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# Hidden thermal structure in Fock space

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The emergence of quantum statistical mechanics from individual pure states of closed many-body systems is under intensive investigations. While most efforts have been put on the impacts of the *direct* interaction (i.e., the usual mutual interaction), here we study systematically and analytically the impacts of the *exchange* interaction, that arises from the particle indistinguishability. We show that this interaction leads an overwhelming number of Fock states to exhibit a structure, that can be resolved only by observables adjusted according to system's dynamical properties and from which thermal distributions emerge. This hidden thermal structure in Fock space is found to be related to the so-called limit shape of random geometric objects in mathematics. The structure enables us to uncover, for both ideal and nonideal Fermi gases, new mechanisms for the emergence of quantum statistical mechanics from individual eigenstates.

There have been increasing evidences [1–8] showing that a closed quantum many-body system can act as its own heat bath, leading to the emergence of equilibrium statistical mechanics from pure states (see Refs. [9–12] for review). Notwithstanding this, the ingredients indispensable for this emergence remain an open problem. It has been shown that the direct interaction, via driving many-body quantum chaos, gives rise to complex structures of eigenstates, from which the Fermi-Dirac (FD) or Bose-Einstein (BE) distribution arises [3, 10, 13, 14]. The need of this interaction conforms to standard statistical mechanics [15]. Whereas the studies of entanglement entropy suggest that without the direct interaction, the distribution arises also [16–19]. That thermal distributions exist in such a broad range of extreme conditions motivates exploring universal routes to their emergence from pure states. Furthermore, the exchange interaction – “a peculiar mutual effect of particles that are in the same quantum state” [15] – is a common ingredient of quantum many-body systems. It is a building block of traditional ensemble-based quantum statistical mechanics, giving rise to the FD (BE) distribution and many intriguing phenomena ranging from the BE condensation to the Haldane-Wu fractional exclusion statistics [20, 21]. Thus in-depth investigations of the exchange interaction in the emergence of statistical mechanics from pure states are of fundamental importance and urgent need.

In this Rapid Communication, for the first time, we study systematically and analytically how the exchange interaction drives thermal equilibrium phenom-

ena at individual pure states. For simplicity we focus on Fermi statistics, and consider  $N(\gg 1)$  indistinguishable fermions confined in a volume [22] for both situations: with and without the direct interaction. Without the direct interaction an ideal Fermi gas results; the exchange interaction endows it with many-body nature. Its eigenstate is a Fock state  $\lambda$ , represented by a pattern of the number of particles occupying a single-particle eigenstate. Three classes of representative single-particle eigenstates are considered, corresponding to distinct quantum motions (Fig. 1): Liouville integrable, chaotic and Anderson localized. To realize the first we put the particle on a torus (A1) or in a one-dimensional harmonic potential (A2), the second in a chaotic cavity (B), and the third in a quasi one-dimensional cavity with scatterers randomly place inside (C). When the direct interaction is switched on, a nonideal Fermi gas results, whose eigenstate  $\Phi$  is a superposition of Fock states. In this work we first uncover a thermal structure hidden in Fock space, and then study its consequences on both ideal and nonideal Fermi gases.

The main results are summarized in words as follows:

- We find that, irrespective of dynamical properties (Liouville integrable, chaotic or Anderson localized) of single-particle motion, for an overwhelming number of Fock states the FD distribution emerges from individual occupation number pattern (cf. Table I), and can be resolved only by appropriate observables (that is, this emergence does not ensure that in a given Fock state, the expectation values of all observables are thermal.). As such this is a hidden thermal structure. Moreover, it has nothing to do with many-body quantum chaos, but is related to the limit shape of random geometric objects [23–27], a subject well explored by mathematicians.

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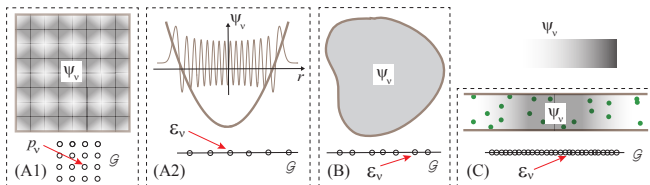


FIG. 1: Schematic representation of the spatial structures of single-particle eigenstate  $\psi_\nu$  and the good quantum number space  $\mathcal{G}$  (empty circles) of Liouville integrable (A1,2), chaotic (B) and Anderson localized (C) motions in distinct setups. Green dots in (C) are scatterers.

- We find that the influence of dynamical properties is to determine whether an observable can resolve the thermal structure. Table I gives the results for the one-particle correlation function  $M_{rr'}$  between two spatial points  $r, r'$ . It shows that the short-ranged (small  $|r-r'|$ ) correlation is always thermal, implying that if a subsystem is small, an individual  $\lambda$ , namely, a many-body eigenstate of ideal Fermi gas, acts as the heat bath of the subsystem, irrespective of dynamical properties. This is in spirit consistent with the results for the reduced density matrix based on the *canonical typicality* [5–7], which makes no reference to system’s constructions. Whereas the long-ranged (large  $|r-r'|$ ) correlation is thermal only if the single-particle motion is chaotic. In this case  $\lambda$  acts as the heat bath of the entire system.
- We find that, without Berry’s conjecture [3, 29], the eigenstate  $\Phi$  of nonideal Fermi gases on a torus exhibits *eigenstate thermalization* [3]. Specifically, we show [Eq. (16)] that the short-ranged correlation at  $\Phi$  is thermal, i.e., governed by the FD distribution, but not the detailed constructions of  $\Phi$ .

