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Streak instability in viscoelastic Couette flow

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Abstract

The secondary instability of nonlinear streaks and transition to turbulence in viscoelastic Couette flow are studied using direct numerical simulations (DNS). Viscoelasticity is modelled using the FENE-P constitutive equations. Both the polymer concentration $\beta$ and Weissenberg number $Wi$ are varied in order to assess their effects on transition at moderate Reynolds number. The base streaks are obtained from nonlinear simulations of the Couette flow response to a streamwise vortex. We select the initial amplitude of the vortex which yields a desired maximum amplitude of the nonlinear streaks during their temporal evolution. The development of streaks in both Newtonian and non-Newtonian flows is primarily due to the action of streamwise vorticity onto the mean shear. In the viscoelastic case, it is also affected by the polymer torque, which opposes the vorticity and becomes more pronounced at large Weissenberg number. Streaks with the same maximum streamwise velocity perturbation can therefore have different total kinetic energy at higher Weissenberg number.

At every streak amplitude of interest, harmonic forcing is introduced along the transverse direction to trigger the secondary instability and breakdown to turbulence. We demonstrate that the critical amplitude of the forcing $A_d$ increases at large Weissenberg number. The degree of stabilization due to elasticity depends on the initial streak intensity, $A_{s,in}$. For weak streaks the critical amplitude for secondary instability is more sensitive to $Wi$ than for strong ones. This is explained by the existence of two different mechanisms that can trigger transition to turbulence. The perturbation to weak streaks is initially stabilized by the polymer torque which acts to oppose the amplification of wall-normal vorticity and, as a result, delays breakdown to turbulence. The secondary instability of strong streaks, on the other hand, is more immune to this stabilizing influence of the polymer.
1. INTRODUCTION

In transitional wall-bounded shear flows, streaks are often a precursor to breakdown to turbulence\(^1\). These streaks are narrow regions of excess/defect streamwise velocity elongated in the streamwise direction. They are generated by streamwise vortices via the so-called lift-up effect, or vorticity tilting\(^2\)–\(^4\). Tilting in this context refers to the generation of wall-normal vorticity due to the perturbation strain rate which tilts the vorticity of the mean flow, i.e. the mean spanwise vorticity, into the wall-normal direction. The large energy amplifications associated with streak growth, of the order of the square of the Reynolds number, are explained by the strongly non-normal nature of the linearized Navier-Stokes operator for shear flows.

Among the different studies of streak instability in Newtonian fluids, Cossu et al. (2011)\(^5\) considered Couette flow and sought amplitude threshold for the streak breakdown. They concluded that the critical amplitude of the perturbations (sinuous in their case) increases when the streak amplitude decreases. For strong streaks, breakdown is triggered by a secondary modal instability\(^6\)–\(^8\) while for small amplitude streaks transition is triggered by a two-step process in which the vortex tilting mechanism plays a role starting from the transient streamwise vortices induced by the sinuous forcing\(^9\). The streaks are initially distorted by the sinuous perturbation, but after a short period they reach a maximum in their energy and return to a nearly stable state. However, they ultimately reach a higher amplitude and break down to turbulence. This process was previously described by Waleffe\(^10\) for a generic shear flow.

In contrast to the wealth of studies focusing on Newtonian streaks and their secondary instability, there are relatively fewer efforts dedicated to the influence of fluid elasticity on (i) the growth of the streaks and (ii) their secondary instability. Whether elasticity is stabilizing or destabilizing to shear flows depends on the particular flow configuration and parameters. For example, it can promote or suppress absolute instability in spatially developing mixing layers\(^11\) and jets\(^12\). The linear analysis by Jovanović and Kumar\(^13\) showed that polymer stretching in elasticity-dominated flows can lead to streaks that are phenomenologically similar to those generated by lift-up in inertial Newtonian flows\(^13,14\). Whatever their origin, when streaks are present they can introduce streamline curvature that can be host for new elastic instabilities\(^15\). In direct numerical simulations of bypass transition in
polymeric channel flow, Agarwal, Brandt and Zaki (2014)\textsuperscript{16} found that the polymers weaken the primary streaks and prolong the transition process.

Page & Zaki (2014)\textsuperscript{17} analyzed the linear evolution of streaks in polymeric Couette flow. They identified three classes of streaks: (i) the quasi-Newtonian class includes streaks whose evolution collapses onto the Newtonian behavior when relaxation is either very fast or slow, (ii) the elastic class, in which the streaks can reach very large amplitude even in the absence of inertia\textsuperscript{18,19}, (iii) a class of streaks that undergo cycles of re-energization within an envelope of decay, and which takes place when the solvent diffusion and relaxation timescales are commensurate.

The secondary instability of streaks in viscoelastic flow has not been examined, and the present work aims to address this gap. We analyze the canonical configuration of viscoelastic Couette flow distorted by a primary streak, and examine its secondary instability. We first analyze the non-linear evolution of the primary streaks generated by introducing a streamwise vortex. Afterwards, we assess the secondary instability of the streaks by focusing on the transition to turbulence triggered by a sinuous disturbance.

The paper is organized as follows. In section 2 we introduce the governing equations, the numerical method and the simulation setup. The amplification of the primary streaks is studied in section 3, followed by the transition to turbulence due to the streak instability in section 4. Finally, conclusions are drawn in section 5.

