



CHORUS

This is the accepted manuscript made available via CHORUS. The article has been published as:

Secular dynamics of long-range interacting particles on a sphere in the axisymmetric limit

Jean-Baptiste Fouvry, Ben Bar-Or, and Pierre-Henri Chavanis

Phys. Rev. E **99**, 032101 — Published 5 March 2019

DOI: [10.1103/PhysRevE.99.032101](https://doi.org/10.1103/PhysRevE.99.032101)

Secular dynamics of long-range interacting particles on a sphere in the axisymmetric limit

Jean-Baptiste Fouvry* and Ben Bar-Or
Institute for Advanced Study, Princeton, NJ, 08540, USA

Pierre-Henri Chavanis
Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, France

We investigate the secular dynamics of long-range interacting particles moving on a sphere, in the limit of an axisymmetric mean field potential. We show that this system can be described by the general kinetic equation, the inhomogeneous Balescu–Lenard equation. We use this approach to compute long-term diffusion coefficients, that are compared with direct simulations. Finally, we show how the scaling of the system’s relaxation rate with the number of particles fundamentally depends on the underlying frequency profile. This clarifies why systems with a monotonic profile undergo a kinetic blocking and cannot relax as a whole under $1/N$ resonant effects. Because of its general form, this framework can describe the dynamics of globally coupled classical Heisenberg spins, long-range couplings in liquid crystals, or the orbital inclination evolution of stars in nearly Keplerian systems.

I. INTRODUCTION

Long-range interacting systems generically undergo an evolution in two stages. First, a fast (collisionless) violent relaxation [1] during which the system reaches a quasistationary state (a steady state of the collisionless Boltzmann equation) and the system is dynamically frozen under the mean field dynamics. Then, as a consequence of the finite number of particles, the system undergoes a slow (collisional) relaxation that drives it towards thermodynamical equilibrium. This second stage is generically described by the Balescu–Lenard (BL) equation [2, 3], recently generalized to inhomogeneous systems [4, 5]. These formalisms can account simultaneously for inhomogeneity (i.e. non-trivial orbital structures), collective effects (i.e., spontaneous amplification of perturbations) and non-local resonant couplings.

In this letter, we focus our attention on one such long-range interacting system, namely the problem of long-range coupled particles evolving on a sphere. Because of its general form, this system is of relevance in various physical setups ranging from spin dynamics to stellar systems (see Section II). Here, we show how in the axisymmetric limit, the generic methods of inhomogeneous kinetic theory can be applied, and accordingly derive the associated kinetic equation. In addition to allowing for quantitative predictions of the system’s diffusion coefficients, we clarify how this theory predicts the dependence of the system’s relaxation rate with the number of particles, and the important role played by the frequency profile in that respect.

The paper is organized as follows. In Section II, we present the considered model. Placing ourselves within the axisymmetric limit, we derive in Section III the ap-

propriate Balescu–Lenard equation describing the long-term evolution of that system. In Section IV, we present applications of this formalism to recover the system’s diffusion coefficients as well as the scaling of the relaxation rate with the number of particles. Finally, we conclude in Section V.

II. THE MODEL

We consider a set of N particles evolving on a sphere of unit radius, and denote the spherical coordinates with (ϕ, ϑ) . To any location on the sphere, we associate a normal vector $\mathbf{L} = \mathbf{L}(\phi, \vartheta)$. The specific Hamiltonian of the system is

$$H = \mu \sum_{i < j} U(\mathbf{L}_i \cdot \mathbf{L}_j) + \sum_i U_{\text{ext}}(\mathbf{L}_i), \quad (1)$$

where $\mu = M_{\text{tot}}/N$ is the individual mass of the particles, $U(\mathbf{L}, \mathbf{L}') = U(\mathbf{L} \cdot \mathbf{L}')$ is the pairwise interaction, and $U_{\text{ext}}(\mathbf{L})$ is an imposed external potential. The pairwise interaction is developed in Legendre Polynomials as

$$\begin{aligned} U(\mathbf{L}_i \cdot \mathbf{L}_j) &= - \sum_{\ell} \alpha_{\ell} P_{\ell}(\mathbf{L}_i \cdot \mathbf{L}_j) \\ &= - \sum_{\ell, m} \alpha_{\ell} b_{\ell} Y_{\ell}^m(\mathbf{L}_i) Y_{\ell}^{m*}(\mathbf{L}_j); \quad b_{\ell} = \frac{4\pi}{2\ell + 1}, \end{aligned} \quad (2)$$

where we used the addition theorem, and introduced the spherical harmonics $Y_{\ell}^m(\mathbf{L}) = K_{\ell}^m P_{\ell}^m(u) e^{im\phi}$, where $P_{\ell}^m(u)$ is the associated Legendre functions [6], and $K_{\ell}^m = \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2}$. The spherical harmonics are normalized as $\int d\mathbf{L} Y_{\ell}^m Y_{\ell'}^{m'*} = \delta_{\ell}^{\ell'} \delta_m^{m'}$, with the unit volume $d\mathbf{L} = d\vartheta \sin(\vartheta) d\phi$. The canonical coordinates associated with this two-dimensional phase space are $\mathbf{w} = (\phi, \cos(\vartheta) = u)$, and the equations of motion for particle i are $\dot{\phi}_i = \partial H / \partial u_i$ and $\dot{u}_i = -\partial H / \partial \phi_i$. We recast

*Hubble Fellow

these equations as

$$\frac{d\mathbf{L}_i}{dt} = \sum_{\ell, m} M_\ell^m(t) \mathbf{X}_\ell^m(\mathbf{L}_i) + \mathbf{X}_{\text{ext}}(\mathbf{L}_i), \quad (3)$$

where $\mathbf{X}_\ell^m(\mathbf{L}) = \mathbf{L} \times \partial Y_\ell^m / \partial \mathbf{L}$ are the vector spherical harmonics,

$$M_\ell^m(t) = \mu \alpha_\ell b_\ell \sum_j Y_\ell^{m*}(\mathbf{L}_j(t)) \quad (4)$$

are the system's instantaneous magnetizations and $\mathbf{X}_{\text{ext}}(\mathbf{L}) = -\mathbf{L} \times \partial U_{\text{ext}} / \partial \mathbf{L}$ captures the contribution from the external potential.