Our findings suggest that the thermal structure hidden in the Fock space, arising from the exchange interaction namely the particle indistinguishability, is a basis of the emergence of thermal equilibrium phenomena from pure states. In particular, they indicate new mechanisms for the eigenstate thermalization.

TABLE I: Structures of individual Fock state  $\lambda$  resolved by spatial correlation function  $M_{rr'}$  of distinct ranges.

eigenstate $\psi_\nu$	short-ranged [28]		long-ranged	
	structure resolved	expectation value	structure resolved	expectation value
integrable (A1,2)	FD	thermal [Eq. (11)]	$\{n_\nu\}$	athermal [Eq. (1)]
chaotic (B)	FD	thermal [Eq. (11)]	FD	thermal [Eq. (11)]
localized (C)	FD	thermal [Eq. (13)]	$\{n_\nu\}$	athermal [Eq. (1)]

Observable-resolved structure  $\Lambda(\lambda)$  of individual  $\lambda$  – An

individual Fock state  $\lambda$  is a pattern  $\{n_\nu\}$ , where  $n_\nu (=0,1)$  is the occupation number at single-particle eigenstate  $\psi_\nu$ .  $\nu$  denotes the complete set of good quantum numbers associated with the single-particle motion, which refers to the eigenmomentum  $p_\nu$  for free motion (A1) and to the eigenenergy  $\varepsilon_\nu$  for harmonic oscillation (A2), chaotic motion (B) and Anderson localization (C). Given a system all  $\nu$  constitute a space, denoted as  $\mathcal{G}$  (Fig. 1). We will show that there are intimate relations between resolving the fine structures of  $\{n_\nu\}$  by observables and the emerging of FD distribution from individual many-body eigenstates of ideal or nonideal Fermi gases. To this end we first illustrate in this part how distinct observables resolve structures of individual  $\lambda$  at different scales of  $\mathcal{G}$ .

We take a family of basic observables, namely, the one-particle correlation function  $M_{rr'}$  at different ranges of  $|r-r'|$ . At  $\lambda$  the correlation function is

$$M_{rr'} \equiv \langle \lambda | c_r^\dagger c_r | \lambda \rangle = \sum_\nu n_\nu C_\nu(r, r'). \quad (1)$$

Here  $c_r$  ( $c_r^\dagger$ ) is the annihilation (creation) operator at  $r$ , and  $C_\nu(r, r') \equiv \psi_\nu(r) \psi_\nu^*(r')$  is the autocorrelation of  $\psi_\nu(r)$ .

- (i) If  $C_\nu(r, r')$  varies slowly with  $\nu$  (Fig. 2, left):

$$C_\nu(r, r') \approx C_{\nu'}(r, r'), \quad \text{for nearest } \nu, \nu', \quad (2)$$

then  $\mathcal{G}$  has a “natural” decomposition into many subspaces  $\mathcal{G}_m$  (Fig. 2, left). [We are not aware of generic conditions for Eq. (2). Thus we will justify it and derive the conditions for distinct dynamical systems later.] In each  $\mathcal{G}_m$ ,  $C_\nu(r, r')$  fixed) and  $\varepsilon_\nu$  are approximately a constant, denoted as  $C_m(r, r')$  and  $\varepsilon_m$ , respectively, i.e.,

$$\mathcal{G} = \oplus_m \mathcal{G}_m, \quad \forall \nu \in \mathcal{G}_m : C_\nu(r, r') \approx C_m(r, r'), \varepsilon_\nu \approx \varepsilon_m. \quad (3)$$

By Eq. (2) the number of elements of  $\mathcal{G}_m$ , denoted as  $G_m$ , is  $\gg 1$  [30]. Using the decomposition (3) we obtain:

$$\sum_\nu n_\nu C_\nu(r, r') = \sum_m C_m(r, r') \sum_{\nu \in \mathcal{G}_m} n_\nu. \quad (4)$$

With the help of this result we reduce Eq. (1) to

$$M_{rr'} = \sum_m N_m C_m(r, r'), \quad N_m = \sum_{\nu \in \mathcal{G}_m} n_\nu. \quad (5)$$

Therefore, provided that Eq. (2) holds,  $M_{rr'}$  cannot resolve  $n_\nu$  at a specific  $\nu$ ; rather, it resolves a less fine structure  $\{N_m\} \equiv \Lambda(\lambda)$ , which is constrained by [31]:

$$\sum_m N_m = N (= \sum_\nu n_\nu), \quad \sum_m N_m \varepsilon_m \approx E (= \sum_\nu n_\nu \varepsilon_\nu). \quad (6)$$

(ii) If  $C_\nu(r, r')$  varies rapidly with  $\nu$  (Fig. 2, right), then neither the decomposition (3) nor the reduction (5) follows. As  $M_{rr'}$  is given by Eq. (1), a fine tuning in the pattern  $\{n_\nu\}$  can lead to a significant change in  $M_{rr'}$ . That is,  $M_{rr'}$  can resolve the fine structure of  $\{n_\nu\}$ .