2. GOVERNING EQUATIONS AND SIMULATIONS SET-UP

2.1. Governing equations

The incompressible dimensionless Navier-Stokes equations for viscoelastic flow take the form,

\[
\frac{\partial u}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \beta \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1 - \beta}{Re} \frac{\partial \tau_{ij}}{x_j} \tag{1}
\]

\[
\frac{\partial u_i}{\partial x_i} = 0 \tag{2}
\]

where \(Re\) is the Reynolds number, \(\beta\) is the ratio of the solvent to the total viscosity and \((1 - \beta)\) is effectively a measure of the polymer concentration, and \(\tau_{ij}\) is the polymer stress. This stress accounts for the interaction between the solvent and the polymer, and depends
on the conformation of the polymer chains $c_{ij}$. The relationship between the polymer stress and the conformation for a FENE-P fluid is,

$$\tau_{ij} = \frac{1}{Wi} \left( \frac{c_{ij}}{\psi} - \delta_{ij} \right)$$

where the Weissenberg number $Wi$ is the ratio of the polymer relaxation and the flow timescales, $\psi \equiv 1 - \frac{c_{kk}}{L^2}$ is the Peterlin function, $a \equiv 1 - \frac{3}{L^2}$, and $L$ the maximal extensibility of the polymers\textsuperscript{20}. Finally, the conformation tensor satisfies the evolution equation,

$$\frac{\partial c_{ij}}{\partial t} + u_k \frac{\partial c_{ij}}{\partial x_k} = c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k} - \tau_{ij},$$

which includes advection by the velocity field, stretching due to the strain exerted on the polymer chains, and relaxation due to the elastic nature of the polymer.

Zhou & Akhavan\textsuperscript{21} compared results for the pre-averaged models with those for the FENE chain. They found the FENE-P closure to be in qualitative agreement. Furthermore, Stone et al.\textsuperscript{22} found the FENE-P model to qualitatively capture the features of passive-bead spring chains in a complex turbulent flow field modelled using Brownian dynamics. For these reasons the model has been used extensively to simulate viscoelastic shear flows\textsuperscript{23–25}. However appropriate computational methods must be adopted to ensure stable and accurate simulations.

### 2.2. Numerical method

The hyperbolic nature of the conformation tensor evolution equation creates severe gradients in the conformation field, which can lead to numerical instabilities. Several numerical methods have been proposed to ensure stability and accuracy. For example, upwind schemes along with artificial diffusivity increase the stability of the numerical solution of hyperbolic equations\textsuperscript{26}. In simulations of polymeric fluids with a maximum extensibility constraint, numerical errors can lead to predictions of polymer extensions that exceed their bounds. An implicit method to evaluate the conformation tensor equation, however, can resolve this problem\textsuperscript{27}.

The numerical method used in this work for the solution of the governing equations follows the approach by Min et al.\textsuperscript{28,29}. We adopt a control-volume formulation for the spatial discretisation, which has been widely tested for accurate simulations of transitional and turbulent flows\textsuperscript{8,30}. The equations are advanced in time using a fractional-step algorithm where
an implicit scheme (i.e. Crank-Nicolson) is adopted for the diffusion and the polymer stress terms, while the advection term is treated explicitly. A third-order upstream central scheme is used to compute the spatial derivatives of the conformation tensor in the longitudinal direction. A local artificial diffusivity is added at locations where the conformation tensor loses its positive definiteness to ensure numerical stability. The additional term is \( \kappa \Delta_k^2 \frac{\partial^2 c_{ij}}{\partial x_k} \) where \( \Delta_k \) is the local grid spacing in the \( k \)-direction. The value of the coefficient \( \kappa \) should be sufficiently small. For the simulations presented here the choice \( \kappa = 10^{-3} \) guarantees that the artificial diffusivity is inactive during the disturbance linear evolution, and is restricted to less than 10% of the grid nodes during transition.

2.3. Simulation set-up

We study dimensionless Couette flow (see figure 1). The height of the domain is \( L_y = 2 \) and the velocities at the walls are \( U(\pm 1) = \pm U_w = \pm 1 \). The base velocity profile is \( U(y, 0, 0) \) with corresponding conformation tensor \( C_{xx} = \frac{1}{\psi(C_{kk})} \left( 1 + \frac{2W_i^2}{\alpha \psi(C_{kk})^2} \right) \), \( C_{xy} = C_{yx} = \frac{W_i}{\alpha \psi(C_{kk})} \) and \( C_{yy} = C_{zz} = \frac{1}{\psi(C_{kk})} \). Note that the other components are zero (see appendix A for further details). The reference scales are the half-height of the domain \( L_{ref} = L_y/2 \), the total viscosity of the fluid \( \mu_{ref} = \mu_S + \mu_P \) and the velocity at the upper wall \( U_{ref} = U_w \). Based on these scales, the Reynolds number is \( Re \equiv \frac{U_w L_y}{2 \nu} = 400 \). Only one case with a different \( Re \) will be analyzed. We will vary both the Weissenberg number \( Wi \) and the polymer concentration \( \beta \) while the maximal extensibility of the polymers \( L \) is held constant, \( L = 100 \). Another important parameter is the extensibility, or the maximum value of the Trouton ratio. In the present work, its influence is represented by changes in \( \beta \) since \( L \) is constant.

2.3.1. Initial and boundary conditions

We will report two classes of simulations: the first focuses on the evolution of the primary streaks (section 3) and the second examines the transition to turbulence (section 4).

For the study of streak amplification, the initial condition is a Couette profile plus a streamwise vortex selected from the eigenspectrum of the Orr-Sommerfeld (O-S) equations.
Figure 1: Schematic of the computational setup.

