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# On turbulent flow over a long flat plate with uniform roughness

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For turbulent boundary-layer flow under a uniform free-stream speed  $U_{\infty}$  over a plate of length L, covered with uniform roughness of nominal sand-grain scale  $k_s$ , the physical behaviors underlying two distinguished limits at large  $Re_L \equiv U_{\infty}L/\nu$  are explored: the fullyrough wall flow where  $k_s/L$  is fixed and the long-plate limit where  $Re_k \equiv U_{\infty}k_s/\nu$  is fixed. For the fully-rough limit it is shown that not only is the drag coefficient  $C_D$  independent of  $Re_L$  but that universal skin-friction coefficient  $C_f$  and normalised boundary-layer thickness  $\delta/k_s$  can be found that depends only on  $k_s/x$ , where x is the downstream distance. In the long-plate limit, it is shown that the flow becomes asymptotically smooth at huge  $Re_L$  at a rate that depends on  $Re_k$ . Comparisons with wind-tunnel and field data are made.

# I. INTRODUCTION

The large Reynolds number limit for internal flows such as turbulent flow through a long, rough-wall pipe are well characterized through the experiments of Nikuradse [1] and expressed via the Moody diagram [2]. For pipe and open-channel flow, when the flow is fully developed, turbulent, and fully rough in the sense that  $k_s^+ \equiv k_s u_\tau/\nu$  (where  $k_s$  is the nominal sand-grain scale,  $u_\tau = \sqrt{\tau_w/\rho}$  and  $\tau_w$  is the wall drag per unit area for the rough surface) is sufficiently large, the time-averaged friction factor or skin-friction coefficient becomes asymptotically independent of Reynolds number at large values, and depends only on the ratio of some measure of the roughness scale to either the pipe radius or the channel half height. Traditionally the concept of sand-grain roughness has been utilized for a given surface by determining an equivalent sand-grain roughness scale for which the skin-friction matches classical experimental sand-grain surface measurements [1]. For transitionally rough flows, where typically  $100 > k_s^+ > 5$ , the Hama [3] velocity correction can depend strongly on the surface roughness profile; see Jiménez [4].

There has been less attention to the effects of surface roughness at very large Reynolds number for canonical external flows such as the zero-pressure gradient flat plate boundary layer in the presence of uniform or variable roughness. For turbulent flow over a long flat plate of length L and free-stream speed  $U_{\infty}$ , calculations of the drag coefficient were reported by Prandtl and Schlichting [5, 6]. They used a piecewise model of roughness variation with  $k_s^+$  containing regions corresponding to smooth, transitional and fully rough flow. The large  $Re_L \equiv U_{\infty}L/\nu$  limit has been subsequently considered by several authors [7–10]. Granville [7] found that the integrated drag coefficient  $C_D \equiv (1/L) \int_0^L C_f dx$  was independent of  $Re_L$  when this was large. The universality of mean-velocity profiles, skin friction, and some integral parameters for boundary layers over a variety of rough surfaces was studied analytically, assuming fully rough conditions, and experimentally by Castro [9]. Presently we revisit this flow using an approach that is rather simpler than the methods of Prandtl and Schlichting and Granville and is designed to provide an analytical interpretation of interesting and perhaps unexpected flow behavior in the large Reynolds number limit.

# II. FLOW OVER A ROUGH FLAT PLATE

We consider flow of a turbulent boundary layer over a flat plate of length L with stream-wise distance from the leading edge  $0 \le x \le L$ . The plate surface is covered with roughness whose height distribution above a mean value z = 0 can be described as a random function of x and span-wise distance y, that is isotropic and homogeneous. For the purposes of this analysis we consider the roughness to be characterized hydrodynamically by a single length-scale  $k_s$  which is identified as the equivalent sand-grain roughness. In order to obtain an integrated description of smooth to fully rough wall flow, a Colebrook-type roughness function is used to describe the effects of roughness from the smooth through the transitional and fully rough-wall flow regimes. A principal parameter will be  $Re_L$ . This will be assumed to be sufficiently large that the prior laminar boundary layer and laminar-turbulent transition regions, typically in the range  $0 \le Re_x \le 5 \times 10^5$ , can be ignored.