Here we make two remarks. First, the decomposition (3) resembles some ideas of von Neumann [32] regarding the fundamentals of statistical mechanics of closed quantum systems. Specifically, that observables can induce

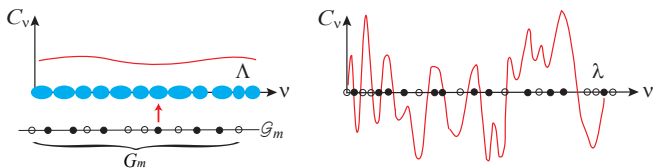


FIG. 2: Left: When  $C_\nu$  varies slowly with  $\nu$ , a decomposition of the space  $\mathcal{G}$  into subspaces  $\mathcal{G}_m$  (blue cells) results. Consequently,  $M_{rr'}$  can resolve only the structure  $\Lambda$  less fine than  $\lambda$ . Right: When  $C_\nu$  varies rapidly with  $\nu$ , the decomposition does not follow and  $M_{rr'}$  can resolve the fine structure of  $\Lambda$ . Solid (empty) circles denote (un)occupied eigenstates  $\nu$ .

the decomposition of the space of quantum states is a basis of von Neumann's analysis [32]. However, his decomposition refers to the Hilbert space spanned by the eigenstates of the entire system, which are  $\lambda$  for an ideal Fermi gas and  $\Phi$  for a nonideal Fermi gas. Whereas the decomposition (3) refers to  $\mathcal{G}$ . Secondly, although  $\Lambda$  looks similar to the "macroscopic state" of Landau [33], there are conceptual differences. Notably, as discussed  $\Lambda$  is resolved only by proper observables, whereas the macroscopic state is independent of observables.

*Emergence of thermal structures from  $\Lambda(\lambda)$*  – A question naturally is: What does the structure  $\Lambda(\lambda)$  look like? To study this problem we note that by definition of  $\Lambda$ , distinct  $\lambda$  [constrained by Eq. (6)] can correspond to the same structure  $\Lambda$ . The number of  $\lambda$  corresponding to  $\Lambda$  is given by  $\prod_m \frac{G_m!}{N_m!(G_m - N_m)!} \equiv W[\Lambda]$ . From this expression we see that  $W$  has a sharp peak at some  $\Lambda^* \equiv \{N_m^*\}$ . Physically, this means that an overwhelming number of  $\lambda$  have the same observable-resolved structure  $\Lambda^*$ .

Now we can show that the thermodynamic relation emerges from an individual  $\lambda$  satisfying  $\Lambda[\lambda] = \Lambda^*$ : this is in contrast to standard statistical mechanics where thermodynamics is built upon an ensemble. By definition,

$$\left. \frac{\partial S}{\partial N_m} \right|_{\Lambda = \Lambda^*} = \alpha + \beta \varepsilon_m, \quad S \equiv \ln W[\Lambda]. \quad (7)$$

Here  $\alpha, \beta$  are the Lagrange multipliers. They depend on  $N, E$ , and so do  $N_m^*$  and  $W[\Lambda^*]$ . Taking this and Eqs. (6) and (7) into account, we find that

$$\left. \frac{\partial S}{\partial E} \right|_{\Lambda = \Lambda^*} = \left. \frac{\partial}{\partial E} \sum_m N_m (\alpha + \beta \varepsilon_m) \right|_{\Lambda = \Lambda^*} = \beta, \quad (8)$$

where in deriving the last equality we have used the fact that  $N, E$  are independent variables. Similarly, we have

$$\left. \frac{\partial S}{\partial N} \right|_{\Lambda = \Lambda^*} = \left. \frac{\partial}{\partial N} \sum_m N_m (\alpha + \beta \varepsilon_m) \right|_{\Lambda = \Lambda^*} = \alpha. \quad (9)$$

Thus  $S = \ln W[\Lambda^*]$  gives the thermodynamic entropy [22],  $\beta$  the inverse thermodynamic temperature  $\frac{1}{T}$ , and  $-\frac{\alpha}{\beta}$  the chemical potential  $\mu$ . So Eqs. (8) and (9) reduce to  $\left. \frac{\partial S}{\partial E} \right|_{\Lambda = \Lambda^*} = \frac{1}{T}$  and  $\left. \frac{\partial S}{\partial N} \right|_{\Lambda = \Lambda^*} = -\frac{\mu}{T}$ . Note that these relations are independent of the explicit form of  $\Lambda^*$ .

