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Phys. Rev. E **98**, 052305 — Published 14 November 2018
DOI: 10.1103/PhysRevE.98.052305

Percolation of Hierarchical Networks and Networks of Networks

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(Dated: September 27, 2018)

Much work has been devoted to studying percolation of networks and interdependent networks under varying levels of failures. Researchers have considered many different realistic network structures such as scale-free networks, spatial networks, and more. However, thus far no study has analyzed a system of hierarchical community structure of many networks. For example, infrastructure across cities are likely distributed with nodes tightly connected within small neighborhoods, somewhat less connected across the whole city, and with even less connections between cities. Furthermore, while previous work identified interconnected nodes, those nodes with links outside their neighborhood, to be more likely to be attacked or to fail, in a hierarchical structure nodes can be interconnected at different layers (between neighborhoods, between cities, etc.). We consider here the case where the nodes with interconnections at the highest level of the hierarchy are most likely to fail, followed by those with interconnections at the next level, etc. This is because nodes at higher levels of the hierarchy have the longest length links as well as having more flow passing through them. We develop an analytic solution for percolation of both single and interdependent networks of this structure and verify our theory through simulations. We find that depending on the number of levels in the hierarchy there may be multiple transitions in the giant component (fraction of connected nodes), as the network separates at the various levels. Our results show that these multiple jumps are a feature of hierarchical networks and can affect the vulnerability of infrastructure networks.

I. INTRODUCTION

The robustness of infrastructure systems can be understood through the frameworks of complex networks, percolation, and interdependent networks [1–12]. The initial research on network robustness was later expanded to include various network structures such as different degree distributions [13–15], clustering [16–18], spatial embedding [19–21], and quite recently, community structure [22, 23]. Additional research has considered various types of attacks on these networks such as degree-based attacks [24, 25], localized attacks [26, 27], and attacks based on nodes linking across communities [22, 23, 28].

Despite these advances, there remain network structures that are likely relevant for robustness that have not yet been studied. Among these is a hierarchical structure, which we will study here, where communities connect loosely with one another to form larger communities, that then connect to one another, and so on [29–32] (See Fig. 1). In the context of infrastructure robustness these hierarchical modules are likely describing real neighborhoods overlapping to form cities, which then overlap to form states, etc. which are then interconnected among themselves.

Furthermore, in this model, the nodes at the highest level of the hierarchy (e.g. between states) are likely more vulnerable to failure or attack than those at the next highest level, which are in turn more vulnerable than those at an even lower level etc. This is because the nodes at higher levels have longer distance links between them which are more likely to fail or be attacked [33] and also have higher betweenness [22] which yields additional load on them [34, 35]. Moreover, recent work by da Cunha et al. [28] showed that attacks on these 'interconnected nodes' are an optimal form of attack on the US power grid, an infrastructure system of critical interest.

II. FAILURE AND ATTACK ON COMPLEX NETWORKS

We will now provide a short review on the analytic methods for studying the effects of various attacks on complex networks. In the next sections we will make use of these methods to find an analytic solution describing the fractional size of the giant component in our model under attack. We recall the definitions from Callaway et al. [25] for the generating function of a variable x

$$G(x) = \sum_{k=0}^{\infty} P_i(k)x^k, \tag{1}$$

where k is a number of links and $P_i(k)$ is the likelihood that a node has exactly k links.

For targeted attack, they also define

$$F_0(x) = \sum_{k=0}^{\infty} r_k P_i(k)x^k, \tag{2}$$

where the symbols are as before, except that r_k represents the likelihood that a node with exactly k links fails.

Next, the generating function of the branching process, $F_1(x)$ is given by

$$F_1(x) = F_0'(x)/G'(1) \tag{3}$$

where the $F_0'(x)$ means the derivative of $F_0(x)$ with respect to x and likewise for $G'(1)$.