Equation (3) is the exact evolution equation of this problem. The Hamiltonian from Eq. (1) encompasses a wide class of long-range interacting systems: (i) $U = -\alpha_1 P_1$ describes globally coupled classical Heisenberg spins [7, 8], (ii) $U = -\alpha_2 P_2$ is the Maier-Saupe model for liquid crystals [9, 10], (iii) $U = -\sum_\ell \alpha_{2\ell} P_{2\ell}$ captures the process of *vector resonant relaxation* in galactic nuclei [11–13] (up to additional conserved quantities).

In the coming section, we consider the most general setup, but place ourselves within the axisymmetric limit, i.e. where the mean field Hamiltonian is invariant w.r.t. ϕ . We show how the general kinetic theory of long-range interacting systems (see Appendix A) can straightforwardly be applied to this regime, and accordingly derive the associated evolution equation.¹

III. THE BALESKU–LENARD EQUATION

Let us assume that the system is characterized by a mean distribution function (DF), $F(\mathbf{L})$, normalized so that $\int d\mathbf{L} F = M_{\text{tot}}$, with $M_{\text{tot}} = 1$ the total mass of the system. Following Eq. (1), the mean specific Hamiltonian of a particle in that system reads

$$\begin{aligned} H_0(\mathbf{L}) &= \int d\mathbf{L}' U(\mathbf{L} \cdot \mathbf{L}') F(\mathbf{L}') + U_{\text{ext}}(\mathbf{L}) \\ &= \sum_\ell h_\ell P_\ell(u) + U_{\text{ext}}(u), \end{aligned} \quad (5)$$

where in the second line, we assumed that the system's DF and the external potential are axisymmetric, i.e., $F(\mathbf{L}) = F(u)$ and $U_{\text{ext}}(\mathbf{L}) = U_{\text{ext}}(u)$, and introduced the coefficients $h_\ell = -2\pi\alpha_\ell \int du' P_\ell(u') F(u')$. The associated orbital frequency $\Omega(u) = dH_0/du$ naturally follows from Eq. (5). For axisymmetric configurations, we have $H_0(\mathbf{L}) = H_0(u)$. Therefore, the Poisson bracket satisfies $[H_0(u), F(u)] = 0$, i.e., any axisymmetric DF is a steady

state for the mean field dynamics. In addition, the mean Hamiltonian is integrable, as the action $J = u$ is conserved along the mean motion, while the associated angle $\theta = \phi$, evolves linearly in time with the frequency $\Omega(u)$.

Investigating the long-term evolution of such a quasi-stationary steady state amounts to investigating the slow distortion of the system's mean DF, $F(u)$ (assumed to remain linearly stable and axisymmetric throughout its evolution). Following the general kinetic theory of long-range interacting integrable systems (briefly reproduced in Appendix A), deriving the kinetic equation for $F(u)$ is immediate. One only needs to proceed by analogies as we detail below.

The interaction potential can be written under the separable form $U(\mathbf{L} \cdot \mathbf{L}') = -\sum_p \psi^{(p)}(\mathbf{L}) \psi^{(p)*}(\mathbf{L}')$, where the potential basis elements are

$$\psi^{(p)}(\mathbf{L}) = C_{\ell p} Y_{\ell p}^{m p}(\mathbf{L}), \quad C_\ell = \sqrt{\alpha_\ell b_\ell}. \quad (6)$$

Their Fourier transform w.r.t. the angle $\theta = \phi$ reads

$$\psi_k^{(p)}(u) = \int \frac{d\phi}{2\pi} e^{-ik\phi} C_{\ell p} Y_{\ell p}^{m p}(u, \phi) = \delta_k^{m p} c_{\ell p}^{m p}(u), \quad (7)$$

with the coefficient $c_\ell^m(u) = C_\ell K_\ell^m P_\ell^m(u)$. Injected in Eq. (A7), the system's response matrix becomes

$$\widehat{\mathbf{M}}_{pq}(\omega) = 2\pi \delta_{m p}^{m q} \int du \frac{m^p \partial F / \partial u}{\omega - m^p \Omega(u)} c_{\ell p}^{m p*}(u) c_{\ell q}^{m q}(u). \quad (8)$$

A system is then said to be linearly unstable if there exists a complex frequency $\omega = \omega_0 + i\eta$ (with $\eta > 0$), for which $\widehat{\mathbf{M}}(\omega)$ admits an eigenvalue equal to 1. In that case, the system supports an unstable mode of pattern speed ω_0 , and growth rate η [see Section 5.3 in 14]. As such, $\widehat{\mathbf{M}}(\omega)$ generically describes the system's linear stability around the mean axisymmetric state $F(u)$, as captured by the linearized collisionless Boltzmann equation $\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F, \delta H(\delta F)] = 0$, with $[F, H] = \frac{\partial F}{\partial \phi} \frac{\partial H}{\partial u} - \frac{\partial F}{\partial u} \frac{\partial H}{\partial \phi}$ the usual Poisson bracket. In the present context, Eq. (8) generalizes the stability criteria put forward in [7, 8] (see Appendix B).

Following Eq. (A6), the system's *dressed* susceptibility coefficients read

$$\psi_{kk'}^d(u, u', \omega) = -\delta_k^{k'} \sum_{\ell, \ell' \geq |k|} c_\ell^k(u) c_{\ell'}^{k'*}(u') [\mathbf{I}_k - \widehat{\mathbf{M}}_k(\omega)]_{\ell \ell'}^{-1}, \quad (9)$$

where

$$[\mathbf{I}_k]_{\ell \ell'} = \delta_\ell^{\ell'}; \quad [\widehat{\mathbf{M}}_k(\omega)]_{\ell \ell'} = \widehat{\mathbf{M}}_{[\ell, k], [\ell', k]}(\omega). \quad (10)$$

Assuming that the system is linearly stable, and that the frequency profile is non-degenerate (i.e. $\partial \Omega / \partial u = 0$ only in isolated points), the long-term evolution of this axisymmetric system is characterized by the inhomogeneous BL equation (see Eq. (A1)), that reads here

$$\begin{aligned} \frac{\partial F}{\partial t} &= 2\pi^2 \mu \frac{\partial}{\partial u} \left[\int du' |\psi_{\text{tot}}^d(u, u', \Omega(u))|^2 \delta_D(\Omega(u) - \Omega(u')) \right. \\ &\quad \left. \times \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial u'} \right) F(u) F(u') \right], \end{aligned} \quad (11)$$

¹ We make the correspondence with Appendix A by noting that $\mathbf{L} = (\phi, u)$ plays the role of $\mathbf{w} = (\theta, J)$, ϕ the role of the angle θ , and u the role of the action J .