#### A. Mean velocity profile

We assume that the velocity profile within the boundary layer at any stream-wise station x is given by the classical log-wake relationship

$$\frac{u}{u_{\tau}} = \frac{1}{\kappa} \log\left(\frac{z \, u_{\tau}}{\nu}\right) + \frac{\Pi}{\kappa} W\left(\frac{z}{\delta}\right) - \Delta U^{+}\left(\frac{k_{s} \, u_{\tau}}{\nu}\right) + A,\tag{1}$$

where  $\kappa$  is the Kármán constant, z a suitably defined wall-normal distance, A an offset constant, W the wake function,  $\Pi$  the Coles wake factor [11]. In (1), both  $u_{\tau}$  and  $\delta$  are functions of x and  $\Delta U(k_s^+)$  is a roughness function that quantifies the effect of surface roughness on the mean velocity profile. Equation (1) does not include a description of the mean velocity variation in the viscous sub-layer, the buffer layer or a possible roughness sub-layer. The contribution to mass and momentum transport across the whole boundary layer from these regions is expected to be small when  $Re_L$  is large, and so will not be included in the analysis to follow. We assume a Colebrook form for transitionally rough conditions

$$\Delta U^+(k_s^+) = \frac{1}{\kappa} \log\left(1 + \beta k_s^+\right),\tag{2}$$

When  $k_s^+ \to \infty$  the choice  $\beta = e^{\kappa (A-B)}$  allows matching to the usual fully-rough form  $\Delta U^+(k_s^+) = (1/\kappa) \log (k_s^+) + A - B$ .

The length scale  $\delta$  is defined such that

$$S \equiv \frac{U_{\infty}}{u_{\tau}} \equiv \sqrt{\frac{2}{C_f}} = \frac{1}{\kappa} \log\left(\frac{\delta u_{\tau}}{\nu}\right) + \frac{2\Pi}{\kappa} - \Delta U^+\left(\frac{k_s u_{\tau}}{\nu}\right) + A,\tag{3}$$

where W(1) = 2 by definition. For simplicity we use a simple model of the wake function  $W(z/\delta) = 2 \sin^2 [\pi z/(2\delta)]$ . Using  $k_s u_\tau/\nu = (k_s/x) Re_x/S$  and  $\delta u_\tau/\nu \equiv Re_\delta/S$ , where  $Re_\delta \equiv U_\infty \delta/\nu$ ,  $Re_k \equiv U_\infty k_s/\nu$  and  $Re_x \equiv U_\infty x/\nu = Re_k x/k_s$ , we can solve for  $Re_\delta$  from (3) as

$$Re_{\delta} = S e^{\kappa (S-A) - 2\Pi} + Re_k e^{\kappa (S-B) - 2\Pi}.$$
(4)

## B. Momentum thickness and Kármán integral relation

The momentum thickness  $\theta$  is

$$\theta = \int_0^\delta \frac{u}{U_\infty} \left( 1 - \frac{u}{U_\infty} \right) \, dz. \tag{5}$$

Using (1) in (5) and integrating then gives

$$Re_{\theta} = Re_{\delta}(S) F(S), \quad F = \frac{\kappa S - 2 - \frac{3}{2} \Pi^2 + \Pi(\kappa S - Q)}{\kappa^2 S^2}$$
(6)

where  $Q = (2/\pi)[\pi + \text{Si}(\pi)]$  and  $\text{Si}(\pi) \equiv \int_0^{\pi} [(\sin z)/z] dz = 1.85194$  and hence Q = 3.17898. As noted earlier the above ignores the contribution of the viscous sub-layer, the buffer layer and/or a roughness sublayer.