Given the above, the distribution of sizes of clusters of connected nodes reached by following a randomly chosen edge is given by

$$H_1(x) = 1 - F_1(x) + xF_1\left(H_1(x)\right). \tag{4}$$

Similarly the distribution for sizes of clusters from a randomly chosen node is given by

$$H_0(x) = 1 - F_0(1) + xF_0\left(H_1(x)\right). \tag{5}$$

It was noted that above the percolation threshold, this refers to the sizes of clusters that are not in the giant component and thus $H_0(1)$ gives the fraction of nodes that are not in the giant component [25].

The fraction of nodes in the giant component, P_∞, can thus be found by

$$P_\infty(x) = 1 - H_0(1) = F_0(1) - F_0(u), \tag{6}$$

where u is given by

$$u = 1 - F_1(1) + F_1(u). \tag{7}$$

When $1 - p$ fraction of nodes are removed randomly from an Erdős-Rényi network (Poission distribution of links) with average degree $<k>$ per node it is found that $G(x) = e^{<k>(x-1)}$ [36] and $F_0(x) = pG(x)$ where p is the fraction of surviving nodes. Continuing the derivation leads to the classical result from Erdős-Rényi, $P_\infty = p(1 - e^{-<k>P_\infty})$.

For the case of modular networks with Poisson distributed inter- and intra- connections; where the average degree of interconnections is k_{inter}, average degree of intraconnections is k_{intra}, and r is the fraction of interconnected nodes that survive, the generating functions are as follows, [22]

$$G(x) = e^{(k_{\text{intra}}+k_{\text{inter}})(x-1)} \tag{8}$$

$$F_0(x) = e^{k_{\text{intra}}(x-1)-k_{\text{inter}}}(1-r) + rG(x) \tag{9}$$

$$F_1(x) = F_0'(x)/G'(1). \tag{10}$$

Following the above derivation, leads to the following formula for P_∞ in modular networks [22]

$$P_\infty = \begin{cases} e^{-k_{\text{inter}}}(1-r)(1-e^{-k_{\text{intra}}P_\infty}) + r(1-e^{-(k_{\text{intra}}+k_{\text{inter}})P_\infty}) & 0 < r < 1 \\ \frac{p}{m}(1-e^{-mk_{\text{intra}}P_\infty}) & r = 0, \end{cases} \tag{11}$$

where k_{inter} is the average degree of interconnections, k_{intra} is the average degree of intraconnections, and r is the fraction of interconnected nodes that remain. Once $r = 0$, all interconnected nodes are removed and the model continues by removing the remaining nodes randomly.

Note that for $r = 0$, the value of P_∞ is divided by m since at this point the modules are separated and thus the fraction of nodes in the giant component is scaled by 1 over the number of modules. We note that the earlier study [22] found that the network may segregate into separate modules before collapsing, or it may collapse all together as a single network. It has also been found that if nodes are targeted entirely randomly (i.e. no preference for attacking interconnected nodes) then P_∞ is the same as for an Erdős-Rényi network with degree $k_{\text{inter}} + k_{\text{intra}}$. [22]

More details on the above derivations can be found in [22, 23, 25].

III. MODEL

We now develop and analyze a stochastic block model [37–40] with overlap among the various blocks (modules). We first define a vector $\vec{m}$ describing the number of distinct modules or communities (blocks) at each layer of the hierarchy. At the first layer we always consider the entire network as a single community, thus $m_1 = 1$. The next layer, m_2 counts how many modules are at the second layer. Next is the total number of modules at the third layer, followed by the number of modules at the fourth layer, etc. We also assume for simplicity that all of the m_j modules in layer j are broken down into the same fixed number of m_{j+1} modules. For example, for the network shown in Fig. 1, we define $\vec{m} = [1, 3, 12, 36]$, since