(see Page & Zaki\textsuperscript{17}) so that the initial condition reads

$$u_i \equiv [y, 0, 0] + K[0, v_0(y, z), w_0(y, z)]$$

where $v_0(y, z)$ and $w_0(y, z)$ are the vertical and spanwise velocities

$$v_0(y, z) = \cos(k_z z) \Re\{\hat{v}(y)\} - \sin(k_z z) \Im\{\hat{v}(y)\}$$

$$w_0(y, z) = \cos(k_z z) \Re\{\hat{w}(y)\} - \sin(k_z z) \Im\{\hat{w}(y)\}$$

and $K$ controls the vortex amplitude. Note that the quantities $\hat{a}(y)$ represent the linear O-S eigenmodes computed using the linear solver described by Zhang \textit{et al.}\textsuperscript{33}. The initial polymer conformation tensor is also defined by the sum of the base state and the O-S eigensolution

$$c_{xx,0}(y, z) = C_{xx} + K \cos(k_z z) \Re\{\hat{c}_{xx}(y)\} - K \sin(k_z z) \Im\{\hat{c}_{xx}(y)\}$$

$$c_{yy,0}(y, z) = C_{yy} + K \cos(k_z z) \Re\{\hat{c}_{yy}(y)\} - K \sin(k_z z) \Im\{\hat{c}_{yy}(y)\}$$

$$c_{xy,0}(y, z) = C_{xy} + K \cos(k_z z) \Re\{\hat{c}_{xy}(y)\} - K \sin(k_z z) \Im\{\hat{c}_{xy}(y)\}$$

$$c_{zz,0}(y, z) = C_{zz} \quad \text{and} \quad c_{xz}(y, z) = c_{yz}(y, z) = 0.$$

For the second part of the study, the focus is placed on the secondary instability of the streak and breakdown to turbulence. Following the work by Cossu \textit{et al.}\textsuperscript{5} a sinuous secondary disturbance is introduced in the spanwise velocity when the primary streaks reach the highest amplitude. The secondary disturbance is

$$w_d(x, y) = (1 - y^2) \sin(\alpha_d x)$$

(12)
where $\alpha = 0.7$. This value corresponds to the most unstable streamwise wavelength of the streak secondary instability in both Newtonian and viscoelastic Couette flow. The initial velocity of this second set of simulations is therefore

$$\begin{align*}
(u_i, v_i, w_i) &= (u_s, v_s, w_s) + A_d[0, 0, w_d(x, y)],
\end{align*}$$

where $A_d$ is the amplitude of the secondary instability, and $u_s$, $v_s$ and $w_s$ are the flow velocities from the first set of computations at the peak streak amplitude (see S4 for more details).

Periodic boundary conditions are enforced in the streamwise and spanwise directions, while no slip conditions are prescribed on the upper and lower walls.

### 2.4. Computational domain

Two different domain sizes were adopted for the non-linear evolution of the primary streaks and the transition simulations, respectively. The grid was also adapted to the flow parameters since highly-viscoelastic flows, i.e. high $Wi$ or low $\beta$, can create sharp gradients in the polymer stress. As a result, these configurations required finer grids.

The numerical domain for simulating the primary streaks was $L_x = \pi$, $L_y = 2$ and $L_z = 2.5\pi$. The width $L_z$ was chosen to accommodate the spanwise wavenumber of the initial streamwise vortices, $k_z = 1.6$. In this manner we will have two periods of the mode along the $z$-direction. The number of grid points in each direction depends on the flow under consideration. For Newtonian or low-viscoelastic cases we use $N_x = 8$, $N_y = 64$ and $N_z = 64$. However as the Weissenberg number increases or $\beta$ decreases we increase the number of grid points in the $y$- and $z$-directions up to four times, i.e. $N_y = 256$ and $N_z = 256$, while it was not necessary to increase the number of points along the $x$-direction when dealing with streamwise independent disturbances.

As the sinuous secondary disturbance is introduced we set $L_x$ to contain two streamwise wavelengths of the sinuous mode, i.e. $L_x = \frac{40}{7}\pi$. The domain size and the number of grid points were unchanged in the other two directions. Along the $x$-direction we choose $N_x = 96$ for Newtonian and low-viscoelastic configurations. This value was increased up to fourfold in order to capture the dynamics at higher $Wi$. 
3. EVOLUTION OF THE PRIMARY STREAKS

In this section we discuss the streak amplification in response to a streamwise vortex. As noted in S2.3, the vortex is an eigenmode of the viscoelastic Orr-Sommerfeld equation, and is superimposed onto the Couette profile. The spanwise wavenumber of this mode is \( k_z = 1.6 \) and it corresponds to the optimal disturbance in a Newtonian Couette flow\(^3\). Streamlines of the initial vortex are shown in a cross-stream plane in figure 2a. Due to the lift-up effect, the pair of counter-rotating streamwise vortices creates low- and high-speed streaks in the flow.

3.1. Linear evolution

For low amplitudes of the initial disturbance, its evolution will follow the linear behavior studied in Page & Zaki\(^{17}\). Those authors defined three different regimes for the streamwise vortex mode in an Oldroyd-B fluid: quasi-Newtonian, elastic and re-energization regimes. The range of parameters analyzed in the present work places it in the quasi-Newtonian regime.

In figure 2b the linear evolution of the streak amplitude, i.e. the response to the streamwise vortex, is illustrated in the \( z-t \) plane for a Newtonian fluid. The streaks initially grow and eventually decay after reaching a maximum amplitude at time \( t \approx 55 \). The amplitude of the streak response is defined as,

\[
A_s = \left[ \frac{1}{2} \frac{(u_{\text{max}} - u_{\text{min}})}{U_{\text{ref}}} \right] \times 100,
\]

where \( u_{\text{max}} \) and \( u_{\text{min}} \) are the instantaneous maximum/minimum of the streamwise velocity perturbation over the entire domain, \( u(x, y, z) = u_{\text{tot}}(x, y, z) - y \). The maximum in the streak amplitude also corresponds to a maximum in their total kinetic energy. A similar behavior is observed in viscoelastic flows. We have validated that our numerical simulations are in agreement with the linear response reported in Ref. 17.

