac{\partial \phi}{\partial t} + u_k \frac{\partial \phi}{\partial x_k} = -\frac{\gamma}{2}\phi\frac{\partial u_k}{\partial x_k} + \frac{f}{2\phi} \quad \text{where,} \quad f = \left[-\frac{\partial q_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j}\right]. \tag{25}$$

The Reynolds average of the thermal flux and viscous terms $(f)$ can be written as:

$$\overline{f} = \left[-\frac{\partial \overline{q_k}}{\partial x_k} + \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}}\right]. \tag{26}$$

The last term in eq. (25) can be simplified as:

$$\frac{f}{2\phi} = \frac{f}{2\overline{\phi}} - \frac{f\phi'}{2\overline{\phi}\phi} \quad \text{which gives,} \quad \overline{\left(\frac{f}{2\phi}\right)} = \frac{\overline{f}}{2\overline{\phi}} - \frac{1}{2\overline{\phi}}\overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{27}$$

Reynolds-averaging eq. (25) gives an equation for $\overline{\phi}$ as:

$$\frac{\partial \overline{\phi}}{\partial t} + \tilde{U}_k \frac{\partial \overline{\phi}}{\partial x_k} + \overline{u''_k}\frac{\partial \overline{\phi}}{\partial x_k} + \overline{u''_k\frac{\partial \phi'}{\partial x_k}} = -\frac{\gamma}{2}\left[\overline{\phi}\frac{\partial \tilde{U}_k}{\partial x_k} + \overline{\phi}\frac{\partial \overline{u''_k}}{\partial x_k} + \overline{\phi'\frac{\partial u''_k}{\partial x_k}}\right] + \frac{\overline{f}}{2\overline{\phi}} - \frac{1}{2\overline{\phi}}\overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{28}$$

Multiplying eq. (28) by $2\overline{\phi}$ yields the governing equation for the mean field contribution of mean internal energy ($e_m$). Using $\overline{U}_k = \tilde{U}_k + \overline{u''_k}$ and eq. (26), the simplified equation for $e_m$ becomes:

$$\frac{\partial e_m}{\partial t} + \frac{\partial (e_m \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k}\left[\overline{\phi}\,\overline{\phi}\,\overline{u''_k} + 2\overline{\phi}\,\overline{\phi' u''_k} + \overline{q_k}\right] = \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}} - (\gamma-1)\overline{\phi}\,\overline{\phi}\frac{\partial \overline{U}_k}{\partial x_k} + 2\overline{u''_k\phi'}\frac{\partial \overline{\phi}}{\partial x_k} + (2-\gamma)\overline{\phi}\overline{\phi'\frac{\partial u''_k}{\partial x_k}} - \overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{29}$$

The above equation is further simplified using the expression in eq. (12) and the definitions in eq. (24):

$$\frac{\partial e_m}{\partial t} + \frac{\partial (e_m \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k}\left[\overline{\phi}\,\overline{\phi}\,\overline{u''_k} + 2\overline{\phi}\,\overline{\phi' u''_k} + \overline{q_k}\right] = \overline{\tau_{ij}}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}} - \overline{p}_m\frac{\partial \overline{U}_k}{\partial x_k} + 2\overline{u''_k\phi'}\frac{\partial \overline{\phi}}{\partial x_k} + \frac{(2-\gamma)}{2(\gamma-1)}\overline{p'_m\frac{\partial u'_k}{\partial x_k}} - \overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{30}$$

We can now determine the $\phi'$ equation by subtracting eq. (28) from eq. (25):

$$\frac{\partial \phi'}{\partial t} + \tilde{U}_k\frac{\partial \phi'}{\partial x_k} + u''_k\frac{\partial \overline{\phi}}{\partial x_k} + u''_k\frac{\partial \phi'}{\partial x_k} - \overline{u''_k}\frac{\partial \overline{\phi}}{\partial x_k} - \overline{u''_k\frac{\partial \phi'}{\partial x_k}} = -\frac{\gamma}{2}\left[\phi'\frac{\partial \tilde{U}_k}{\partial x_k} + \overline{\phi}\frac{\partial u''_k}{\partial x_k} + \phi'\frac{\partial u''_k}{\partial x_k} - \overline{\phi}\frac{\partial \overline{u''_k}}{\partial x_k} - \overline{\phi'\frac{\partial u''_k}{\partial x_k}}\right] + \left(\frac{f}{2\phi}\right)'. \tag{31}$$

To obtain the governing equation for turbulent internal energy ($e_t$) we multiply eq. (31) with $2\phi'$ and take average:

$$\frac{\partial e_t}{\partial t} + \tilde{U}_k\frac{\partial e_t}{\partial x_k} + 2\overline{\phi' u''_k}\frac{\partial \overline{\phi}}{\partial x_k} + \overline{u''_k\frac{\partial \phi'\phi'}{\partial x_k}} = -\gamma\left[\overline{\phi'\phi'}\frac{\partial \tilde{U}_k}{\partial x_k} + \overline{\phi}\,\overline{\phi'\frac{\partial u''_k}{\partial x_k}} + \overline{\phi'\phi'\frac{\partial u''_k}{\partial x_k}}\right] + \overline{\left(\frac{f}{\phi}\right)'\phi'}. \tag{32}$$