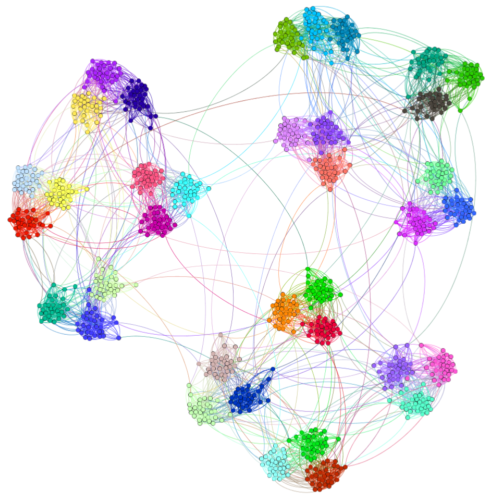

FIG. 1. **Model Illustration.** For this realization, the model has 4 hierarchical layers. At the top layer there are 3 modules, each of which is broken down into 4 modules, each of which is then broken down into 3 modules, which are not broken down further. We could describe this configuration of hierarchical modules by the vector $\vec{m} = [1, 3, 12, 36]$

the top layer is a connected graph, at the next layer we have three modules, then a total of 12 modules (i.e. each of the three is broken down into four smaller modules), and finally 36, since each of the 12 modules is broken down into three additional ones.

We next define the vector $\vec{k}$, which describes the average degree between nodes connected at each layer of the network. Thus, if at the highest layer the nodes have an average of 0.1 links to nodes in other modules, this will be the first entry, k_1 in $\vec{k}$. If the average degree at the next layer is 0.3 then that will be the second entry, k_2 etc. We assume that the entries of $\vec{k}$ should be strictly increasing since we expect there to be more links within communities at a lower layer than at a higher layer (e.g., neighborhoods are more tightly connected than cities).

In our model, we carry out a targeted attack on the nodes of the network, assuming that nodes that are interconnected at the top level are most likely to fail followed by those connected at the second level, etc. To do so, we must determine how many nodes are connected at each level and convert from the survival likelihood of nodes at a given layer, r_i, to the overall survival likelihood, p. First, we must estimate the fraction of nodes that are connected at level i. We note that the distribution of links at each layer is Poisson with the likelihood of a node to have k links being given by $P(k) = k_i^k e^{-k_i}/k!$ where k_i is the average degree at layer i. We can then find the likelihood of not having any links as $P(0) = e^{-k_i}$ and thus the likelihood of having at least one link (being interconnected) is $1-e^{-k_i}$. We define r_i as the survival probability of interconnected nodes at layer i. For the top layer of interconnections we can find how the $1-p$ overall fraction of nodes removed from the network, corresponds to the $1-r_1$ fraction of interconnected nodes removed, using [22],

$$p = r_1(1 - e^{-k_1}) + e^{-k_1}. \tag{12}$$

This equation can be understood by recognizing that the likelihood of a node *not* having an interlink (i.e. not be interconnected) at layer 1 is given by e^{-k_1} and therefore the likelihood of being interconnected is $1 - e^{-k_1}$. Since r_1 is the fraction of interconnected nodes that survive, the overall survival probability is given by multiplying r_1 by the fraction of nodes that are interconnected, and adding the fraction of nodes that are not interconnected (since all non-interconnected nodes survive the attack).

After we have removed all interconnected nodes at the top level, we then begin removing interconnected nodes at the next level of the hierarchy. In order to convert from the survival probability of nodes at this next level, r_2, to the overall survival probability p, we must take into account that some of the nodes removed at the previous layer were likely interconnected at this layer too, but as they have already been removed we must remove them from our calculation.