3.2. Non-linear evolution

Upon further increase of the initial amplitude of the streamwise vortex, \( K \), the growth of the streaks is no longer linear. At nonlinear amplitudes, the streaks distort the base-flow profile, which in turn alters their growth rate. In addition, the generation of higher harmonics alters the shape of the disturbance field.
Figure 2: (a) End view showing streamlines of the initial streamwise vortex. (b) Top view of the linear streak response, shown by contours of the streamwise velocity perturbation $-0.0071 \leq u' \leq 0.0071$.

First, we define the root mean square (rms) of a generic observable $\phi_{\text{rms}} = \sqrt{\overline{\phi^2}}$, where $\overline{\phi^2} = \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \phi^2(x,y,z) dx dy dz$. The evolution of the amplitude $A_s$ and the root mean square (rms) streamwise velocity $u_{\text{rms}} = \sqrt{u^2}$ are reported in figure 3 for different Weissenberg numbers and constant concentration, $\beta = 0.6$. In the figure, the Newtonian flow is indicated by the lightest colour and increasing Weissenberg numbers are shown by darker lines; the darkest line corresponds to $Wi = 15$. As demonstrated by figure 3a, the initial vortex amplitude $K$ has been adjusted in order to have an equal maximum streak amplitude, $A_s = 48.13$, for all the viscoelastic conditions to be considered later. However the maximum of $u_{\text{rms}}$ is not equal among all configurations; it decreases with increasing elasticity. The difference can be attributed to the change in the streak shape with elasticity.

The mechanism causing the primary streak growth is vorticity tilting which depends on the strain rate $\frac{\partial v}{\partial z}$ of the initial vortex. A measure of the strength of this process can be obtained from the streamwise vorticity. In addition, Page & Zaki$^{35}$ demonstrated that, in viscoelastic flows, a torque is exerted by the polymers onto the fluid $\chi = \nabla \wedge (\nabla \cdot \tau)$ which has a resistive influence.

In figure 4 we illustrate the instantaneous streamwise vorticity (grayscale) and polymer torque (red and blue scales) at $t = 50$ for $Wi = 10$ and $\beta = 0.6$. The positive streamwise vorticity (white) is opposed by negative values of the torque (blue), while the negative vorticity (red) is opposed by a positive torque (blue), in agreement with previous results$^{35}$. Figure 5 shows the time evolution of the rms-streamwise vorticity $\omega_{x,\text{rms}}$ and polymer torque $\chi_{x,\text{rms}}$ for different values of $Wi$ at $\beta = 0.6$. The streamwise vorticity decreases in time for
Figure 3: Nonlinear evolution of the primary streak (a) $A_s$ and (b) rms-streamwise velocity $u_{rms}$ for different $Wi$.

Figure 4: Streamwise vorticity (grayscale) and polymer torque (red and blue lines) for $Wi = 10$, $\beta = 0.6$ at $t = 50$.

all cases, at faster rate when increasing the flow elasticity. While the streamwise torque also decays in time, its strength is appreciably increased with elasticity. This means that the polymers are weakening the streamwise vortices similarly to what was observed in the near-wall region in exact coherent structures of viscoelastic shear flows$^{36,37}$.

Figures 4 and 5 together can explain the reduction of the energy of the streak with increasing polymer relaxation time, $Wi$. Streak amplification is resisted by the streamwise polymer torque and this effect becomes more pronounced at higher $Wi$.  
Figure 5: Temporal evolution of the (a) $\text{rms}$-streamwise vorticity $\omega_{x,rms}$ and (b) $\text{rms}$-streamwise polymer torque $\chi_{x,rms}$ for different value of $Wi$ for $\beta = 0.6$. Line colors retain the same designation as in figure 3.

4. TRANSITION TO TURBULENCE

In this section we analyze the effect of viscoelasticity on the secondary instability of streaks, which precedes breakdown to turbulence.

4.1. Secondary sinuous disturbance

As noted in section 2.3.1, a secondary sinuous disturbance is introduced in the spanwise momentum equation as a Dirac delta in time when the streak reaches its maximum amplitude (see eq. 12), similar to Cossu et al. (2011). The wavelength $\alpha_d$ of this sinuous disturbance has to be carefully chosen. To this end, we have first conducted numerical simulations in longer domains (up to $L_x = 20\pi$) with a random disturbance of amplitude $A_d = 4$ to find the most unstable streamwise wavelength. The wavenumber of the emerging instability was found to be approximately $\alpha_x = 0.7$. Additional tests were performed to check the dependence of this value on (i) the length of the domain $L_x$, (ii) the amplitude of the disturbance $A_d$ and (iii) the viscoelasticity parameters ($Wi$ and $\beta$), without significant variations. The wavelength of the sinuous forcing was therefore set to $\alpha_d = 0.7$.

The temporal evolution of the streamwise velocity disturbance, $u_{rms}$, for different $Wi$
Figure 6: The temporal evolution of the streamwise velocity disturbance, $u_{\text{rms}}$, for different $Wi$. Dashed curves show the evolution of the primary streaks created by the streamwise independent initial condition. Solid lines mark the disturbance evolution after the introduction of a sinuous forcing with amplitude $A_d = 12$. The amplitude of the streaks is $A_{s,in} = 22.58$ when the sinuous forcing is introduced. Note that for $Wi = 15$ the streaks do not break down to turbulence.

is depicted in figure 6 for the two types of simulations performed here. The evolution of the primary streaks created by the streamwise independent initial condition is indicated by dashed lines, and the disturbance evolution after the introduction of the sinuous forcing is marked by a continuous line. The sinuous forcing reinforces the disturbance which continues to grow instead of decaying as was the case in section 3.2. Transition to turbulence depends on the amplitude of the forcing $A_d$: for lower amplitudes the streaks eventually decay after reaching a maximum energy.