The last term in eq. (32) can be simplified as:

$$\overline{\left(\frac{f}{\phi}\right)'\phi'} = \overline{\left[\overline{\left(\frac{f}{\phi}\right)}\phi' + \left(\frac{f}{\phi}\right)'\phi'\right]} = \overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{33}$$

Simplifying eq. (32) and using the relation, $\tilde{U}_k = \overline{U}_k - \overline{u''_k}$ we get,

$$\frac{\partial e_t}{\partial t} + \frac{\partial (e_t \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k}[\overline{\phi'\phi' u''_k}] = -(\gamma-1)\overline{\phi'\phi'}\frac{\partial \overline{U}_k}{\partial x_k} - (\gamma-1)\overline{(\phi'\phi' + 2\overline{\phi}\phi' - \overline{\phi'\phi'})\frac{\partial u''_k}{\partial x_k}} - 2\overline{u''_k\phi'}\frac{\partial \overline{\phi}}{\partial x_k} - (2-\gamma)\overline{\phi}\overline{\phi'\frac{\partial u''_k}{\partial x_k}} + \overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{34}$$

Now we can use the relations in eqs. (12), (22) and (24) to obtain the simplified form of the above equation:

$$\frac{\partial e_t}{\partial t} + \frac{\partial (e_t \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k}[\overline{\phi'\phi' u''_k}] = -\overline{p}_t\frac{\partial \overline{U}_k}{\partial x_k} - \overline{p'\frac{\partial u'_k}{\partial x_k}} - 2\overline{u''_k\phi'}\frac{\partial \overline{\phi}}{\partial x_k} - \frac{(2-\gamma)}{2(\gamma-1)}\overline{p'_m\frac{\partial u'_k}{\partial x_k}} + \overline{\left(\frac{f\phi'}{\phi}\right)}. \tag{35}$$

The triadic interactions of turbulent internal energy cascade can now be analyzed similar to turbulent kinetic energy via $\frac{\partial}{\partial x_k}(\phi'\phi' u''_k)$. The equations for $\overline{\phi}$, $e_m$, $\phi'$ and $e_t$ in eqs. (28), (30), (31) and (35) constitute the framework required to describe internal energy interactions.

The mean total energy ($\overline{E} = K_m + k + e_m + e_t$) obtained through this framework can be written as:

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x_k}\left[\overline{E}\tilde{U}_k + \overline{\rho u''_i u''_k}\tilde{U}_i + \frac{1}{2}\overline{\rho u''_i u''_i u''_k} + \overline{p u_k} - \overline{\tau_{ik} u_k} + \overline{q_k} + \frac{\overline{p u''_k}}{\gamma - 1}\right] = 0. \tag{36}$$

The mean total energy ($\overline{E}$) of a system changes only due to energy flux from the outside. Therefore, $\overline{E}$ for an isolated system remains constant. This is an important property of the chosen variables.

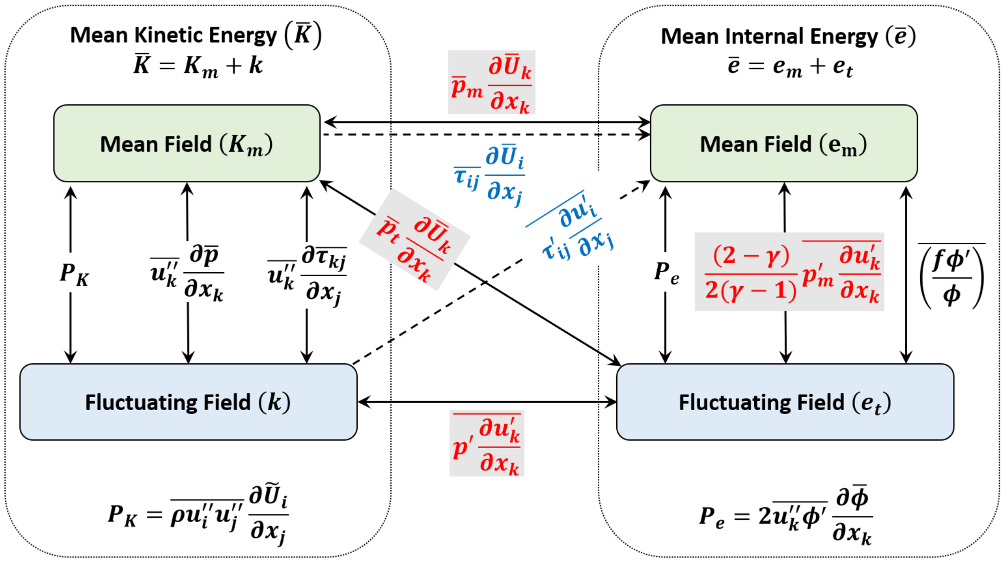


FIG. 1: Interactions between kinetic and internal energies in the nonlinear regime using eqs. (10,13,30,35)

We now discuss important aspects of kinetic–internal energy interactions and summarize the exchanges amongst $K_m$, $k$, $e_m$ and $e_t$ (figure 1).