We do so by first finding the value of p for removing all interconnected nodes of each respective layer. For the first layer, this is when $r_1 = 0$ and thus the value of p at which all interconnected nodes in layer 1 are removed is $p_{co_1} = e^{-k_1}$, where we have defined p_{co_1} as the *cutoff* value for layer 1. For the next layer the cutoff at which all interconnected nodes are removed is given by recognizing that the fraction of interconnected nodes at this next layer is $1 - e^{-k_2}$, but we have already removed $1 - p_{co_1}$ fraction of nodes. We must recognize that some nodes are also likely to be interconnected at *both* layers and we must make sure not to double count them. We can thus find, p_{co_2} using the inclusion-exclusion principle as

$$\begin{aligned} p_{co_2} &= 1 - \left(1 - e^{-k_2} + 1 - e^{-k_1} - (1 - e^{-k_2})(1 - e^{-k_1})\right) \\ &= e^{-k_2 - k_1}, \end{aligned} \tag{13}$$

where in the top line $(1 - e^{-k_1})$ is the fraction of interconnected nodes at layer 1 and $(1 - e^{-k_2})$ is the fraction of interconnected nodes at layer 2 and we subtract $(1-e^{-k_2})(1-e^{-k_1})$, which is the fraction of interconnected nodes at both layers that were double counted. To get the cutoff where all interconnected nodes at either layer 1 or 2 are removed, we take one minus the fraction of nodes that are interconnected at either of our 2 layers. Simplifying terms gives the bottom line of Eq. (13).

Another more simple way to arrive at Eq. (13) is to note that we now remove all nodes with an interlink at either the first layer or the second layer. The total degree at these layers is just the sum $k_1 + k_2$. Given as we can immediately recognize based on the Poisson distribution for $k_1 + k_2$, that $e^{-(k_1+k_2)}$ is the fraction of nodes that will not be connected at either layer, and thus if we remove all nodes that are interconnected at either of these layers, we will be left with exactly a fraction $e^{-(k_1+k_2)}$ of nodes.

We can continue along the above lines to recognize that as we move further down the layers, we must include an additional k_i for each layer i. Thus, for a given layer i, the cutoff value of p for which all interconnected nodes at that layer are removed, is

$$p_{co_i} = e^{-\sum_{j=1}^{i} k_j}. \quad (14)$$

Having solved the case where all nodes of a given layer are removed, we can now consider the values of p for which only some fraction, $0 < r_i < 1$, of nodes at layer i survive. We can then convert from r_i, the survival probability in layer i (after having removed all nodes in higher layers), to p using

$$p = r_i(p_{co_{i-1}} - p_{co_i}) + p_{co_i}, \quad (15)$$

which can be understood by noting that p_{co_i} fraction of nodes will always survive since they are not interconnected at layer i (or any higher layer) and $r_i(p_{co_{i-1}} - p_{co_i})$ fraction of interconnected nodes at layer i survive.

IV. ANALYTIC THEORY FOR A SINGLE HIERARCHICAL NETWORK

We will now present a theory for hierarchical networks, generalizing the results in Eq. (11).

For the top layer, we can make use of the previous results on modular networks by setting our average interconnected degree to the degree at the layer we are attacking, namely k_1. To determine the intra degree, we note that all connections below the layer we are attacking are randomly distributed and thus the average intra degree will be replaced with the sum of the degrees below the layer we are attacking, $\sum_{i=2}^{l} k_i$, where l is the number of layers. We thus obtain

$$P_\infty = e^{-k_1}(1 - r_1)\left(1 - e^{-(\sum_{i=2}^{l} k_i)P_\infty}\right) + r_1\left(1 - e^{-(\sum_{i=1}^{l} k_i)P_\infty}\right) \qquad p_{co_0} < p. \quad (16)$$

Note that the result in Eq. (16) is only accurate until we have removed all the nodes that are interconnected at the top layer, i.e. so long as $p_{co_0} < p$. Also note that Eq. (16) is the same as Eq. (11) when $r > 0$, with $k_{\text{inter}} = k_1$ and $k_{\text{intra}} = \sum_{i=2}^{l} k_i$.