4.2. Neutral curves

The transition scenario just discussed is bound to depend on $A_{s,in}$, the amplitude of the primary streaks at the time when the sinuous forcing is introduced, and $A_d$ the amplitude of this sinuous disturbance. Here we will evaluate the critical amplitude $A_{d,c}$ for which the
streaks break down and initiate the turbulent regime. The results are illustrated in figures 7a,c for different values of Weissenberg number and concentration. In figure 7a the polymer concentration is set to $\beta = 0.6$ and the curves are plotted for different $Wi$. The data reveal that the streaks breakdown is favoured for $Wi = 5$. This is more clearly seen in figure 7b where we rescale the neutral curves displayed in (a) with the Newtonian values. Indeed, the disturbance amplitude necessary to trigger transition for $Wi = 5$ is reduced by 20% relative to the Newtonian case. This destabilization coincides with the regime where the exact coherent state is favoured in viscoelastic Couette flow ($Wi < 7$) and where larger drag has been reported\(^{36}\). As $Wi$ increases, however, the polymers have a stabilizing effect.

Interestingly at $Wi = 15$ we find a different behavior for low- and high-amplitude streaks. The presence of the polymer additives has always a stabilizing effect and transition occurs for higher disturbance amplitudes. However, the stabilization is significant for lower amplitude streaks, $A_{s,in} < 0.35$, with up to 100% increase of the threshold amplitude, and vanishes at higher values of $A_{s,in}$. We also highlight the non-monotonic dependence of the transition thresholds on the Weissenberg number; see inset in figure 7a where the critical disturbance amplitude $A_{d,c}$ is displayed versus $Wi$ for $A_{s,in} = 22.58$.

In figure 7c we trace the neutral curves with varying $\beta$ while keeping the Weissenberg number constant ($Wi = 5$). The influence of viscoelasticity is always destabilizing. In figure 7d the neutral curves from (c) are normalized using the Newtonian values. For $\beta = 0.5$ the sinuous forcing strength is reduced up to $20-30\%$ depending on the initial streak amplitude. In general we observe that the degree of destabilization is stronger for large $A_{s,in}$.

### 4.3. Streaks breakdown mechanisms

The significantly different behavior observed at small and large $A_{s,in}$ is evident when comparing figs. 6 and 8. These two figures report the temporal evolution of $u_{rms}$ for two different streak amplitudes: $A_{s,in} = 22.58$ and $A_{s,in} = 48.13$. For the latter case, the disturbance starts growing after the introduction of the sinuous forcing and the breakdown to turbulence follows directly if the disturbance is of sufficient amplitude. In contrast, for lower amplitude streaks, $A_{s,in} = 22.58$ (fig. 6), a distinct two-stage process is observed, similar to that described in previous studies\(^{5,6,10}\). In figure 9 we report (a) the rms of the streamwise and wall-normal components of the vorticity and (b) the rms of the three
Figure 7: (a) Neutral curves in the $(A_{s,in}, A_d)$-plane for different values of $Wi$. The polymer concentration is $\beta = 0.6$. Inset of (a) $A_{d,c}$ vs $Wi$ for $A_{s,in} = 22.58$. (b) Neutral curves normalized by the Newtonian case. (c) Neutral curves in the plane $(A_{s,in}, A_d)$-plane for different values of $\beta$. The Weissenberg number is $Wi = 5$. (d) Neutral curves normalized by the Newtonian case.

components of velocity. The perturbation due to the instantaneously imposed sinuous forcing (associated with the rms spanwise velocity, see fig. 9b) initially decays, yet causes the growth of streamwise vortices (associated with streamwise vorticity and wall-normal velocity) that in turn induce the growth and breakdown of streaks. In other words, the streamwise periodic vorticity induced by the tilting of the initial wall-normal vorticity disturbance generates streaks of finite length, alternating in a periodic fashion in the streamwise direction. The
Figure 8: The temporal evolution of the streamwise velocity disturbance, $u_{rms}$, for different $Wi$. Dashed curves show the evolution of the primary streaks created by the streamwise independent initial condition. Solid lines mark the disturbance evolution after the introduction of a sinuous forcing with amplitude $A_d = 2.4$. The amplitude of the streaks is $A_{s,in} = 48.13$ when the sinuous forcing is introduced.

Streaks induced by these vortices also have a finite length and their breakdown to turbulence is associated with the interaction between a downstream low-speed region and an upstream high-speed region that generates strongly inflectional wall-normal profiles$^{8,30,38,39}$.

These two different mechanisms are visualized in figs. 10 and 11 where we display contours of $u$ in the wall-parallel $x$-$z$ plane for the Newtonian case at different times, $t' = t - t_d$ where $t_d$ is the time at which the instantaneous sinuous forcing is applied. For $A_{s,in} = 48.13$ the primary streaks monotonously deform under the effect of the forcing while growing in amplitude (10b,c) and eventually break down to turbulence (10d). For small initial amplitudes, the streaks first bend (11b) but eventually return to be almost straight (11c). Finally, the streaks are deformed again (11d) and eventually break down to turbulence under the effect of the growth of the new streamwise vortices created by the forcing. It is important to highlight that the two mechanisms do not qualitatively vary when viscoelasticity is introduced.
Figure 9: The temporal evolution of (a) the streamwise and wall-normal \textit{rms}-vorticity and (b) the streamwise, wall-normal and spanwise \textit{rms}-velocity for $Wi = 0$ and $A_{s,in} = 22.58$. Dashed curves show the evolution of the primary streaks created by the streamwise independent initial condition. Solid lines mark the disturbance evolution after the introduction of a sinuous forcing with amplitude $A_d = 12$. The amplitude of the streaks is $A_{s,in} = 22.58$ when the sinuous forcing is introduced.