where we introduced the total dressed susceptibility coefficients $\psi_{\text{tot}}^{\text{d}}(u, u', \omega)$ as

$$|\psi_{\text{tot}}^{\text{d}}(u, u', \omega)|^2 = 2 \sum_{k \geq 1} k |\psi_{kk}^{\text{d}}(u, u', k\omega)|^2. \quad (12)$$

In Eq. (11), we emphasize the absence of a sum on resonance vectors, owing to the Kronecker symbol in Eq. (9). Collective effects can be switched off by imposing $\widehat{\mathbf{M}}_{pq}(\omega) = 0$ (i.e. replacing the dressed susceptibility coefficients, $\psi_{kk'}^{\text{d}}(u, u', \omega)$, by their bare analogs, $\psi_{kk'}(u, u')$, see Eq. (A3)), which leads to the inhomogeneous Landau equation [15]. Finally, we recall that Eq. (11) can be rewritten as a Fokker–Planck equation

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial u} [D_1(u) F(u)] + \frac{1}{2} \frac{\partial^2}{\partial u^2} [D_2(u) F(u)], \quad (13)$$

with the first- and second-order diffusion coefficients

$$D_2(u) = (2\pi)^2 \mu \int du' |\psi_{\text{tot}}^{\text{d}}|^2 \delta_{\text{D}}(\Omega(u) - \Omega(u')) F(u'), \quad (14)$$

$$D_1(u) = \frac{1}{2} \frac{\partial D_2}{\partial u} + 2\pi^2 \mu \int du' |\psi_{\text{tot}}^{\text{d}}|^2 \delta_{\text{D}}(\Omega(u) - \Omega(u')) \frac{\partial F}{\partial u'}.$$

In practice, for a given value of u , one can carry out the integral $\int du'$ in Eq. (11) by finding the resonant actions u_* satisfying $\Omega(u_*) = \Omega(u)$, which allows for the replacement $\delta_{\text{D}}(\Omega(u) - \Omega(u_*)) = \sum_{u_*} \delta_{\text{D}}(u - u_*) / |\partial\Omega/\partial u|_{u=u_*}$.

Because of its prefactor $\mu = M_{\text{tot}}/N$, the BL equation describes the system's long-term self-consistent evolution computed at first-order in the $1/N$ effects, accounting for the amplification by collective effects. Here, it is important to note that (i) the orbital space is one dimensional (cf. the one dimensional integral $\int du'$ in Eq. (11)), (ii) the symmetry of the interaction imposes 1:1 resonances (cf. the absence of sums over (k, k') in Eq. (11)). As a consequence, if the system's mean frequency profile, $u \mapsto \Omega(u)$, is monotonic, the resonance condition $\delta_{\text{D}}(\Omega(u) - \Omega(u'))$ only allows for local resonances, i.e. $u' = u$, leading to zero flux and $\partial F/\partial t = 0$. In that case, the system cannot relax under $1/N$ effects [8, 16–18]. It undergoes a so-called kinetic blocking [19], and can only relax under weaker finite- N effects associated with higher-order correlations. Still, even if the flux vanishes, the diffusion coefficients $D_1(u)$ and $D_2(u)$ remain non-zero. Conversely, for a non-monotonic frequency profile, non-local resonances, $u' \neq u$, are allowed, the flux is non-zero, and the system can relax at the order $1/N$. We illustrate these various effects in the coming section. Finally, we emphasize that the Boltzmann DF, $F \propto e^{-\beta H_0(u)}$ is always a stationary solution of the BL equation. Yet, the fact that for kinetically blocked systems any axisymmetric DF is a stationary solution of the BL equation does not imply that these states remain stationary when higher order correlation effects are accounted for.

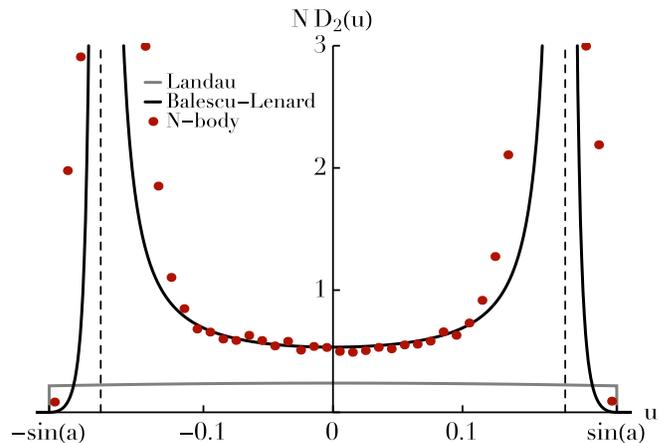


FIG. 1: Illustration of the second-order diffusion coefficient, $N D_2(u)$, for the waterbag DF from Eq. (16), as predicted by Eq. (14), in the absence (Landau) or presence (BL) of collective effects, and compared with N -body simulations (that naturally include collective effects). As a result of the presence of neutral modes (see Eq. (B6)), the BL diffusion coefficient locally diverge, as indicated by the vertical dashed lines.

IV. APPLICATION

We now illustrate the previous formalism, and compare it with direct N -body simulations (whose details are presented in Appendix C).

Following [7], we first consider a system driven by interactions of the form

$$U(x) = -\alpha_1 P_1(x); \quad U_{\text{ext}}(u) = D_{\text{ext}} u^2, \quad (15)$$

with $\alpha_1 = 1$, $D_{\text{ext}} = 15$, and $P_1(x) = x$. In that case, Eq. (5) gives that $\Omega(u)$ is a first degree polynomial in u , i.e. the frequency profile is monotonic. As for the system's DF, we consider a waterbag DF

$$F(u) = C \Theta(\sin(a) - |u|), \quad (16)$$

with $\Theta(x)$ the Heaviside function, and C a normalization constant. We pick $\epsilon = D_{\text{ext}} \sin^2(a)/3 = 0.24$, for which the system is linearly stable [7]. The gradient of this DF involves Dirac deltas, which makes the computation of the response matrix immediate, as one can get rid of the integral from Eq. (8), and we refer to Eq. (B3) for the associated explicit expression. Yet, because of these infinite gradients, the system also supports neutral modes (i.e. modes with zero growth rates [20]), which lead to localized divergences in the system's diffusion coefficients, as detailed in Eq. (B6). In Fig. 1, we illustrate the BL prediction for such diverging diffusion coefficients as well as measurements from direct N -body simulations (using the procedure described in Appendix C).