Once we have removed all interconnected nodes in the first layer, we then move on to removing nodes that are interconnected at the second layer. In this case, the average degree of interconnections is now k_2 and the average degree of intraconnections is $\sum_{i=3}^{l} k_i$. Furthermore, we must recall that the survival probability has already dropped to p_{co_1}, which is distributed randomly from the perspective of these lower layers. We also must note that at this point the network is already split into m_2 separate modules and nodes that are interconnected at layer 2 survive with a probability of only r_2. Accounting for this

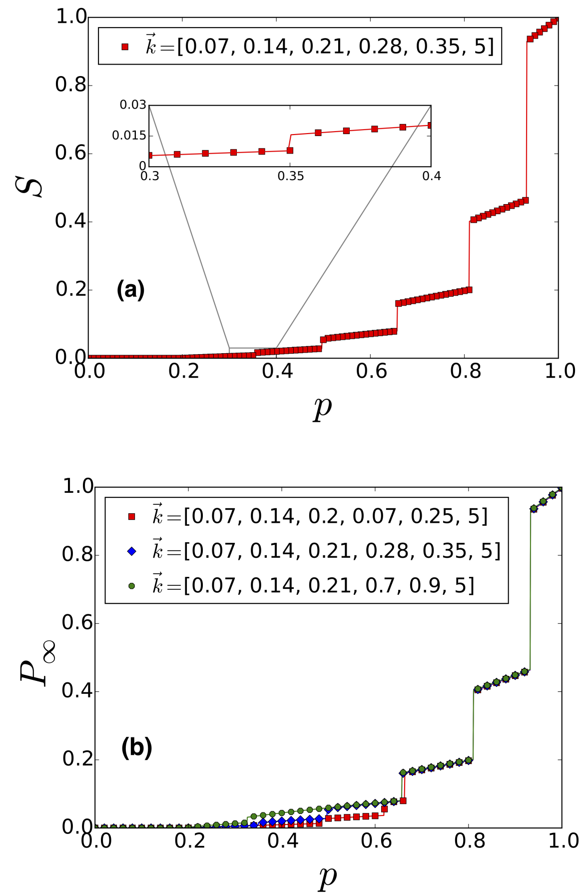

FIG. 2. **Comparison between simulation results and theory.** In both figures we have six layers, where each layer splits a module into two other modules, thus there are $\vec{m} = [1, 2, 4, 8, 16, 32]$ modules at each layer, with the average degrees between nodes at each layer given by the vector $\vec{k}$ in the legend. We have different values for the degree at each layer as given in the legends of **(a)** and **(b)**. The lines represent the theory of Eq. (18) and the points are simulations averaged over 10 runs on networks of $N = 10^6$ nodes.

gives

$$m_2 P_\infty = p_{co_1}\Big[e^{-k_2}(1-r_2)\left(1-e^{-(\sum_{i=3}^{l} k_i)m_2 P_\infty}\right)$$
$$+ r_2\left(1-e^{-(\sum_{i=2}^{l} k_i)m_2 P_\infty}\right)\Big], \qquad p_{co_2} < p < p_{co_1}. \quad (17)$$

Note that this equation is only accurate for values of p for which all interconnected nodes at layer 1 are removed, but not all interconnected nodes at layer 2 are removed, i.e., $p_{co_2} < p < p_{co_1}$.

We can generalize the above results for all values of p to find

$$m_j P_\infty = p_{co_{j-1}}\Big[e^{-k_j}(1-r_j)\left(1-e^{-(\sum_{i=j+1}^{l} k_i)m_j P_\infty}\right)$$
$$+ r_j\left(1-e^{-(\sum_{i=j}^{l} k_i)m_j P_\infty}\right)\Big], \qquad p_{co_j} < p < p_{co_{j-1}}. \quad (18)$$

To apply Eq. (18) for a given p one must first examine the p_{co_j} values to determine between which two cutoffs p is, and then convert p to a corresponding r_j value. Finally one plugs r_j, $p_{co_{j-1}}$ and the other parameter values into Eq. (18).