4.4. Effect of the viscoelasticity

The previous section illustrated the two mechanisms that regulate the transition to turbulence for low- and high-amplitude streaks. Here we aim to explain the different degrees of stabilization that are observed at large $Wi$ for low- (S4.4.1) and high-amplitude (S4.4.2) streaks.
4.4.1. Small initial primary streak amplitudes

To understand the effect of polymers we consider the perturbation energy budget. The transport equation for the Reynolds stress in a viscoelastic flow is given by

$$\frac{\partial u_i' u_j'}{\partial t} = -U_k \frac{\partial u_i' u_j'}{\partial x_k} - \frac{\partial u_i' u_j' u_k'}{\partial x_k} - \left( u_j' \frac{\partial p'}{\partial x_i} + u_i' \frac{\partial p'}{\partial x_j} \right)$$

$$- \left( u_j' u_k' \frac{\partial U_i}{\partial x_k} + u_i' u_k' \frac{\partial U_j}{\partial x_k} \right) + \frac{\beta}{Re} \frac{\partial^2 u_i' u_j'}{\partial x_k^2}$$

$$- \left( \frac{2}{Re} \frac{\partial u_i' \partial u_j'}{\partial x_k \partial x_k} + \frac{1}{Re} \frac{\partial \tau_{ik}'}{\partial x_k} + \frac{\partial \tau_{jk}'}{\partial x_k} \right)$$

where $A_{ij}$ is advection by the mean flow, $Q_{ij}$ is the transport by the velocity fluctuations, $R_{ij}$ is the pressure redistribution, $P_{ij}$ is the production against the mean shear, $D_{ij}$ is viscous...
Figure 11: Instantaneous contours of the streamwise velocity disturbance \( u \) in the \( x-z \) plane for the Newtonian case. The initial streak amplitude is equal to \( A_{s,in} = 22.58 \) while the sinuous disturbance amplitude is \( A_d = 12 \). The contours are evaluated at: (a) \( t' = 0 \), (b) \( t' = 20 \), (c) \( t' = 60 \), (d) \( t' = 120 \).

diffusion, \( \epsilon_{ij} \) is dissipation and \( W_{ij} \) is the polymer work. The evolution equation for the perturbation energy is obtained by setting \( i = j \) in eq. (14). First, by examining the terms in the energy budget we note that the influence of viscoelasticity is most pronounced in: the energy production \( P_{ij} \), the viscous dissipation \( \epsilon_{ij} \) and the polymer work \( W_{ij} \). These will therefore be the focus of the following analysis.

We first examine the \( \overline{u'u'} \)-energy budget since the streaks are associated with the streamwise velocity. In particular in figure 12 (a) the energy production \( P_{xx} \), (b) viscous dissipation \( \epsilon_{xx} \), (c) polymer work \( W_{xx} \) and (d) their sum are plotted for different \( Wi \). The maximum of the production is enhanced by the polymers, while the dissipation is damped. Most interestingly, the polymer work has a damping effect (i.e. the polymers extract energy from the flow) whose magnitude decreases when increasing Weissenberg number. The peak of the sum of the three components is however increasing with \( Wi \), which indicates that the polymeric solution enhances the streak energy growth. Viscoelasticity thus has a destabilizing influence. While this trend is in agreement with the observations at low \( Wi \), it cannot
explain the behaviour at large $Wi$. A similar approach explains the destabilization due to an increase in the polymer concentration observed (figs. 7c,d) as a result of enhanced production.

The second, and important growth of the streaks observed when the sinuous forcing is introduced at small $A_{s,in}$ is due to the tilting of the wall-normal vorticity as explained in section 4.3. We therefore examine the effect of viscoelasticity on the evolution of the wall-normal vorticity, in order to explain the stabilization shown for large $Wi$ in figure 7. The contours of the wall-normal vorticity (grayscale) at $t' = 85$ for $Wi = 10$, $\beta = 0.6$, 

Figure 12: Temporal evolution of the (a) energy production $P_{xx}$, (b) dissipation $\epsilon_{xx}$, (c) polymer work $W_{xx}$ and (d) their sum for $A_{s,in} = 22.58$ and $A_d = 12$. 


Figure 13: Wall-normal vorticity (grayscale) and polymer torque (red and blue lines) for $Wi = 10$, $\beta = 0.6$, $A_{s,in} = 22.58$ and $A_d = 12$ at $t' = 85$ in the (a) $x$-$z$ and (b) $z$-$y$ planes.

Figure 14: Temporal evolution of the wall-normal component of the (a) rms-vorticity and (b) rms-polymer torque for $A_s = 22.58$, $A_d = 12$ and $\beta = 0.6$ at different $Wi$. $A_{s,in} = 22.58$ and $A_d = 12$ in the (a) $x$-$z$ and (b) $z$-$y$ planes are illustrated in figure 13. Also plotted is the wall-normal polymer torque (see section 3.2) in red and blue.