For a zero pressure-gradient boundary layer, the Kármán integral relation can be written as

$$\frac{d\,Re_{\theta}}{d\,Re_{x}} = \frac{1}{S^{2}},\tag{7}$$

where  $Re_{\theta} \equiv U_{\infty} \theta / \nu$ . Considering  $Re_{\theta} = Re_{\theta}(S(Re_x))$ , using the chain rule for differentiation and (6), and integrating with  $Re_k$  and other parameters fixed, we find that

$$Re_{x} - Re_{x_{0}} = \int_{S_{0}}^{S} S^{\prime 2} \frac{d}{dS^{\prime}} \left[ Re_{\delta}(S^{\prime}) F(S^{\prime}) \right] dS^{\prime}.$$
 (8)

The choice  $S_0 = 0$  gives a divergent integral. At the cost of considerable complexity, this can be resolved by matching to a prior laminar boundary layer at some transition point  $Re_{x_0} = O(10^5)$ . Instead, we use a simple cutoff  $S_0 = O(1)$ . Further, the contribution from the limit of integration  $S = S_0$  can also be shown to be small when  $Re_x \gg O(10^5)$ . Since this can be expected to have negligible effect on integrated quantities when  $Re_L = O(10^8)$ , this will also be neglected. Alternative methods for handling the singularity have been used: see for example Castro [9] equation (3.2).

Integrating (8) with  $Re_k$  fixed and neglecting the contribution from  $S = S_0$ , using  $Re_k = (k_s/x) Re_x$  then gives

$$Re_x = \frac{\mathrm{e}^{\kappa \, (S-A)-2\,\Pi}}{\kappa^3} \, K_1(S) + Re_x \left(\frac{k_s}{x}\right) \, \frac{\mathrm{e}^{\kappa \, (S-B)-2\,\Pi}}{\kappa^2} \, K_2(S),\tag{9}$$

$$K_1 = \Pi^2 \left(3 - \frac{3}{2}\kappa S\right) + \Pi \left(2 - 2\kappa S + \kappa^2 S^2 + Q \left(2 - \kappa S\right)\right) + (6 - 4\kappa S + \kappa^2 S^2), \tag{10}$$

$$K_2 = (\kappa S - 4 - \Pi (2 - \kappa S + Q) - \frac{3}{2} \Pi^2) + (4 + 2 \Pi Q + 3 \Pi^2) \operatorname{Ei}(\kappa S) e^{-\kappa S},$$
(11)

where  $\operatorname{Ei}(x) = -\int_{-x}^{\infty} e^{-t}/t \, dt$  is the exponential integral. If  $\kappa, A, B, \Pi$  are specified, (9) provides a relation between  $(Re_x, k_s/x, S)$  or alternatively  $(Re_x, Re_k, S)$ . Calculations for specific cases are straightforward. All discussed subsequently were performed using the symbolic manipulator Mathematica which provides special function capability for accurate calculation of  $\operatorname{Ei}(x)$ . As a check, some particular cases were calculated using asymptotic forms of  $\operatorname{Ei}(x)$ . Figure 1 shows resulting solutions from (9) for lines of constant  $Re_k$  (black curves) and lines of constant  $\epsilon = k_s/x$ (blue curves) using  $\kappa = 0.384$ , A = 4.17, B = 8.5 and  $\Pi = 0.53$ .

Note that for a homogeneously distributed roughness of unvarying  $k_s$  along a flat plate, the black lines denote fixed unit Reynolds number  $(U_{\infty}/\nu)$  and increasing x, while the blue curves denote fixed x and increasing unit Reynolds number.