We compare in Fig 2, the theory of Eq. (18) and simulations of a corresponding network, observing excellent agreement between them. In the figure we observe multiple discontinuities in P_∞ as a function of p. Such multiple transitions have previously been observed in a few different models that considered bootstrap percolation or percolation on interdependent networks [41–43], however to the best of our knowledge multiple (more than 2) transitions have not been observed under targeted attack with ordinary percolation as in our model. As all interconnected nodes in a particular layer are removed, the system experiences a discontinuous jump. We note that the number of layers minus one serves as an upper bound on the number of potential jumps, as there may not be more jumps than that, however there may be *fewer* jumps depending on the parameters. This generalizes the results of [22] where only for certain sets of parameter values did the network separate into distinct communities before collapsing entirely, while for other parameter values there was only a single collapse where all the communities failed in tandem. In the next section we assess the number of discontinuous jumps for a given set of parameters. While one could of course examine this by plotting the results of Eq. (18) and then observing how many jumps take place, we provide more intuition on the underlying process by considering the number of jumps explicitly via analytical considerations.

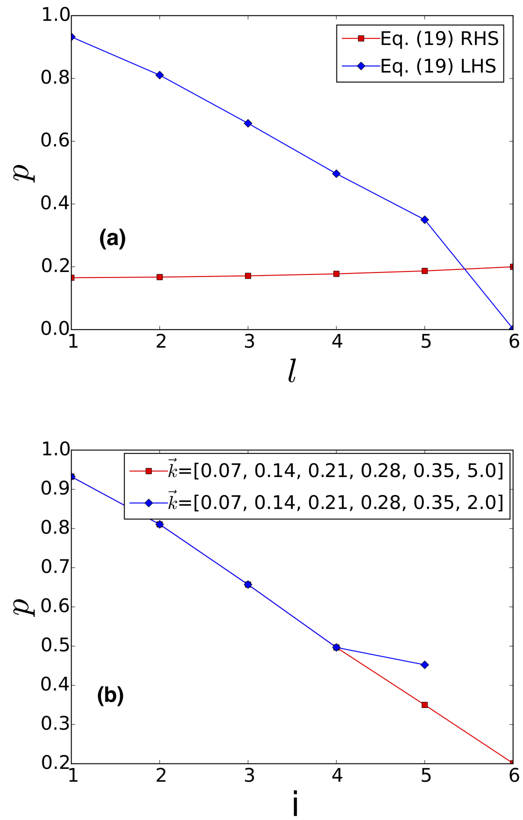

FIG. 3. **(a)** The RHS and LHS of Eq. (19) for the network of Fig. 2a. We observe that for the first $l \le 5$ layers the value of p for which all interconnected nodes are removed, LHS of Eq. (19), is greater than the value at which the network intra-connectivity breaks down (RHS of Eq. (19)). However for the 6th layer this is no longer true and we observe the continuous percolation transition of a random network. **(b)** The values of the ith critical points for both the network described in Fig. 2a and another network which has a smaller degree at its bottom layer. We note how changing the degree at the bottom layer effects the number of jumps since for the network with a lowest layer average degree of 2, before removing all interconnected nodes at the 5th layer, the network already breaks apart.

A. Number of Abrupt Jumps

Here we evaluate the expected number of jumps that will take place using our analytic theory from above. The key insight is to notice that jumps will occur when all interconnected nodes at a particular layer are removed, yet there remain enough total surviving nodes such that the network at the next lower layer remains connected. For the limiting case where all nodes are connected at a particular layer, then the network will collapse before lower layers are reached since all nodes will already have been removed.

This condition can be expressed mathematically by recognizing that we need the value of p_c, the critical threshold of the remaining intralinks to be lower than the value of p for which all interconnected nodes at a given layer i are removed. The classical result of Erdős-Rényi informs us that there will remain a giant component so long as $p > \frac{1}{<k>}$, where $<k>$ is the average degree, which for our case is $<k>= \sum_{j=i+1}^{l} k_j$ or the sum of the degrees at all lower levels. The point at which all interconnected nodes at layer i are removed is given by the cutoff value defined in Eq. (14). Overall the condition that there will be a jump once all interconnected nodes at layer i are removed is