Keeping the same interactions as in Eq. (15), one can avoid the presence of neutral modes by considering a

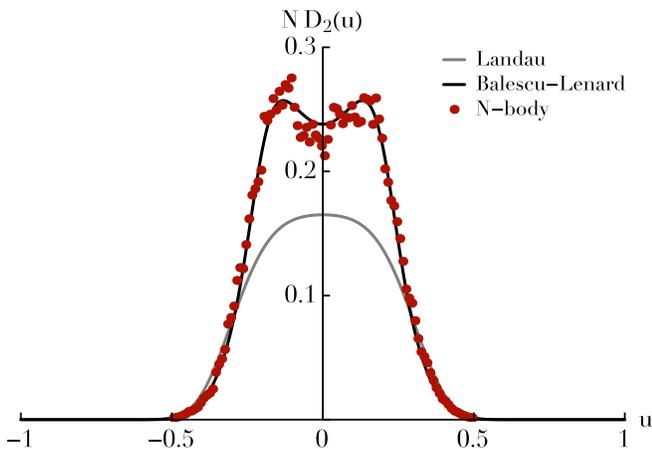


FIG. 2: Same as in Fig. 1, but for the DF from Eq. (17).

smooth DF, for example

$$F(u) = C e^{-(u/\sigma)^4}, \quad (17)$$

with $\sigma = 0.35$ and C a normalization constant. In Appendix D, we present our implementation of the matrix method, and check that such a system is linearly stable (see Fig. 6). In Fig. 2, we illustrate the BL diffusion coefficients and the associated N -body measurements for that system.

Glancing at Eq. (11), we argued that a system with a monotonic frequency profile undergoes a kinetic blocking and cannot relax under $1/N$ effects. We illustrate this in Fig. 3 for the waterbag DF from Eq. (16). Following [18], the dependence of the relaxation rate with N is estimated through the quantity $m_4(N, t) = \{(u - \{u\})^4\}$, with $\{x\} = \sum_i x_i/N$ the average over all the particles of a given realization. For a given N , the time series of $m_4(N, t)$ is averaged over 100 realizations, as illustrated in the top panel of Fig. 3. Finally, for a given threshold value \bar{m}_4 , we determine the crossing time \bar{t}_N such that $m_4(N, \bar{t}_N) = \bar{m}_4$. Should the BL equation (11) have a non-vanishing flux, one expects the scaling $\bar{t}_N \propto N$. The dependence of $N \mapsto \bar{t}_N$ for the waterbag DF is illustrated in the bottom panel of Fig. 3. In the range $6 \times 10^2 \leq N \leq 32 \times 10^2$, we measure the scaling $\bar{t}_N \propto N^{1.92 \pm 0.09}$, which is expected to converge to N^2 for larger values of N [18]. This system indeed suffers from a kinetic blocking because of the impossibility of non-local resonant couplings for a monotonic frequency profile.

In order to recover the scaling predicted by the BL equation (while assuming that $U_{\text{ext}}(u) = D_{\text{ext}}u^2$ as in Eq. (15)), one has to consider a model in which higher harmonics ($\ell = 3$ or higher) contribute to the pairwise interaction. To illustrate this point, we finally consider a system driven by interactions of the form

$$U(x) = -\alpha_1 P_1(x) - \alpha_3 P_3(x); \quad U_{\text{ext}}(u) = D_{\text{ext}}u^2, \quad (18)$$

with $\alpha_1 = \alpha_3 = 1$, $D_{\text{ext}} = -1/2$, and $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. In that case, Eq. (5) gives that $\Omega(u)$ is a non-monotonic

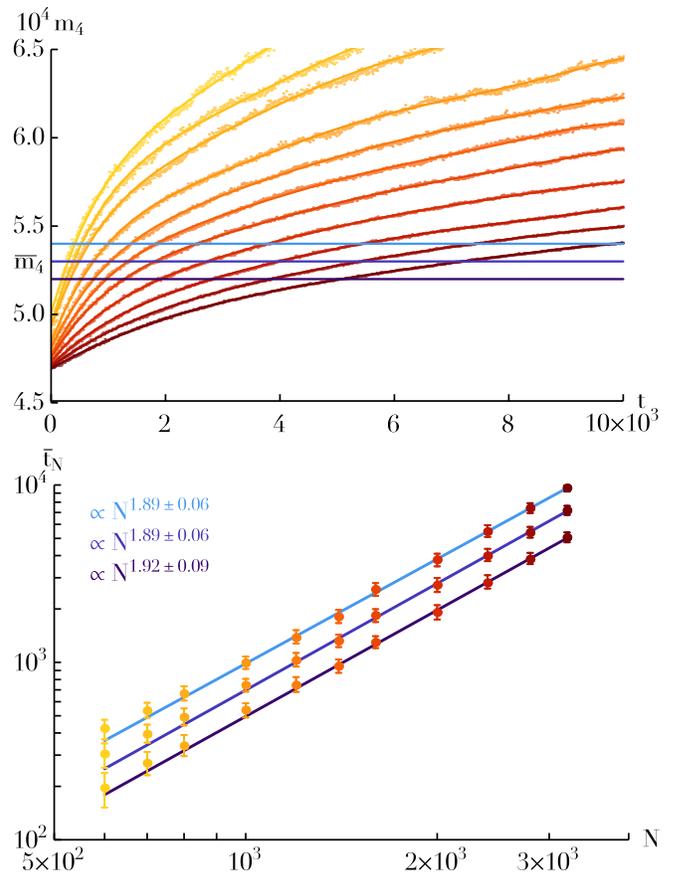


FIG. 3: Dependence of the relaxation rate with the number of particles for the waterbag system from Eq. (16) that has a monotonic frequency profile yielding a kinetic blocking. *Top panel*: Time dependence of $m_4(N, t)$ for simulations with $N \in \{6, 7, 8, 10, 12, 14, 16, 20, 24, 28, 32\} \times 10^2$ (from light to dark colors) averaged over 100 realizations (dots), and the associated fits (curves). The horizontal lines represent the threshold values \bar{m}_4 used to measure the crossing times. *Bottom panel*: Dependence of the crossing time \bar{t}_N with the number of particles for different \bar{m}_4 (light to dark colors). Errors bars for the crossing times were estimated by performing 200 bootstrap resamplings over the realizations available: colored dots represent the median value, and error bars the 10% and 90% confidence levels. Errors on the power-law fits were estimated by fitting each bootstrap resamplings with a power-law, while the plotted fits are the best fit for the median values.