FIG. 1: Local friction coefficient  $C_f$  versus  $Re_x$  for fixed  $\epsilon = k_s/x$  and fixed  $Re_k$ . Blue solid lines; top to bottom:  $\epsilon = 10^{-3}$ ,  $3 \times 10^{-4}$ ,  $10^{-4}$ ,  $3 \times 10^{-5}$ ,  $10^{-5}$ ,  $3 \times 10^{-6}$ ,  $10^{-6}$ . Black lines; top to bottom:  $Re_k = 3 \times 10^4$ ,  $10^4$ ,  $3 \times 10^3$ ,  $10^3$ ,  $3 \times 10^2$ ,  $10^2$ , 30. Red curve shows the smooth wall ( $Re_k = 0$ ).

#### C. Drag coefficient

The drag coefficient for a plate of length L is

$$C_D = \frac{1}{L} \int_0^L C_f \, dx = \frac{1}{L} \int_0^L \frac{2}{S^2} \, dx = \frac{2}{Re_L} \int_0^{Re_L} \frac{1}{S^2} dRe_x. \tag{12}$$

Using (7) this can be written in the form

$$C_D = \frac{2}{Re_L} \int_0^{Re_L} \frac{dRe_\theta}{dRe_x} dRe_x = 2 \frac{Re_\theta(S_L)}{Re_L},$$
(13)

where we have used  $\theta(x = 0) = 0$  and defined  $S_L \equiv S(Re_L)$ . Utilizing (6) and again using that  $Re_k = (k_s/L) Re_L$  we obtain an explicit formula for  $C_D$ 

$$C_D = 2\frac{1}{Re_L} \left( S_L e^{\kappa (S_L - A) - 2\Pi} + Re_L \left( \frac{k_s}{L} \right) e^{\kappa (S_L - B) - 2\Pi} \right) \frac{\kappa S_L - 2 - \frac{3}{2} \Pi^2 + \Pi(\kappa S_L - Q)}{\kappa^2 S_L^2}.$$
(14)

# **III. TWO LIMITING CASES**

Two distinct limits are of interest. These are referred to as the *fully rough-wall* and *long-plate limits* respectively. For most practical applications at large but finite  $Re_L = O(10^8 - 10^{10})$  and typical values of  $k_s/L$ , the rough-wall limit is of most interest. We consider the kinematic viscosity  $\nu$  to be fixed.

## A. Fully rough-wall flow



FIG. 2: Schematic demonstrating development of boundary layer along a hull for various different unit Reynolds number, under (a) smooth and (b) fully rough conditions. Lighter shaded profiles denote higher free-stream velocities. Homogenously distributed roughness of constant  $k_s$  is assumed for the fully rough case. Boundary layer growth is exaggerated and indicative only.

First let  $Re_x \to \infty$  by increasing the unit Reynolds number  $U_{\infty}/\nu$  at fixed  $\epsilon$ . Here  $\epsilon Re_x \to \infty$ and the first term on the right-hand side of (9) can then be neglected. This gives the rough-wall limit

$$\frac{1}{\epsilon} \equiv \frac{x}{k_s} = \frac{\mathrm{e}^{\kappa\,(S-B)-2\,\Pi}}{\kappa^2} \,K_2(S),\tag{15}$$

This equation shows explicitly that in the fully rough limit, S is a function of  $k_s/x$  only, or since  $S^2 = 2/C_f$ , the local skin friction coefficient is only a function of  $k_s/x$  and is independent of  $Re_x$ . Solutions to (15) are plotted on figure 1 as blue dashed lines.