$$e^{-\sum_{j=1}^{i} k_j} \geq \frac{1}{\sum_{j=i+1}^{l} k_j}. \quad (19)$$

We note that assuming all $k_i > 0$, then $e^{-\sum_{j=1}^{i} k_j}$ is strictly decreasing as i increases. Furthermore, the above assumption also implies that $1/\sum_{j=i+1}^{l} k_j$ is strictly increasing as i increases (since the denominator must decrease as there are fewer k_j terms). Therefore, once the condition of Eq. (19) is first violated for a particular layer, we know that it will continue to break down for lower layers and thus we can be sure that our number of jumps is the number of layers for which Eq. (19) is valid.

We plot the two sides of Eq. (19) in Fig. 3a, where for $l \leq 5$ we see that the left-hand-side (LHS) of the equation is larger than the right-hand-side (RHS). Comparing to the number of jumps in Fig. 2a we see that the network indeed experiences 5 abrupt jumps as expected (see inset for the 5th jump).

B. The p Values of the Jumps

Having predicted above the number of jumps that the network will undergo, we can now analyze the multiple values of p_c, the critical thresholds at which the transitions occur. We first note that so long as the LHS of Eq. (19) is greater than the RHS of the same equation, there will be a transition at the point of the LHS of the equation. After these i transitions, there will be one final $i+1$st transition which will be continuous as opposed to abrupt. We can find the point at which this final transition occurs by generalizing Eq. (21) from [23], which gave a form for the critical transition point of a modular network that has experienced both targeted attack of the type proposed here and also random failures on all nodes. The formula found there was

$$r_c^2(p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}}) + r_c(p_{\text{rand}} k_{\text{inter}} + p_{\text{rand}} k_{\text{intra}} - p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}}) + (p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - 1) = 0, \quad (20)$$

where p_{rand} represents the survival probability due to the random failures, r_c represents the critical threshold, and k_{intra} (k_{inter}) represents the degrees of intra (inter) connected nodes, respectively.

For our case of hierarchical networks, the random failures are represented by the attacks on nodes that were interconnected at higher layers. Overall the probability of surviving the random attacks is p_{co_i} thus p_{rand} is replaced by p_{co_i}. Furthermore, the inter degree k_{inter} is now given by k_{i+1} and the intra degree k_{intra} will be $\sum_{j=i+2}^{l} k_j$. Thus, we can find the point of transition, which we will call r_{i+1}, by solving Eq. (21), which is a quadratic formula for r_{i+1},

$$r_{i+1}^2 \left(p_{co_i}^2 \left(\sum_{j=i+2}^{l} k_j \right) k_{i+1} e^{-k_{i+1}} \right) + r_{i+1} \left(p_{co_i} k_{i+1} + p_{co_i} \left(\sum_{j=i+2}^{l} k_j \right) - p_{co_i} \left(\sum_{j=i+2}^{l} k_j \right) e^{-k_{i+1}} - p_{co_i}^2 \left(\sum_{j=i+2}^{l} k_j \right) k_{i+1} e^{-k_{i+1}} \right) + \left(p_{co_i} \left(\sum_{j=i+2}^{l} k_j \right) e^{-k_{i+1}} - 1 \right) = 0. \quad (21)$$

After finding r_{i+1} we can convert it to a value of p using Eq. (15). We note a slight subtlety in this system, in that even for the case where the hierarchical network is completely isolated at the lowest level we do not precisely recover the critical threshold of a random network with $k = k_{i+1}$. This is since we are targeting only those nodes which have at least one link. This leads to a slight correction where we obtain $r_{i+1} = 1/k_{i+1}$ (and then convert this to a value for the last p_c), rather than obtaining the usual $p_c = 1/k_{i+1}$. In most cases this correction will be quite small as for any reasonable value of k at the lowest level, there will be very few nodes that do not have even a single link. For example, for the case of the network referred to by the top line of the legend in Fig. 3b, the transition for the 6th layer takes place at $p_c \approx 0.201$ as opposed to $1/k_6 = 0.2$. Nonetheless, it is worth noting this discrepancy.