second degree polynomial in u . We choose the system's axisymmetric DF to be

$$F(u) = C e^{-(u-u_0)^2/(2\sigma^2)}, \quad (19)$$

with $u_0 = 0.2$ and $\sigma = 0.1$, and illustrate it in Fig. 4. Following Appendix D, we checked that such a system is linearly stable. In Fig. 5, we estimate the scaling of the system's relaxation with the number of particles. In the range $6 \times 10^2 \leq N \leq 32 \times 10^2$, we measure a scaling of the form $\bar{t}_N \propto N^{1.1 \pm 0.06}$, that is in sensible agreement with

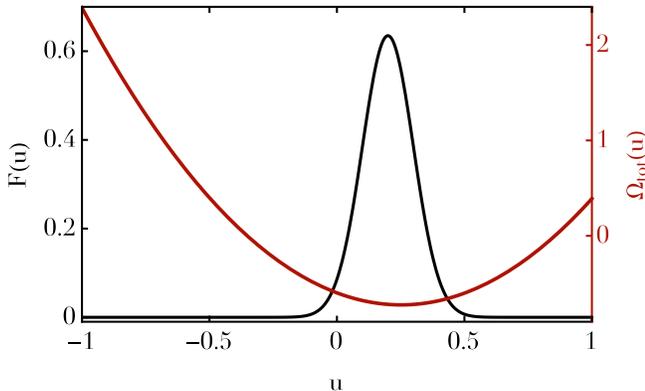


FIG. 4: Illustration of the DF from Eq. (19) and the associated non-monotonic frequency profile $u \mapsto \Omega(u)$. In the region of the DF's maximum, the resonance condition $\Omega(u') = \Omega(u)$ has two solutions, allowing for non-local resonant couplings.

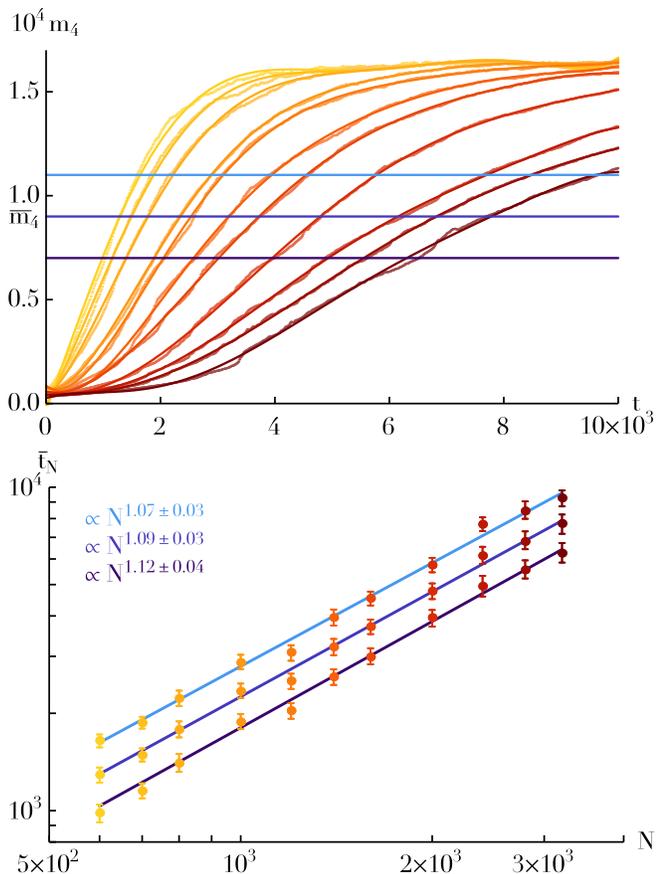


FIG. 5: Same as in Fig. 3, but for the DF from Eq. (19) that has a non-monotonic frequency profile, preventing any kinetic blocking.

the prediction from the BL equation. Because this system can support non-local orbital resonances, it relaxes much more efficiently than kinetically-blocked systems.

V. CONCLUSION

The inhomogeneous BL equation is being increasingly used to constrain complex dynamical regimes, such as the 1D Hamiltonian Mean Field (HMF) model [21], 2D razor-thin stellar disks [22], or 3D stellar systems with or without central mass [23, 24].

In the present letter, we illustrated how the same method may be applied to characterize the dynamics of long-range coupled particles on a sphere in the axisymmetric limit. Once one has recognized that this system's evolution equations are formally identical to the ones of a long-range interacting integrable system, the derivation of the kinetic theory becomes straightforward. In the present case, the reduced number of dimensions of phase space imposes additional geometrical constraints to the system's dynamics, e.g. allowing only for 1:1 resonance. We detailed how in the presence of a monotonic frequency profile, the system is submitted to a kinetic blocking and cannot relax under $1/N$ effects, a behavior already encountered in the context of axisymmetrically distributed point vortices [19].² This blocking gets lifted in the presence of a non-monotonic frequency profile, for which non-local resonant couplings are possible.

To emphasize the strength of the BL formalism, we presented quantitative comparisons with direct numerical simulations, recovering both the individual diffusion coefficients, as well as the expected scaling of the relaxation rate with the number of particles.

Despite its recent success, the kinetic theory of long-range interacting systems still asks for more developments, in particular to describe systems with fully degenerate frequency profiles (i.e. $\Omega(\mathbf{J}) = 0$, e.g. in the isotropic limit of the present system), or to obtain the $1/N^2$ kinetic equation for systems undergoing a kinetic blocking [17, 18].

² Because of the absence of a (quadratic) kinetic energy term in the Hamiltonian from Eq. (1), the present model shares some similarities with 2D point vortices, e.g. the existence of negative temperature statistical equilibria, or a similar BL equation for axisymmetric distributions of point vortices. Depending on the harmonic indices present in the interaction potential, the shape of $\Omega(u)$ for the present model can be independent of time, while in the vortex case it depends on time as it is obtained self-consistently from the system's density profile [19]. In particular, point vortices systems can still undergo a kinetic blocking even if the frequency profile is initially non-monotonic, provided that it becomes monotonic during the evolution.

Acknowledgments

JBF acknowledges support from Program number HST-HF2-51374 which was provided by NASA through a grant from the Space Telescope Science Institute, which is operated by the Association of Universities for Research in Astronomy, Incorporated, under NASA contract NAS5-26555. BB is supported by membership from Martin A. and Helen Chooljian at the Institute for Advanced Study.

Appendix A: The inhomogeneous Balescu–Lenard equation

In this Appendix, we repeat the main results regarding the inhomogeneous BL equation, first derived in [4, 5]. These results are used in the main text to concisely derive the kinetic equation for the problem at hand.