Furthermore, by substituting the explicit form of  $W[\Lambda]$  into Eq. (7) we obtain

$$N_m^*/G_m = (e^{\frac{\varepsilon_m - \mu}{T}} + 1)^{-1} \equiv f_{\text{FD}}(\varepsilon_m). \quad (10)$$

So  $\Lambda^*$  is determined by FD distribution  $f_{\text{FD}}$ , i.e., is a thermal structure. Unlike standard textbooks [15], here  $f_{\text{FD}}$  refers to individual  $\lambda$ , not an ensemble. The emergence of FD (BE) distribution from pure states has recently appeared as a new fundamental aspect of statistical mechanics [14, 16, 17]. Most importantly,  $f_{\text{FD}}$  can be resolved only if Eq. (2) holds, whereas in textbooks thermal distributions have nothing to do with observables.

*Probing hidden thermal structure  $\Lambda^*$*  – Let the Fock space constrained by Eq. (6) be equipped with a uniform probability measure, and  $\lambda$  be drawn randomly from this measure. Equation (5) and the analysis above suggest that  $M_{rr'}$  has a *typical* value (with respect to this measure), because an overwhelming number of  $\lambda$  satisfy  $\Lambda(\lambda) = \Lambda^*$ . Combining Eqs. (5) and (10) we find that, provided that Eq. (2) holds, this typical value is

$$M_{rr'} = \int d\mu(\nu) (e^{\frac{\varepsilon_\nu - \mu}{T}} + 1)^{-1} C_\nu(r, r'). \quad (11)$$

Here  $d\mu(\nu)$  gives the number of single-particle eigenstates in  $d\nu$ . The left-hand side of Eq. (11) is the expectation value of  $c_r^\dagger c_r$  at  $\lambda$ , while the right-hand side is the thermal average of  $C_\nu$ . Note that the latter is determined by thermodynamic quantities  $T, \mu$ , and thus fine tunings of  $\{n_\nu\}$  do not change the value of  $M_{rr'}$ . This implies that for an overwhelming number of (but not all)  $\lambda$  constrained by Eq. (6),  $M_{rr'}$  takes the same value – the onset of eigenstate thermalization [2, 3, 8] of ideal Fermi gases. Moreover, Eq. (11) provides a guide of probing  $\Lambda^*$ .

**For highly-excited  $\lambda$ , the thermal de Broglie wavelength is much smaller than the mean distance between nearest particles. So the FD distribution in Eq. (11) can be well approximated by the Maxwell-Boltzmann (MB) distribution. In the Supplemental Materials [34] we show that the MB distribution appearing from  $M_{rr'}$  results from the quantum entanglement of indistinguishable particles. Provided particles are *distinguishable*, this entanglement does not exist (since neither the exchange nor direct interaction exists), and unlike Eq. (11) the MB distribution cannot emerge from the one-particle correlation function [34]. This scenario is fundamentally from standard statistical physics: the former refers to a pure quantum state while the latter to an ensemble.**

The remainder is to find the conditions under which Eq. (2) holds. Below we consider the single-particle quantum motions in Fig. 1 separately, and show that *precisely at this point, dynamical properties make significant differences* (see Table I for a summary of the results below). In essence, distinct motions give rise to distinct spatial structures of  $\psi_\nu$  (Fig. 1) and thus the autocorrelation  $C_\nu(r, r')$  of  $\psi_\nu$  displays distinct dependences on  $\nu$ .

(A1) With the substitution of  $\psi_\nu(r) = \frac{e^{ip_\nu r}}{L}$  ( $L$  the torus size),  $C_\nu \sim e^{ip_\nu \cdot (r - r')}$ . (i) For  $|r - r'| \ll L$ , since the difference between nearest neighbors:  $p_\nu \cdot p_{\nu'}$  is  $\mathcal{O}(L^{-1})$ , we have  $|(p_\nu - p_{\nu'}) \cdot (r - r')| \ll 1$ . From this we find that  $p_\nu \cdot (r - r')$  varies slowly with  $\nu$ , and justify Eq. (2). Thus we have Eq. (11), i.e., the short-ranged  $M_{rr'}$  is thermal. (ii) For  $|r - r'| = \mathcal{O}(L)$ , we have  $|(p_\nu - p_{\nu'}) \cdot (r - r')| = \mathcal{O}(1)$ .

Thus  $p_\nu \cdot (r-r')$  varies rapidly with  $\nu$  and Eq. (2) breaks down. So the long-ranged  $M_{rr'}$  is given by Eq. (1), i.e., athermal, and cannot be used to probe  $\Lambda^*$ .

(A2) The eigenvalue  $\varepsilon_\nu = \nu + \frac{1}{2}$  and corresponding eigenstate  $\psi_\nu(r) = \frac{\pi^{-1/4}}{\sqrt{2\nu}} e^{-\frac{r^2}{2}} H_\nu(r)$ , where  $H_\nu$  is the Hermite polynomial. For  $N \gg 1$  most fermions occupy highly excited single-particle eigenstates. Thus the sum in Eq. (1) is dominated by large  $\nu$ , for which  $\psi_\nu(r) \sim \cos(\sqrt{2\varepsilon_\nu}r)$ . Substituting this asymptotic expression into  $C_\nu$  and repeating the discussions on (A1), we find that  $M_{rr'}$  is thermal for  $|r-r'| \ll \sqrt{E/N}$  and athermal otherwise.