**$(K_m - k)$ Interactions:** The most important interaction between mean–turbulent kinetic energy is production of turbulent kinetic energy $(k)$ via $\overline{\rho u''_i u''_j} \frac{\partial \tilde{U}_i}{\partial x_j}$ as seen in eqs. (10) and (13). Production draws energy from the mean flow and deposits it into the fluctuating field. Other interactions appear as a consequence of compressibility and Favre-averaging. Both pressure work $(\overline{u''_k} \frac{\partial \bar{p}}{\partial x_k})$ and viscous action $(\overline{u''_k} \frac{\partial \overline{\tau_{kj}}}{\partial x_j})$ can lead to two-way exchange between the mean and fluctuating fields. For an incompressible flow, only production remains active among all $(K_m - k)$ interactions.

**$(e_m - e_t)$ Interactions:** The terms $2\overline{u''_k \phi'} \frac{\partial \bar{\phi}}{\partial x_k}$, $\frac{(2-\gamma)}{2(\gamma-1)} \overline{p'_m \frac{\partial u'_k}{\partial x_k}}$ and $(\overline{\frac{f\phi'}{\phi}})$ in eqs. (30) and (35) lead to exchange between the mean field internal energy $(e_m)$ and the turbulent internal energy $(e_t)$. The first term is analogous to production in the kinetic energy equations. The second term causes exchange due to the linear or rapid component of fluctuating field pressure-dilatation mechanism while the last term is exchange due to viscous and thermal flux action.

**$(K_m - e_m)$ Interactions:** The mean field kinetic–internal energy interact via the linear or rapid component of mean field pressure-dilatation $(\bar{p}_m \frac{\partial \bar{U}_k}{\partial x_k})$ and viscous action $(\overline{\tau_{ij}} \frac{\partial \bar{U}_i}{\partial x_j})$ as evident in eqs. (10) and (30). The pressure-dilatation action causes a two-way exchange and is represented with a double solid arrow in figure 1. On the other hand, viscous action dissipates mean field kinetic energy to mean field internal energy and is shown with a dashed one-way arrow. As mentioned by Huang *et al.* [16], the Reynolds-averaged velocity field $(\bar{U}_i)$ contributes towards the internal–kinetic exchange and not the Favre-averaged mean field $(\tilde{U}_i)$.

**$(K_m - e_t)$ Interactions:** The mean field kinetic energy and turbulent internal energy only interact via the nonlinear or slow component of mean field pressure-dilatation $(\bar{p}_t \frac{\partial \bar{U}_k}{\partial x_k})$. This exchange is identified using eqs. (10) and (35).

**$(k - e_m)$ Interactions:** The turbulent kinetic energy dissipates energy into the mean field internal energy via the fluctuating field viscous action $(\overline{\tau'_{ij} \frac{\partial u'_i}{\partial x_j}})$ as seen from eqs. (13) and (30).

**$(k - e_t)$ Interactions:** The exchange between turbulent kinetic and turbulent internal energy is caused via the fluctuating field pressure-dilatation mechanism $(\overline{p' \frac{\partial u'_k}{\partial x_k}})$ as evident in eqs. (13) and (35). This exchange includes contribution of both linear (or rapid) and nonlinear (or slow) pressure fluctuations. It is also synchronous with $(e_m - e_t)$ exchange via the linear or rapid component of fluctuating field pressure-dilatation.

In conclusion, we have established a framework for examining internal energy of any fluid flow governed by the compressible Navier-Stokes equations. Analogous to fluid velocity, a suitable thermodynamic state variable for internal energy is identified $(\phi \sim \sqrt{p})$. Defining internal energy in terms of $\phi$ allows partitioning of the mean internal energy of the flow into the mean field and perturbation field contributions, similar to mean kinetic energy. It is also established that spectral behavior of turbulent internal energy can be rigorously examined using $\phi'$. Governing equations for $e_m$

and $e_t$ are derived to identify key internal–kinetic energy interactions. It is anticipated that this framework will be important for analysis and modeling of energy dynamics in high-speed compressible transition and turbulent flows.

## ACKNOWLEDGMENTS

The authors acknowledge AFOSR for their financial support through the project 'Nonlinear growth and breakdown toward turbulence in Hypersonic Boundary layers: Investigation of fundamental physical mechanisms' – Grant number FA9550-16-1-0360.


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