We generically consider an Hamiltonian system in $2d$ dimensions, and write the phase space canonical coordinates as $\mathbf{w} = (\boldsymbol{\theta}, \mathbf{J})$, respectively the angle and action coordinates [14]. The system is assumed to be in an integrable steady state, and following Jeans' theorem [25], it can be described by a DF of the form, $F(\mathbf{w}) = F(\mathbf{J})$, that we normalize as $\int d\mathbf{w} F = M_{\text{tot}}$. We denote the mean (integrable) potential as $H_0(\mathbf{w}) = H_0(\mathbf{J})$, with the associated orbital frequencies $\boldsymbol{\Omega}(\mathbf{J}) = \partial H_0 / \partial \mathbf{J}$. The system comprises N particles of mass $\mu = M_{\text{tot}}/N$, coupled one to another through a long-range interaction potential $U(\mathbf{w}, \mathbf{w}')$.

As a result of the finite number of particles, the system's orbital structure gets slowly distorted. To first order in $1/N$ this dynamics is described by the inhomogeneous BL equation

$$\begin{aligned} \frac{\partial F(\mathbf{J})}{\partial t} &= \pi(2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}, \mathbf{k}'} \mathbf{k} \int d\mathbf{J}' |\psi_{\mathbf{k}\mathbf{k}'}^d(\mathbf{J}, \mathbf{J}', \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}))|^2 \right. \\ &\times \delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \boldsymbol{\Omega}(\mathbf{J}')) \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}} - \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J}) F(\mathbf{J}') \left. \right]. \end{aligned} \quad (\text{A1})$$

Following Kalnajs matrix method [26], one introduces a biorthogonal basis of potentials and densities $(\psi^{(p)}(\mathbf{w}), \rho^p(\mathbf{w}))$, with the convention

$$\begin{aligned} \psi^{(p)}(\mathbf{w}) &= \int d\mathbf{w}' U(\mathbf{w}, \mathbf{w}') \rho^{(p)}(\mathbf{w}'), \\ \int d\mathbf{w} \rho^{(p)}(\mathbf{w}) \psi^{(q)*}(\mathbf{w}) &= -\delta_p^q. \end{aligned} \quad (\text{A2})$$

The pairwise interaction, $U(\mathbf{w}, \mathbf{w}')$, can then be cast under the separable form

$$U(\mathbf{w}, \mathbf{w}') = - \sum_p \psi^{(p)}(\mathbf{w}) \psi^{(p)*}(\mathbf{w}'). \quad (\text{A3})$$

The separable decomposition from Eq. (A3) is a fundamental equation that determines what are the *natural*

basis elements appropriate for a given problem, e.g. as in Eq. (6). The bare susceptibility coefficients, $\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}')$, are the Fourier transform of the pairwise interaction $U(\mathbf{w}, \mathbf{w}')$ w.r.t. $(\boldsymbol{\theta}, \boldsymbol{\theta}')$, namely

$$U(\mathbf{w}, \mathbf{w}') = \sum_p \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^d} \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') e^{i(\mathbf{k} \cdot \boldsymbol{\theta} - \mathbf{k}' \cdot \boldsymbol{\theta}')}, \quad (\text{A4})$$

$$\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') = \int \frac{d\boldsymbol{\theta}}{(2\pi)^d} \frac{d\boldsymbol{\theta}'}{(2\pi)^d} U(\boldsymbol{\theta}, \mathbf{J}, \boldsymbol{\theta}', \mathbf{J}') e^{-i(\mathbf{k} \cdot \boldsymbol{\theta} - \mathbf{k}' \cdot \boldsymbol{\theta}')},$$

and Eq. (A1) with $\psi_{\mathbf{k}\mathbf{k}'}^d(\mathbf{J}, \mathbf{J}', \omega) \rightarrow \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}')$ is the inhomogeneous Landau equation. One can write

$$\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') = - \sum_p \psi_{\mathbf{k}}^{(p)}(\mathbf{J}) \psi_{\mathbf{k}'}^{(p)*}(\mathbf{J}'), \quad (\text{A5})$$

with $\psi_{\mathbf{k}}^{(p)}(\mathbf{J}) = \int d\boldsymbol{\theta} / (2\pi)^d \psi^{(p)}(\mathbf{w}) e^{-i\mathbf{k} \cdot \boldsymbol{\theta}}$ standing for the Fourier transform of the basis elements. To account for collective effects (i.e. the spontaneous amplification of fluctuations), one replaces the bare susceptibility coefficients by their dressed analogs

$$\psi_{\mathbf{k}\mathbf{k}'}^d(\mathbf{J}, \mathbf{J}', \omega) = - \sum_{p,q} \psi_{\mathbf{k}}^{(p)}(\mathbf{J}) [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]_{pq}^{-1} \psi_{\mathbf{k}'}^{(q)*}(\mathbf{J}'), \quad (\text{A6})$$

as introduced in the BL Eq. (A1). Finally, in Eq. (A6), the response matrix of a long-range integrable system is given by

$$\widehat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}'}^{(q)}(\mathbf{J}'), \quad (\text{A7})$$

and its effective numerical computation is briefly illustrated in Appendix D. We note that (i) Eq. (A5) can be obtained from Eq. (A6) by taking $\widehat{\mathbf{M}} = 0$ (i.e. switching off collective effects); (ii) the substitution of Eq. (A3) into Eq. (A2) leads to an identity; (iii) Eq. (A3) can be obtained from Eqs. (A4) and (A5) showing the unicity of this expression. As emphasized after Eq. (8), the response matrix, $\widehat{\mathbf{M}}(\omega)$, characterizes the linear stability of the system.