(B) To calculate  $C_\nu$  we consider (i) large and (ii) small  $\varepsilon_\nu$  separately. For (i) we perform the Wigner transformation:  $C_\nu(r, r') \equiv \int dp e^{-i(r-r') \cdot p} \Psi_\nu(q, p)$  with  $q \equiv \frac{1}{2}(r+r')$ , and adopt Berry's conjecture for single-particle chaotic motion [29]:  $\Psi_\nu(q, p) = \frac{\delta(\varepsilon_\nu - H(q, p))}{\iint dq dp \delta(\varepsilon_\nu - H(q, p))}$ , with  $H$  being the Hamiltonian. This conjecture implies that  $|\psi_\nu(r)|$  is homogeneous on large scales. Unlike Ref. [3], here the conjecture is not made for many-particle motion. Using the conjecture we obtain  $C_\nu \sim f(\frac{|r-r'|}{\lambda_{\varepsilon_\nu}})$ , with  $\lambda_{\varepsilon_\nu}$  being the de Broglie wavelength at energy  $\varepsilon_\nu$ . The function  $f(x)$  oscillates in  $x$ , whose explicit form is unimportant. For nearest  $\nu, \nu'$  and for any  $r, r'$ , we have

$$|r-r'|(\lambda_{\varepsilon_\nu}^{-1} - \lambda_{\varepsilon_{\nu'}}^{-1}) \sim (|r-r'|/L)(\Delta/\varepsilon_\nu)^{1/2} \ll 1, \quad (12)$$

with  $\Delta$  being the level spacing and  $L$  the cavity size. From this we find that  $f(\frac{|r-r'|}{\lambda_{\varepsilon_\nu}})$  is the same for nearest  $\nu, \nu'$ . Thus Eq. (2) is justified. For (ii) we do not expect Berry's conjecture to hold, since it is based on the semiclassical approximation. So Eq. (2) breaks down in general. But, the number of particles occupying low-lying single-particle states is  $\ll N$ . Thus their contributions to the sum in Eq. (1) are negligible, and the breakdown of Eq. (2) has no effects on  $M_{rr'}$ . So both short- and long-ranged  $M_{rr'}$  are thermal and Eq. (11) follows.

(C) The Anderson localization [35–37] implies that the eigenvalues  $\{\varepsilon_\nu\}$  are discrete and dense, and  $\psi_\nu$  exhibits exponential localization in the longitudinal direction (Fig. 1). Moreover, the localization center has a singular dependence on  $\nu$ : as  $\nu$  approaches  $\nu'$  the distance between localization centers of  $\psi_\nu$  and  $\psi_{\nu'}$  diverges. In addition, the localization length varies with  $\nu$ . As a result, (i) if  $|r-r'|$  is sufficiently large,  $C_\nu$  varies rapidly with  $\nu$ . Thus Eq. (2) breaks down and the long-ranged  $M_{rr'}$  is athermal. (ii) For  $r, r'$  in the same localization volume, the sum in Eq. (1) is dominated by the subset of  $\{\varepsilon_\nu\}$  that corresponds to this volume. Since each localization volume is an effective chaotic cavity, we can repeat the analysis of (B). As a result, we obtain Eq. (11), but with  $d\mu$  replaced by  $d\mu_{\text{loc}}$  which gives the number of eigenstates in a localization volume and the interval  $d\nu$ :

$$M_{rr'} = \int d\mu_{\text{loc}}(\nu) (e^{\frac{\varepsilon_\nu - \mu}{T}} + 1)^{-1} C_\nu(r, r'). \quad (13)$$

So we can use the short-ranged correlation to probe  $\Lambda^*$ .

*New mechanism for eigenstate thermalization in non-ideal Fermi gas* – Now we switch on the direct hard-

sphere interaction between particles. For simplicity we consider particles on a torus. This system is essentially the same as what was studied in Ref. [3]. An eigenstate  $\Phi$  of this system, corresponding to the eigenenergy  $E$ , is a superposition of  $\lambda \in \Omega_{N,E}$ , where  $\Omega_{N,E}$  is composed of all  $\lambda$  satisfying  $\sum_\nu n_\nu = N$  and  $\sum_\nu n_\nu \varepsilon_\nu = E$  ( $\varepsilon_\nu = \frac{p_\nu^2}{2}$ ):

$$|\Phi\rangle = \sum_{\lambda \in \Omega_{N,E}} C_\lambda |\lambda\rangle, \quad \sum_{\lambda \in \Omega_{N,E}} |C_\lambda|^2 = 1. \quad (14)$$