Similarly, the rough-wall limit of (4) can be taken by letting  $\epsilon Re_x \to \infty$  with  $\epsilon$  fixed to give  $\delta = k_s \exp[\kappa (S - B) - 2\Pi]$  or equivalently the well known form,

$$S = \frac{1}{\kappa} \left( \log \left( \frac{\delta}{k_s} \right) + 2 \Pi \right) + B.$$
(16)

Recall that (15) has already shown that in the rough wall limit, S (and hence  $C_f$ ) are invariant with unit Reynolds number at fixed  $k_s/x$  (see blue dashed curves on figure 1). Combining this result with (16), demonstrates that in the fully rough limit, for fixed  $k_s/x$ ,  $\delta/x$  must also be constant, i.e.

$$\frac{\delta}{x} = \frac{\delta}{k_s} \left( S\left(\frac{k_s}{x}\right) \right) \cdot \frac{k_s}{x} = G\left(\frac{k_s}{x}\right). \tag{17}$$

In practise, this suggests that for a flat plate (or say a ship's hull) homogeneously covered with roughness of height  $k_s$ , under fully rough conditions the boundary layer thickness at some fixed distance downstream from the leading edge ( $\delta$ ), must be invariant with unit Reynolds number. This result is somewhat counterintuitive, since for the smooth surface we know that  $\delta$  remains a function of  $U_{\infty}/\nu$  (Note that for the smooth wall, with  $k_s/x = 0$  in (9), we find that S is a function of  $Re_x$ , and hence from (3) with  $\Delta U^+ = 0$  we see that  $\delta$  is a function of x and  $Re_x$ ). This result is illustrated schematically for the ship case in figure 2; A smooth hull yields boundary layer profiles that are a function of freestream velocity (plot a), while under fully rough conditions the profiles are invariant with unit Reynolds number (b).

This result, while implicit in Granville [7] (who shows that  $C_D$  depends only on  $L/k_s$ ), is perhaps not widely known in the broad turbulence research community, but equations (15) and (16) offer a succinct explicit demonstration of this. It is possible to find proof of this tendency in the literature. Figure 3 shows data from Squire et al. [12] for smooth and rough surfaces (P36 grit sandpaper). Figure 3(b) shows the boundary layer thickness at x = 21.7 m downstream of the inlet to the working section ( $\delta_{21.7}$ ) for both the smooth and rough surfaces as a function of the freestream velocity  $U_{\infty}$ . Since the rough surface is not altered, these data are at fixed  $k_s/x$ . Note that for the smooth surface, the boundary layer thickness at x = 21.7 m, decreases as a function  $U_{\infty}$ . However, for the rough surface, once the fully rough limit is approached ( $k_s^+$  is shown on figure 3a), the boundary layer thickness becomes invariant with unit Reynolds number ( $U_{\infty}$  in this case), confirming the result from (17). As a validation of the formulation presented here, the blue dashed line on figure 3 shows the corresponding numerical solutions obtained by letting  $k_s/x = 1.96 \times 10^{-3}/21.7$  in (9) and (4). It should be noted that the experiments of Squire et al. [12] are unique in the sense that (i) they studied a high Reynolds number boundary layer



FIG. 3: (b) Variation of boundary layer thickness at x = 21.7 m with free-stream velocity as reported by Squire et al. [12] for the Melbourne University High Reynolds Number Boundary Layer Wind Tunnel HRNBLWT for (star symbols) rough test surface, consisting of P36 grit sandpaper, with a determined  $k_s = 1.96mm$ ; (circle symbols) smooth surface. Gray curves show fits to the data. Blue dashed curve shows numerical solutions from (9) and (4) evaluated for  $k_s/x = 1.96 \times 10^{-3}/21.7 = 9.03 \times 10^{-5}$ . (a) shows  $k_s^+$  for the rough surface corresponding to  $U_{\infty}$ .

(we note from Figure 1 that  $Re_x \approx O(10^7)$  is required to observe constant  $C_f$  at fixed  $k_s/x$ ); (ii) they used an independent and accurate measurement of  $C_f$  using a floating plate drag balance; (iii) had a sufficiently small blockage  $k/\delta$  such that assumptions of outer layer similarity (and assumptions about the logarithmic form of the mean velocity profile) were unlikely to be violated; (iv) employed testing at fixed x and multiple different unit Reynolds numbers  $U_{\infty}/\nu$ ; (v) presented boundary layer thickness data in tabulated form. However, there are other rough wall studies in developing turbulent boundary layers where constant  $C_f$  as a function of  $Re_x$  can be approximately observed. Specifically here we note that Schultz & Flack [13] show  $C_f$  becoming nominally constant with  $Re_x$  for both their uniform spheres and uniform spheres with grit cases, and in these cases (particularly the latter), it is clear  $\delta$  seems to be tending to a constant (in particular when compared to the variation of  $\delta$  with  $Re_x$  for the smooth surface). Similar tendencies are also observed for the 220-grit and 60-grit sandpapers studied in Schultz & Flack [14].