Appendix B: The case of the Heisenberg spins

In this Appendix, we consider the case where the system's interaction is limited to only one harmonic ℓ , so that $U(x) = -\alpha_\ell P_\ell(x)$. This is of particular relevance for classical Heisenberg spins ($\ell = 1$) [7, 8], and the Maier-Saupe model for liquid crystals ($\ell = 2$) [9, 10]. In that limit, the dressed susceptibility coefficients from Eq. (9) become

$$\psi_{k k'}^d(u, u', \omega) = -\delta_k^{k'} \frac{c_\ell^k(u) c_\ell^{k'*}(u')}{\varepsilon_\ell^k(\omega)}, \quad (\text{B1})$$

with $0 < |k| \leq \ell$, and the susceptibility coefficient, $\varepsilon_\ell^k(\omega)$, follows from Eq. (8) reading

$$\varepsilon_\ell^k(\omega) = 1 - 2\pi \int du \frac{k \partial F / \partial u}{\omega - k\Omega(u)} |c_\ell^k(u)|^2. \quad (\text{B2})$$

Let us now follow Eq. (15) and restrict ourselves to the pairwise interaction $U(x) = -\alpha_1 P_1(x)$, with $\alpha_1 = 1$. In that context, Eq. (B2) reduces to

$$\varepsilon_1^{\pm 1}(\omega) = 1 \mp \pi \int du \frac{(1 - u^2) \partial F / \partial u}{\omega \mp \Omega(u)}, \quad (\text{B3})$$

with $\Omega(u) = h_1 + 2D_{\text{ext}}u$ (see Eq. (5) for the definition of h_ℓ). As such, in Eq. (B3), we immediately recover the susceptibility coefficients obtained by a more complex method in [8] (see Eq. (33) therein). For a waterbag DF as in Eq. (16), the expression of the susceptibility coefficients can be further simplified to become

$$\varepsilon_1^{\pm 1}(\omega) = 1 + \frac{D_{\text{ext}}(1 - \sin^2(a))}{\omega^2 - (2D_{\text{ext}}\sin(a))^2} \equiv \varepsilon(\omega). \quad (\text{B4})$$

Such a system is linearly stable if there exists no $\omega = \omega_0 + i\eta$ (with $\eta > 0$) for which $\varepsilon(\omega) = 0$. The constraint $\text{Im}[\varepsilon(\omega)] = 0$, naturally imposes $\omega_0 = 0$. As for the constraint $\text{Re}[\varepsilon(i\eta)] = 0$, and introducing the energy of the system as $\epsilon = D_{\text{ext}}\sin^2(a)/3$, one concludes that the system admits no unstable modes if

$$\kappa = \frac{D_{\text{ext}} - 3\epsilon}{12D_{\text{ext}}\epsilon} \quad (\text{B5})$$

satisfies $\kappa < 1$. Introducing the critical energy $\epsilon_* = D_{\text{ext}}/(3 + 12D_{\text{ext}})$, the system is therefore stable if $\epsilon > \epsilon_*$, and we recover the criterion put forward in [7, 8].

For a waterbag DF from Eq. (16), one can finally compute explicitly the diffusion coefficient from Eq. (14). The frequency, $\Omega(u) = 2D_{\text{ext}}u$, being monotonic, the resonance condition is straightforwardly solved, and Eq. (16) gives

$$D_2(u) = \frac{(2\pi)^2 \mu}{D_{\text{ext}}} \frac{|c_1(u)|^4}{|\varepsilon(\Omega(u))|^2} F(u), \quad (\text{B6})$$

with $c_1(u) = \sqrt{(1 - u^2)/2}$. Introducing $\tilde{\omega} = \omega/\omega_{\text{max}}$, with $\omega_{\text{max}} = (2D_{\text{ext}}\sin(a))$, the inverse of the susceptibility coefficient reads

$$\frac{1}{\varepsilon(\omega)} = \frac{1 - \tilde{\omega}^2}{(1 - \kappa) - \tilde{\omega}^2}, \quad (\text{B7})$$

where we recall that $\kappa < 1$ for a linearly stable system. As a consequence, for $\tilde{\omega} = \sqrt{1 - \kappa}$, the inverse of the susceptibility coefficient becomes infinite, i.e. the system supports a neutral mode [20]. This leads in particular to a divergence of the diffusion coefficient in $u = \pm \sin(a)\sqrt{1 - \kappa}$, as highlighted in Fig. 1. We finally note that in the absence of collective effects (i.e. in the Landau limit), these divergences vanish.

Appendix C: The N -body implementation

In this Appendix, we present our N -body implementation of the problem at hand. Glancing back at the equations of motion Eq. (3), one notes that the velocity vector, $d\mathbf{L}_i/dt$, is expressed only as a function of the current particle's location, $\mathbf{L}_i(t)$, and the instantaneous values of the magnetizations, $M_\ell^m(t)$. There are N such evolution equations, but since magnetizations are shared by all particles, their computation can be done only once per timestep. As a result, the overall complexity of advancing the particles for one timestep scales like $\mathcal{O}(N \ell_{\text{max}}^2)$, with ℓ_{max} the maximum harmonic number appearing in the considered pairwise interaction in Eq. (2).

The heart of the N -body implementation is then (i) to compute efficiently the spherical harmonics (and the vector ones) at the location of the particles, (ii) to compute the magnetizations in Eq. (4), and (iii) to compute the velocity fields in Eq. (3). To compute the spherical harmonics, it is convenient to work with real spherical harmonics. These are computed following [27, see Eq. (6.7.9)] for the renormalized associated Legendre polynomials, and the second-order recurrence relation $\cos(m\phi) = 2\cos(\phi)\cos((m-1)\phi) - \cos((m-2)\phi)$ (similarly for $\sin(m\phi)$) for the azimuthal component. For the (real) vector spherical harmonics, we follow the recurrences presented in [28, see Appendix (B.2)], adapted to the renormalized associated Legendre polynomials. Once all velocity vectors $d\mathbf{L}_i/dt$ are determined, particles are advanced for a timestep h ($= 10^{-3}$ in all our applications) using a fourth-order Runge-Kutta integrator [27, see Eq. (17.1.3)].

To measure diffusion coefficients in N -body simulations, we proceed similarly to [21] in the case of the HMF model. We note that the present model shares some similarities with the HMF model [29], in particular the property of being a decoupled N -body problem, i.e. it can be integrated in $\mathcal{O}(N)$ operations per timestep, rather than $\mathcal{O}(N^2/2)$. Here, to measure diffusion coefficients, we perform $N_{\text{real}} = 200$ different realizations, with $N = 10^5$ particles. At the initial time, particles are divided among action bins of size $\delta u = 10^{-2}$. For each realization and action bin, we determine the time series of the mean square variation of $\Delta u^2(t) = (u(t) - u(0))^2$, averaged over all the particles initially within the bin. For every action bin, these time series are then averaged over all realizations, and considered up to the time where $\langle \Delta u^2(t) \rangle \geq (\delta u)^2$. The diffusion coefficients are obtained finally through a linear fit of these ensemble-averaged series, as in Figs. 1 and 2.