Note that for this system the many-body eigenenergy  $E$  is exactly the total kinetic energy, and the direct interaction enters only into the coefficients  $C_\lambda$ . By simple algebra we find the one-particle correlation function at  $\Phi$ :

$$\begin{aligned} \langle \Phi | c_{r'}^\dagger c_r | \Phi \rangle &= \sum_{\lambda \in \Omega_{N,E}} |C_\lambda|^2 M_{rr'} \\ &+ \sum_{\lambda \in \Omega_{N,E}} C_\lambda^* C_\lambda \sum_{\nu \neq \nu'} \frac{e^{i(p_\nu \cdot r - p_{\nu'} \cdot r')}}{L^2} \langle \lambda | c_\nu^\dagger c_{\nu'} | \lambda \rangle, \end{aligned} \quad (15)$$

where  $M_{rr'} = \frac{1}{L^2} \sum_\nu e^{ip_\nu \cdot (r-r')} \langle \lambda | c_\nu^\dagger c_\nu | \lambda \rangle$ . The left-hand side on Eq. (15) is translationally invariant, i.e., depends on  $r, r'$  via  $r-r'$ , for the system has the translation symmetry. But this invariance is violated by the second term on the right-hand side. Thus this term must vanish, giving  $\langle \Phi | c_{r'}^\dagger c_r | \Phi \rangle = \sum_{\lambda \in \Omega_{N,E}} |C_\lambda|^2 M_{rr'}$ . On the other hand, for generic  $\Phi$ , most weights of  $|C_\lambda|^2$  go to  $\lambda$  satisfying  $\Lambda(\lambda) = \Lambda^*$ , because as shown above  $W[\Lambda]$  has a sharp peak at  $\Lambda^*$  [38]. For these  $\lambda$  and corresponding  $M_{rr'}$  we use the results for (A1) summarized in Table I. In particular,  $M_{rr'}$  takes the typical value (11) for  $|r-r'| \ll L$ , with  $T, \mu$  in Eq. (11) determined by  $N, E$ . As  $M_{rr'}$  are insensitive to the fine structure of  $\lambda$ , it can be further pulled out of the sum:  $\sum_{\lambda \in \Omega_{N,E}} (\dots) M_{rr'}$ , giving

$$\begin{aligned} \langle \Phi | c_{r'}^\dagger c_r | \Phi \rangle &= M_{rr'} \sum_{\lambda \in \Omega_{N,E}} |C_\lambda|^2 = M_{rr'} \\ &= \int \frac{dp}{(2\pi)^2} \frac{e^{ip \cdot (r-r')}}{e^{(p^2/2 - \mu)/T} + 1}. \end{aligned} \quad (16)$$

Thus the eigenstate thermalization is justified for short-ranged one-particle correlation at both low- and high-lying  $\Phi$ . Note that unlike Ref. [3], we did not use Berry's conjecture made for many-particle chaotic motion, which has not yet been proven.

*Relations between  $\Lambda^*$  and the limit shape of random geometric objects* – Finally, we wish to understand from more rigorous viewpoints why the thermal structure  $\Lambda^*$  can arise merely from the particle indistinguishability. Let us view  $\lambda = \{n_\nu\}$  as a geometric object – a collection of “skyscrapers” located at  $\nu$  wherever  $n_\nu = 1$  (see, e.g., Fig. 3). It turns out that despite the shapes of such objects look random, as discovered by mathematicians [23–27], when they are large and rescaled properly, they can concentrate on a smooth and nonrandom limit shape.

To keep quantitative discussions as simple as possible we consider  $N$  indistinguishable free fermions confined in a harmonic potential. The space  $\mathcal{G}$  is the single-particle eigenenergy spectrum, i.e., the set of natural numbers  $\mathbb{N}$ . (The zero energy is irrelevant and ignored.) For a Fock state  $\lambda$ ,  $E = \sum_{\nu=1}^{\infty} \nu n_\nu$ . This maps  $\lambda$  into a partition of integer  $E$  into  $N$  distinct summands, a research area for



which Euler laid down a foundation [39]. To be precise, an eigenenergy  $\nu$ , when its corresponding eigenstate is occupied ( $n_\nu=1$ ), mimics a summand. In 1941 the field of random integer partitions was opened up [40], and in the past few decades such partitions have been found to bear rich structures [25–27]. (One should not confuse this with the old subject of using standard statistical mechanics to study the number of partitions [41].) In particular, the observable  $\varphi_\lambda(t) \equiv \int_t^\infty \sum_{\nu'=1}^\infty n_{\nu'} \delta(\nu' - \nu) d\nu$ , counting at given  $\lambda$  the number of summands  $\geq t$  or, equivalently, the number of particles occupying single-particle eigenstates with eigenenergies  $\geq t$  [42], defines a random stepped curve. As proven rigorously [26], this curve has a limit shape. Our results have intimate relations to this.

For illustrations we consider the case where  $N$  is not fixed, i.e.,  $\mu = 0$ . In this case for sufficiently large  $E$ , Eq. (10) shows that for an overwhelming number of  $\lambda$ ,

$$\varphi_\lambda(t) \stackrel{E \gg 1}{\approx} \int_t^\infty \frac{d\nu}{e^{\nu/T} + 1} = T \ln(1 + e^{-t/T}), \quad T = \frac{\sqrt{12E}}{\pi} \quad (17)$$

$[E = \int_0^\infty \varphi_\lambda(t) dt]$ , in agreement with the theorem [26]:

$$\lim_{E \rightarrow \infty} \mu^E \left\{ \lambda : \left| \frac{1}{\sqrt{E}} \varphi_\lambda(\sqrt{E}t) + s(t) \right| < \epsilon \right\} = 1, \quad \forall \epsilon > 0.$$

Here  $s(t)$  is given by the Vershik curve:  $e^{-\frac{\pi s}{\sqrt{12}}} - e^{-\frac{\pi t}{\sqrt{12}}} = 1$ . This theorem implies that if the set of all partitions  $\lambda$  defined above is equipped with a uniform probability measure  $\mu^E$ , then a typical partition has a limit shape of  $-s$ . Our method, though less rigorous, has the advantage of being applied to more general conditions and systems.