The fact that  $C_f$  and  $\delta/k_s$  are both invariant with unit Reynolds number in the fully rough limit and for fixed  $k_s/x$ , suggests that all mean velocity profiles under these conditions must collapse under the scaling  $z/k_s$  vs.  $U^+$ . Figure 4 demonstrates this, showing with symbols, the data from [12] at fixed  $k_s/x$  (fixed roughness  $k_s = 1.96$  mm, x = 21.7 m) in the fully rough state  $k_s \gtrsim 70$ , corresponding to  $U_{\infty} = 15.2$ , 20.6, 25.4 & 30.4 ms<sup>-1</sup>. The boundary layer thickness  $\delta_{21.7}$  and  $k_s^+$ 



FIG. 4: Velocity profiles at constant  $k_s/x$  in the fully rough condition  $(k_s^+ > 70)$ . Symbols show data from Squire et al. [12] at x = 21.7 m, for  $U_{\infty} = 15.2$ , 20.6, 25.4, 30.4 ms<sup>-1</sup> corresponding to  $k_s^+ > 70$ . Lines show the mean profiles from the present calculations at matched conditions. Lighter shaded profiles and symbols denote higher freestream velocities. Dashed line shows  $U^+ = \frac{1}{\kappa} \log \frac{z}{k} + B$ .

corresponding to these freestream velocities are shown previously in figure 3. The solid lines show the mean profiles predicted from the previous calculations for matched conditions. These profiles are calculated by using  $k_s/x = 1.96 \times 10^{-3}/21.7$  into (9), to yield S as a function of  $U_{\infty}$ , which can then be used in (4) to obtain  $\delta_{21.7}$  and hence into (1) to obtain the mean profiles. The results in Figure 4 suggest very good collapse under this scaling. The only departures are where expected in the viscous near wall dominated profile that has not been modelled here.

For, completeness, the integrated drag coefficient for the entire flat plate of length L can be calculated for the fully rough-wall case by substituting the limit  $Re_L \to \infty$  into (14) to give

$$C_D = 2\left(\frac{k_s}{L}\right) e^{\kappa (S_L - B) - 2\Pi} \frac{\kappa S_L - 2 - \frac{3}{2}\Pi^2 + \Pi(\kappa S_L - Q)}{\kappa^2 S_L^2}.$$
 (18)

The solid lines blue and black curves on Figure 5 show  $C_D$  as a function of  $Re_L$  for both constant  $Re_k$  and constant  $k_s/L$  respectively, as calculated from (14). The blue dashed lines show the fully rough  $C_D$  as given by (18).

# B. Long-plate limit

We now keep  $Re_k = \epsilon Re_x$  fixed and let  $x \to \infty$ , corresponding to keeping the unit Reynolds number  $U_{\infty}/\nu$  and  $k_s$  fixed while increasing the plate length. We refer to this as the long-plate



FIG. 5: Integrated drag coefficient  $C_D$  versus  $Re_L$  for fixed  $\epsilon_L = k_s/L$  and fixed  $Re_k$ . Blue solid lines; top to bottom:  $\epsilon_L = 10^{-3}$ ,  $3 \times 10^{-4}$ ,  $10^{-4}$ ,  $3 \times 10^{-5}$ ,  $10^{-5}$ ,  $3 \times 10^{-6}$ ,  $10^{-6}$ . Black lines; top to bottom:  $Re_k = 3 \times 10^4$ ,  $10^4$ ,  $3 \times 10^3$ ,  $10^3$ ,  $3 \times 10^2$ ,  $10^2$ , 30. Red curve shows the smooth wall ( $Re_k = 0$ ).