Cossu et al.\textsuperscript{5} have shown how the second maximum in $u_{rms}$, which was observed after introducing the sinuous forcing (see figure 6), is strictly related to a growth of the wall-normal vorticity. Figure 13 shows that the polymer torque has a resistive effect on the streaks by opposing the wall-normal vorticity, consistent with the results by Page & Zaki\textsuperscript{35}. To quantify this effect, we report the rms of the wall-normal vorticity $\omega_{y,rms}$ and wall-normal polymer torque $\chi_{y,rms}$ in figure 14 at several Weissenberg numbers. The figure shows that the wall-normal vorticity initially increases with Weissenberg number while it is damped as
the viscoelastic effects start to be significant. On the other hand, the wall-normal polymer torque varies significantly with $Wi$. In particular, the peak value for $Wi = 15$ is more than ten times larger than that for $Wi = 5$. This substantial increase in the polymer torque is the main cause for the stabilization observed for weak initial streaks when increasing the Weissenberg number to $Wi = 15$.

4.4.2. Large initial primary streak amplitudes

For strong initial vortex amplitudes, the primary streaks are sufficiently energetic and unstable that they break down to turbulence directly when subjected to the weak sinuous forcing. This differs from the previous observations for low $A_{s,in}$ where the transition to turbulence is due to a two-stage mechanism and the energy of the primary streaks is not crucial to explain the breakdown. We report in figure 15 the most relevant terms in the kinetic energy balance for a large-amplitude streak, $A_{s,in} = 48.13$ and $A_d = 2.4$, similar to figure 12 for a streak of lower amplitude. The initial energy production (fig. 15a) is damped by the polymers. This is consistent with the observation in section 3.2 that the peak in the average streak energy decreases when increasing $Wi$. The initial destabilization noticed in figure 7a at low $Wi$ is due to the lower dissipation in viscoelastic flows relative to the corresponding Newtonian case, as shown in fig. 15b. The polymer work (cf. fig. 15c) is negative but small during the initial stage. It is therefore not influential in the streak breakdown at high $A_{s,in}$. As in figure fig. 15d, the sum of the three components $P_{xx} + \epsilon_{xx} + W_{xx}$ is a better indicator of whether transition to turbulence takes place.

The decrease in dissipation for the viscoelastic cases is related to the choice of the reference viscosity $\mu_{ref}$ used to calculate the Reynolds number$^{33}$. We have chosen the total viscosity of the fluid as the reference value. If only the solvent viscosity is used, viscous dissipation would not decrease. This point is demonstrated in figure 16a where we have rescaled the viscous dissipation from figure 15b using the solvent viscosity, i.e. dividing the viscous dissipation by $\beta$. The initial viscous dissipation, once rescaled, is not significantly affected by the polymers. The Reynolds number based on the solvent viscosity for the viscoelastic cases in figure 15 is equal to $Re_s = \frac{Re}{\beta} = 666.67$. In figure 16b we are comparing the viscous dissipation of a Newtonian case with $Re = Re_s = 666.67$ to the viscoelastic configurations in figure 15b. The solvent Reynolds number $Re_s$ is constant for all four cases ($Re_s = 666.67$). We note that the
Figure 15: Temporal evolution of the (a) energy production $P_{xx}$, (b) dissipation $\epsilon_{xx}$, (c) polymer work $W_{xx}$ and (d) their sum for $A_{s,in} = 48.13$ and $A_d = 2.4$.

initial dissipation is inappreciably changed. These results point out how the destabilization at large initial streak amplitudes can be significantly affected by changes in the solvent viscosity. This differs from what is observed at small $A_{s,in}$ where the destabilization is created mainly by an increase in the energy production (see S4.4.1).

We have previously seen how the polymer torque has a resistive effect on vorticity. In particular it plays a fundamental role in (i) lowering the energy of the primary streaks (section 3.2) and (ii) hindering transition for small $A_{s,in}$ (section 4.4.1). For large $A_{s,in}$, the streak breakdown is accompanied by an increase of the streamwise vorticity$^5$. The $rms$-
Figure 16: (a) Viscous dissipation from figure 15b rescaled by the solvent viscosity instead of the total viscosity. (b) Comparison between the viscous dissipation of (i) the Newtonian case at $Re = Re_s = 666.67$ and (ii) the viscoelastic configurations from fig. 15b where $Re = 400$. The solvent Reynolds number is constant in all the four cases, $Re_s = 666.67$.

Streamwise (a) vorticity and (b) polymer torque are illustrated in figure 17 for several values of the Weissenberg number, and large streak amplitude $A_s = 48.13$, sinuous forcing $A_d = 2.4$ and $\beta = 0.6$. The figure shows that the maximum of the streamwise vorticity increases with elasticity, while the polymer torque initially decreases with elasticity at short times. Note also that the polymer torque grows rapidly at large $Wi$. In summary, the polymer torque is lower than in the case of weak streaks (see figure 14b). This explains why the streaks breakdown is only slightly retarded in highly-viscoelastic flows at large $A_{s,in}$, in contrast to small $A_{s,in}$.

5. CONCLUSIONS

In this work we have examined the secondary instability of streaks and transition to turbulence in viscoelastic Couette flow. The polymeric solution was modeled using a FENE-P fluid, and the flow evolution evaluated using direct numerical simulations. The base streaks belong to the quasi-Newtonian regime according to the classification by Page & Zaki (2014)\textsuperscript{17}.

We have first studied the impact of elasticity on the non-linear evolution of the streaks.
The results show that the streaks reach a lower average energy with increasing elasticity. This is due to a resistive polymer torque that opposes the streamwise vorticity and, as a result, opposes the lift-up mechanism.