Appendix D: The matrix method

The generic expression of the response matrix is given by Eq. (8). It asks to compute an expression of the form

$$\begin{aligned} \int_{-1}^1 du \frac{g(u)}{h(u) + i\eta} &\simeq \sum_i \int_{-\frac{\delta u}{2}}^{\frac{\delta u}{2}} dx \frac{a_g^i + b_g^i x}{a_h^i + b_h^i x + i\eta} \\ &= \sum_i \frac{a_g^i}{a_h^i} \delta u \aleph_D \left[\frac{b_g^i \delta u}{a_g^i}, \frac{b_h^i \delta u}{a_h^i}, \frac{\eta}{a_h^i} \right], \end{aligned} \quad (\text{D1})$$

where $g(u), h(u)$ are real, and $\eta > 0$ is an imaginary part added to the frequency ω . To get the r.h.s. of Eq. (D1), we followed [22], truncated the integration domain $u \in [-1; 1]$ into K regions, introducing $\delta u = 2/K$, so that the center of each region is $u_i = -1 + \delta u(i - \frac{1}{2})$ with $1 \leq i \leq K$, and performed a linear expansion of the numerator and denominator, so that $a_g^i = g(u_i)$ and $b_g^i = dg(u_i)/du$ (similarly for h). To get the second line, we assumed $a_g, a_h \neq 0$, and introduced the dimensionless

function \aleph_D

$$\aleph_D[b, c, \eta] = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \frac{1 + bx}{1 + cx + i\eta} = G\left[\frac{1}{2}\right] - G\left[-\frac{1}{2}\right], \quad (\text{D2})$$

where for the primitive $G(x)$, one can choose

$$\begin{aligned} G[x] &= \frac{bx}{c} + \frac{b\eta + i(c-b)}{2c^2} \\ &\times \left\{ 2\left(\frac{\pi}{2} - \tan^{-1} \left[\frac{1+cx}{\eta} \right] \right) - i \ln \left[(1+cx)^2 + \eta^2 \right] \right\}. \end{aligned} \quad (\text{D3})$$

We illustrate this method in Fig. 6, by representing the Nyquist contours associated with the system from Eq. (17). This shows that this particular system is linearly stable. Throughout the applications presented in the main text, we truncated the orbital space in $K = 10^4$ elements, and added to the frequency a small imaginary part $\eta = 10^{-8}$ to regularize the resonant denominator. We checked that these choices had no impact on our results.

[1] D. Lynden-Bell, MNRAS **136**, 101 (1967).
[2] R. Balescu, Phys. Fluids **3**, 52 (1960).
[3] A. Lenard, Ann. Phys. **10**, 390 (1960).
[4] J. Heyvaerts, MNRAS **407**, 355 (2010).
[5] P.-H. Chavanis, Physica A **391**, 3680 (2012).
[6] G. Arfken, H. Weber, and F. Harris, *Mathematical Methods for Physicists* (Elsevier Science, 2013).
[7] S. Gupta and D. Mukamel, J. Stat. Mech. **3**, 03015 (2011).
[8] J. Barré and S. Gupta, J. Stat. Mech. **2**, 02017 (2014).
[9] W. Maier and A. Sauppe, ZNatA **13**, 564 (1958).
[10] Z. Roupas, B. Kocsis, and S. Tremaine, ApJ **842**, 90 (2017).
[11] B. Kocsis and S. Tremaine, MNRAS **448**, 3265 (2015).
[12] Á. Szölgvény and B. Kocsis, Phys. Rev. Lett. **121**, 101101 (2018).
[13] Á. Takács and B. Kocsis, ApJ **856**, 113 (2018).
[14] J. Binney and S. Tremaine, *Galactic Dynamics: Second Edition* (Princeton University Press, 2008).
[15] P.-H. Chavanis, A&A **556**, A93 (2013).
[16] P.-H. Chavanis, Eur. Phys. J. Plus **128**, 126 (2013).
[17] T. M. Rocha Filho, A. E. Santana, M. A. Amato, and A. Figueiredo, Phys. Rev. E **90**, 032133 (2014).
[18] C. R. Lourenço and T. M. Rocha Filho, Phys. Rev. E **92**, 012117 (2015).
[19] P.-H. Chavanis and M. Lemou, Eur. Phys. J. B **59**, 217 (2007).
[20] P.-H. Chavanis, J. Vatteville, and F. Bouchet, Eur. Phys. J. B **46**, 61 (2005).
[21] F. P. C. Benetti and B. Marcos, Phys. Rev. E **95**, 022111 (2017).
[22] J.-B. Fouvry, C. Pichon, J. Magorrian, and P.-H. Chavanis, A&A **584**, A129 (2015).
[23] B. Bar-Or and J.-B. Fouvry, ApJ **860**, L23 (2018).
[24] C. Hamilton, J.-B. Fouvry, J. Binney, and C. Pichon, MNRAS **481**, 2041 (2018).

[25] J. Jeans, MNRAS **76**, 70 (1915).

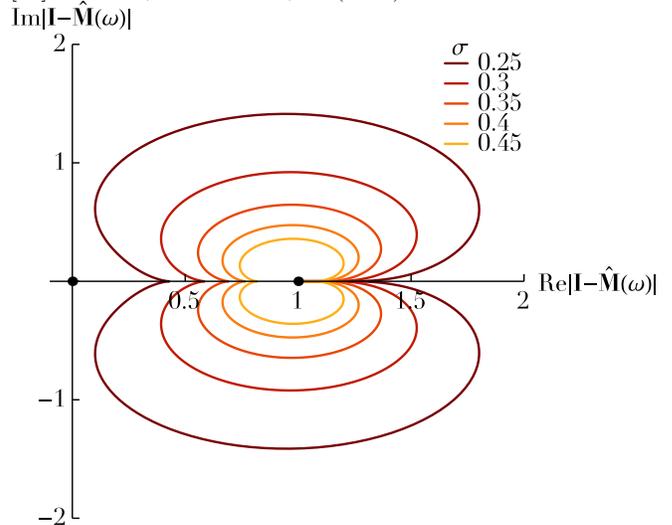


FIG. 6: Illustration of the Nyquist contours $\omega \mapsto \det[\mathbf{I} - \widehat{\mathbf{M}}(\omega, \eta = 10^{-8})]$ for the DF from Eq. (17) with different dispersions σ . None of these contours enclose the origin, indicating that these systems are linearly stable. The smaller the dispersion, the closer is the system to being unstable.

[26] A. J. Kalnajs, ApJ **205**, 745 (1976).
[27] W. Press et al., *Numerical Recipes 3rd Edition* (Cambridge University Press, 2007).
[28] F. Mignard and S. Klioner, A&A **547**, A59 (2012).
[29] M. Antoni and S. Ruffo, Phys. Rev. E **52**, 2361 (1995).