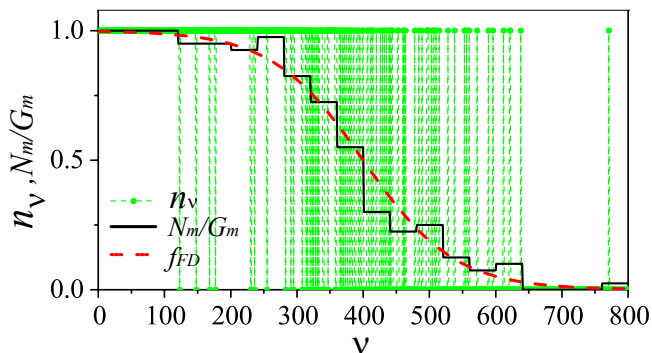


FIG. 3: A Fock state  $\lambda = \{n_\nu\}$  of  $N=400$  fermions of total energy  $E=87800$  confined in a harmonic potential, or a partition of  $E$  into  $N$  distinct summands in number theory, is generated randomly in simulations. For the observable-resolved structure  $\Lambda = \{N_m\}$  ( $G_m=40$ ) of a typical  $\lambda$ ,  $N_m/G_m$  is fitted well by  $f_{FD}$  with  $\mu=400.1$  and  $T=68.84$ .

Indeed, for generic  $N, E$ , for which rigorous results are not available, we have confirmed Eq. (10) numerically. Specifically, we use Monte Carlo method to draw randomly a partition  $\lambda = \{n_\nu\}$  of  $E$  (with  $N$  distinct summands) from the uniform probability measure. As shown in Fig. 3, a typical  $\lambda$  (green dashed curve), though looks random, has a nonrandom  $\Lambda$  (stepped curve) fitted well by  $f_{FD}$  (red dashed curve) [30]. The pattern of  $\{\frac{N_m}{G_m}\}$  concentrates on the smooth curve  $f_{FD}$ . ( $G_m$  corresponds to the scale over which  $C_\nu$  varies and thus to  $M_{rr'}$  with specific ranges of  $r, r'$ .) We see that  $f_{FD}$  is justified not only for an individual  $\lambda$ , but also for  $N$  as small as 400.

Summarizing, we have shown analytically how the exchange interaction namely the particle indistinguishability gives rise to a hidden thermal structure in the Fock space, and opens up a door to the emergence of thermal equilibrium phenomena from eigenstates of many-body systems with or without the direct interaction between particles. Furthermore, we have uncovered a new mechanism for eigenstate thermalization of a nonideal Fermi gas on a torus [Fig. 1(A1)]. It is natural to generalize this result to a nonideal Fermi gas in an Anderson localized cavity [Fig. 1(C)]. This issue is currently under investigations. We expect the outcomes to shed new light on the many-body localization [43–45], especially in view of that the many-body localization is equivalent to localization in the Fock space. In this work, we focused on the kinematic aspect. The dynamical aspect, especially the interplay between relaxation and the hidden thermal structure, is an important issue in future studies. **In a separate work [46], we will show that given a typical non-thermal initial state, under the unitary evolution, the observable will relax to the thermal value discussed here, and the relaxation time is the Ehrenfest time.**

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- [1] R.V. Jensen and R. Shankar, Phys. Rev. Lett. **54**, 1879 (1985).
- [2] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
- [3] M. Srednicki, Phys. Rev. E **50**, 888 (1994).

- [4] V. Zelevinsky, Annu. Rev. Nucl. Part. Sci. **46**, 237 (1996).
- [5] H. Tasaki, Phys. Rev. Lett. **80**, 1373 (1998).
- [6] S. Popescu, A. J. Short, and, A. Winters, Nat. Phys. **2**, 754 (2006).