A streamwise-sinuous disturbance was introduced at the time when the streaks reached their maximum energy. The ensuant secondary instability was promoted (i) at low $Wi$ and (ii) with increasing polymer concentration. However at high $Wi$ a change of trend is observed: transition to turbulence is delayed, and the degree of stabilization depends on the initial streak amplitude $A_{s,in}$. If the streaks are weak we observe a sharp retardation of breakdown, and when they are strong transition is only slightly delayed. The difference is due to a change in the transition mechanism at small and large $A_{s,in}$. At small amplitudes transition takes place via a two-stage process, while at large amplitude the streaks directly break down when perturbed by the weak sinusoidal forcing. In both cases the stabilization at $Wi = 15$ is due to a resistive polymer torque that opposes the growth of streaks. The cause of destabilization at low elasticity, on the other hand, depends on the streak amplitude:

- For small $A_{s,in}$ and weak elasticity, transition is promoted due to an enhanced production. The pronounced perturbation growth leads to stronger, more unstable streaks that break down to turbulence.
- For large $A_{s,in}$ and weak elasticity, the solvent viscosity is effectively reduced thus
promoting streak instability and transition.

Similar arguments explain the destabilization caused by an increase in the polymer concentration.

Our findings can also have bearing on understanding drag reduction in wall-bound polymeric turbulence\textsuperscript{40}. The stabilizing influence of the polymer torque on streaks and its ability to delay breakdown to turbulence is consistent with earlier studies of the more complex, fully turbulent flows. There, the polymer torque was shown to limit the growth of vortical structures, and inhibits the formation of hairpin packets and bursting events\textsuperscript{41,42}. Xi & Graham\textsuperscript{43} attributed the drag reduction in a minimal channel flow to the existence of time intervals of hibernating turbulence during which the streaks remain stable for relatively long times. The frequency and duration of these hibernating states increased at higher Weissenberg number. Analysis of the polymer torque and its role in initiating and sustaining these hibernating states can improve the current understanding of turbulent drag reduction.

It is interesting to consider a flow configuration where streaks appear with various sizes and amplitudes. For example, when a laminar boundary layer is exposed to broadband free-stream noise, Klebanoff streaks with different amplitudes and orientations amplify and their secondary instability\textsuperscript{8,30,39} signals the onset of bypass transition to turbulence. The present analysis shows that the influence of elasticity, be it stabilizing or destabilizing, depends on the flow parameters. In addition, it is important to note that we focused on streaks that belong to the quasi-Newtonian class according to the classification by Page & Zaki (2014). Two other classes have been identified, namely “elastic” and “re-energizing” streaks. In the former, streaks can reach very large amplitudes despite the weak inertia\textsuperscript{18,19}. A similar DNS analysis in this regime could shed light on elastic turbulence\textsuperscript{44,45}. The re-energization regime, on the other hand, occurs when the diffusion and relaxation timescales are commensurate, and is characterized by a cyclical amplification of the base streaks within an envelope of growth and decay. Preliminary direct numerical simulations confirmed that this regime is observed in the full nonlinear problem. A complete study of these streaks and their secondary instability should be the subject of future work.
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Appendix A: Analytical expression for the base-state conformation tensor

In this section, we summarize the solution to eq. (4) for the conformation tensor given a generic, parallel, velocity profile $U(y)$. Using this base-flow ansatz, the conformation tensor can be written as,

\[ c_{xx} = c_{yy} \left[ 1 + 2U'(y)^2 Wi^2 \left( 1 - \frac{3}{L^2} \right) c_{yy} \right] \]  
\[ (A1a) \]

\[ c_{xy} = c_{yx} = U'(y) W i \left( 1 - \frac{3}{L^2} \right) c_{yy} \]  
\[ (A1b) \]

\[ c_{yy} = c_{zz} = 1 - \frac{c_{xx} + 2c_{yy}}{1 - \frac{3}{L^2}}. \]  
\[ (A1c) \]

where $U'(y) = \frac{dU}{dy}$. If $L \to \infty$, the conformation tensor tends to the conformation tensor pertaining an Oldroyd-B fluid: $c_{xx} = 1 + 2U'(y)^2 Wi^2$, $c_{xy} = c_{yx} = U'(y) W i$ and $c_{yy} = 1$. The base solution for the components of the conformation tensor can be obtained by inverting the above system, which yields

\[ C_{xx} = \frac{1}{\psi'(C_{kk})} \left[ 1 + \frac{2U'^2(y)W i^2}{a^2 \psi(C_{kk})^2} \right] \]  
\[ (A2) \]

\[ C_{xy} = \frac{U'(y) W i}{a \psi(C_{kk})^2} \]  
\[ (A3) \]

\[ C_{yy} = C_{zz} = \frac{1}{\psi'(C_{kk})} \]  
\[ (A4) \]

where $f(C_{kk}(j)) = \frac{2}{3} \cosh(\frac{\phi}{2}) + \frac{1}{3}$, $\phi = \text{acosh} \left( \frac{27}{2} \Omega^2 + 1 \right)$, $\Omega = \sqrt{\frac{2}{aL}} \frac{U'(y) W i}{a}$. Dallas et al.\textsuperscript{24} derived a similar expression but with a different choice of the Peterlin function. For the Couette flow studied in the present work $U'(y) = 1$, and therefore,

\[ C_{xx} = \frac{1}{\psi'(C_{kk})} \left[ 1 + \frac{2W i^2}{a^2 \psi(C_{kk})^2} \right] \]  
\[ (A5) \]

\[ C_{xy} = \frac{W i}{a \psi(C_{kk})^2} \]  
\[ (A6) \]
\[ C_{yy} = C_{zz} = \frac{1}{\psi(C_{kk})}. \quad (A7) \]