limit. It was recognized by Prandtl and Schlichting [5] from the trend of their numerical solutions, but they do not provide supporting analysis. The long-plate limit corresponds to moving along a hyperbola with fixed  $Re_k$  in the  $\epsilon - Re_x$  plane:  $\epsilon \to 0$  while  $Re_x \to \infty$  with both  $Re_k$  and the unit Reynolds number  $U_{\infty}/\nu$  remaining constant. It is then not clear a priori which term on the right-hand side of (9) becomes dominant when  $Re_x \to \infty$ , and therefore, whether either a smooth or rough-wall limit is approached. Using that  $\text{Ei}(x) \sim e^x/x$  when  $x \to \infty$ , for  $S \gg 1$ , (9) can be written as

$$Re_{x} \sim \frac{e^{\kappa (S-A)-2\Pi} (1+\Pi) S^{2}}{\kappa} + Re_{k} \frac{e^{\kappa (S-B)-2\Pi} (1+\Pi) S}{\kappa}.$$
 (19)

The first term on the right-hand side is quadratic while the second term is linear in S. Hence as  $Re_x$  is increased at fixed  $Re_k$ , S increases such that when  $S > Re_k$ , the first term become dominant. This is sufficient to show that a smooth-wall limit is approached. An alternative interpretation is that  $k_s^+ \equiv Re_k/S$  decreases monotonically and eventually enters the sub-layer when  $k_s^+ \sim 5$ . This then corresponds to a boundary layer, flowing over roughness of constant scale  $k_s$ , becoming asymptotically smooth when  $x \to \infty$  at fixed  $U_{\infty}/\nu$ .

Figure 1 provides clear evidence of this long plate limit. It is noted that all constant  $Re_k$  curves (black lines) are tending towards the smooth limit (red curve) at high  $Re_x$ . However, this limit is of little practical engineering relevance. If we consider the case of a ship operating at  $U_{\infty} = 8.7$ ms<sup>-1</sup> and  $\nu = 8.97 \times 10^{-7}$  (as considered in [15]), we find that even for  $Re_k = 100$ , corresponding to a very moderate fouling of  $k_s \approx 10 \mu$ m, for the rough wall  $C_f$  to be within 2% of the smooth wall limit requires  $Re_x \approx 3.2 \times 10^{13}$  equating to a plate length in excess of 1000 km.

#### IV. CONCLUDING REMARKS

Although implicit algebraic expressions are here derived for various quantities relevant to a turbulent boundary layer over a uniformly rough flat plate under a uniform free stream, the emphasis here is on interpreting the behaviour of these quantities under two distinguished limits as  $U_{\infty}L/\nu \to \infty$ : i)  $k_s/L$  fixed, called the fully rough limit and ii)  $U_{\infty}k_s/\nu$  fixed, called the long-plate limit. Whilst it is well appreciated that the drag coefficient  $C_D$  approaches a constant in the fully rough limit, it is perhaps underappreciated that both the local skin-friction coefficient  $C_f$  and the roughness-normalised boundary-layer thickness  $\delta/k_s$  approach universal dependencies on  $x/k_s$ . Physically, this behaviour is easily observed when the boundary-layer thickness at a particular station approaches a constant with increasing free-stream speed, an observation we show to be corroborated by data in the literature. In the long-plate limit, it is shown that the flow approaches the behaviour of a smooth wall, essentially because the skin-friction, and hence the friction velocity, has reduced with downstream distance to a sufficiently small value such that  $k_s u_{\tau}/\nu \leq 5$ . However, very large Reynolds numbers are required to observe this limit.

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