- [7] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, *Phys. Rev. Lett.* **96**, 050403 (2006).
- [8] M. Rigol, V. Dunjko, and M. Olshanii, *Nature* **452**, 854 (2008).
- [9] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *Adv. Phys.* **65**, 239 (2016).
- [10] F. Borgonovi, F. M. Izrailev, L. F. Santos, and V. G. Zelevinsky, *Phys. Rep.* **626**, 1 (2016).
- [11] C. Gogolin and J. Eisert, *Rep. Prog. Phys.* **79**, 056001 (2016).
- [12] J. M. Deutsch, *Rep. Prog. Phys.* **81**, 082001 (2018).
- [13] F. Borgonovi, I. Guarneri, F. M. Izrailev, and G. Casati, *Phys. Lett. A* **247**, 140 (1998).
- [14] F. Borgonovi, F. Mattiotti, and F. M. Izrailev, *Phys. Rev. E* **95**, 042135 (2017).
- [15] L. D. Landau and E. M. Lifshitz, *Statistical physics*, Part 1, 3rd Ed. (Butterworth-Heinemann, Oxford, UK, 1980).
- [16] H. H. Lai and K. Yang, *Phys. Rev. B* **98**, 081110(R) (2015).
- [17] S. Nandy, A. Sen, A. Das, and A. Dhar, *Phys. Rev. B* **94**, 245131 (2016).
- [18] X. Li, J. Pixley, D.-L. Deng, S. Ganeshan, and S. D. Sarma, *Phys. Rev. B* **93**, 184204 (2016).
- [19] J. M. Magán, *Phys. Rev. Lett.* **116**, 030401 (2016).
- [20] F. D. M. Haldane, *Phys. Rev. Lett.* **66**, 1529 (1991).
- [21] Y.-S. Wu, *Phys. Rev. Lett.* **73**, 922 (1994).
- [22] The particle mass, the Boltzmann and Planck constants are all set to unity throughout.
- [23] A. Okounkov, *Bull. Amer. Math. Soc.* **53**, 187 (2016).
- [24] A. M. Vershik, *J. Math. Sci.* **119**, 165 (2004).
- [25] A. M. Vershik, *Proceedings of the International Congress of Mathematicians, Zürich, 1994* (Birkhäuser, Basel, 1995).
- [26] A. M. Vershik, *Funct. Anal. Appl.* **30**, 90 (1996).
- [27] For a recent review, see, e.g., Z. G. Su, *Adv. Math.* **45**, 861 (2016).
- [28] For (A2) and (B) it is further required that  $r, r'$  are deep in the bulk.
- [29] M. V. Berry, *J. Phys. A: Math. Gen.* **10**, 2083 (1977).
- [30] The exact locations of the boundary between two nearest  $\mathcal{G}_m$ , i.e., the values of  $G_m$ , are unimportant. Small changes in these values have no physical consequences. This was confirmed by numerical experiments on random partition of integers. Specifically, we have changed  $G_m$  for the partition shown in Fig. 3 and found that  $\Lambda$  is fitted by  $f_{\text{FD}}$  with the same parameters  $T$  and  $\mu$ .
- [31] In fact, for a quantum gas on a  $d$ -dimensional torus we have  $d$  more constraints, imposed by the conservation laws of the total momentum. However, these additional constraints are not essential for this system. Indeed, if we introduce  $d$  Lagrange multipliers:  $(u_1, u_2, \dots, u_d) \equiv u$  corresponding to these constraints, the distribution (10) is modified to be  $(e^{(p^2/2 - u \cdot p - \mu)/T} + 1)^{-1}$ , which is similar to the generalized Gibbs ensemble [9]. This modified distribution can be rewritten as  $(e^{((p-u)^2/2 - \tilde{\mu})/T} + 1)^{-1}$ , where  $\tilde{\mu} = \mu + \frac{1}{2}u^2$ . From this we see that the only physical consequence introduced by the additional constraints is that the torus moves in a velocity  $u$  (with respect to the laboratory reference frame). However, we can always adjust the reference frame in which the torus is at rest.
- [32] J. von Neumann, *Z. Phys.* **57**, 30 (1929).
- [33] See §40 of Ref. [15]: “Any macroscopic state of an ideal gas may be described as follows. Let us distribute all the quantum states of an individual particle of the gas among groups each containing neighbouring states (which, in particular, have neighbouring energy values), both the number of states in each group and the number of particles in these states being still very large.”
- [34] See the Supplemental Materials.
- [35] K. B. Efetov, *Supersymmetry in disorder and chaos* (Cambridge University, Cambridge, UK, 1997).
- [36] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the theory of disordered systems* (Wiley, New York, 1988).
- [37] E. Abrahams, *Fifty years of Anderson localization* (World Scientific, Singapore, 2010).
- [38] Note that there are rare  $\lambda$  which have no thermal structures, i.e.,  $\Lambda(\lambda) \neq \Lambda^*$ . Provided that most weights of  $|C_\lambda|^2$  go to these  $\lambda$ , Eq. (16) ceases to work. But on physical grounds this seems impossible.
- [39] G. E. Andrews, *The theory of partitions* (Addison-Wesley, London-Amsterdam, 1976).
- [40] P. Erdős and J. Lehner, *Duke Math. J.* **8**, 335 (1941).
- [41] F. C. Auluck and D. S. Kothari, *Math. Proc. Cambridge Phil. Soc.* **42**, 272 (1946).
- [42] This observable corresponds to the replacement of  $C_\nu$  in Eq. (1) by  $\theta(\nu - t)$  [ $\theta(x)$  the Heaviside function], which obeys Eq. (2).
- [43] D. Basko, I. Aleiner, and B. L. Altshuler, *Ann. Phys.* **321**, 1126 (2006).
- [44] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, *Phys. Rev. Lett.* **95**, 206603 (2005).
- [45] R. Nandkishore and D. A. Huse, *Ann. Rev. Cond. Mat. Phys.* **6**, 15 (2015).
- [46] C. Tian, K. Yang, and J. Wang, in preparation (see also arXiv: 1606.08371 which is an early short version of this unpublished work).