V. INTERDEPENDENT NETWORKS

Much recent research has also explored the resilience of interdependent networks where the nodes of one network depend on nodes in another network [5, 6, 12, 36, 44–49]. One example is that of a communication network that is interdependent with a power grid, yet more complex interdependencies are also possible [50, 51]. Many of these interdependent networks will likely possess the hierarchical structure described above. Therefore we now extend our theory from a single hierarchical network to the case of networks of interdependent networks (NON).

We will assume that each network in the interdependent system is formed of the same hierarchical structure, i.e. there are the same number of modules at each level. Again, this is realistic since the number of cities, neighborhoods, etc. that exist for the power grid are likely the same as those for a communications network as well as for other infrastructures. Further, we will assume that nodes are dependent on other nodes within their same module at the lowest level. This corresponds to the assumption that nodes are most likely dependent on resources from nodes in their same neighborhood, i.e., a power station depends on a communication tower in the same neighborhood and vice versa.

In the case of interdependent networks formed of n networks with $n > 2$, the structure of the dependencies can take various shapes. Among these are both treelike structures, where networks depend on one another such that their dependencies form a tree, or looplike structures where the dependencies form loops. Here we will consider (i) treelike structures and (ii) a random-regular (RR) network of networks where each network depends on exactly z other networks. Furthermore, one can allow for differing levels of interdependence where only some fraction q of nodes between two networks are interdependent, whereas $1 - q$ fraction of nodes are autonomous with no dependency. This could be the case for example if some communications towers have their own generators for power supply [52].

For the case of interdependent networks, it was noted [23] that the framework described above for failure and attack on complex networks can be extended by noting that each node now has an additional random probability p_{dep} of failure due to the presence of the dependency links. The precise expression for p_{dep} will depend on the number of dependent networks, the amount of the dependencies (q), and the structure of the dependencies (treelike, looplike, etc.). The Eqs. (6) and (7) can be rewritten generally for any p_{dep} as [23]

$$P_\infty(x) = p_{dep}(F_0(1) - F_0(u)), \tag{22}$$

and

$$1 = p_{dep}\frac{F_1(u) - F_1(1)}{1 - u}. \tag{23}$$

A. Treelike Network Formed of Hierarchical Networks

Here we introduce the theory for a network of networks formed of n interdependent networks such that they form a tree. We also note that Eqs. (15)-(16) remain valid as for the single-network case as we still target the nodes using the same procedure.

We assume that all nodes between pairs of interdependent networks have dependency links ($q = 1$). Further, we assume the no-feedback condition, meaning that if node a in network n_1 depends on node b in network n_2, then node b also depends on node a. We restrict the dependencies to being within the smallest community (i.e. the lowest layer of the hierarchy) for each set of interdependent networks. Lastly, we will attack the nodes of only one of the networks and let the attack propagate to the other networks as well as back to the original one. We note that the results are not dependent on the structure of the tree or from which network in the tree the nodes are originally removed [36].

To include the effects of the interdependencies, we must add an additional likelihood of failure based on the interdependence. For a treelike network of n interdependent networks with the dependencies within the neighborhoods, the likelihood of all of a node's interdependent nodes to survive is $p_{dep} = \left(1 - e^{-\left(\sum_{i=j}^{l} k_i\right) m_j P_\infty}\right)^n$ [23, 36]. For a node to survive, it must survive and its dependent nodes must survive thus p_{dep} is multiplied by our previous result from Eq. (18). This gives us the following solution for the treelike NON

$$m_j P_\infty = p_{co_{j-1}}\left[e^{-k_j}(1-r_j)\left(1-e^{-\left(\sum_{i=j+1}^{l} k_i\right)m_j P_\infty}\right)\right.$$
$$\left. + r_j\left(1-e^{-\left(\sum_{i=j}^{l} k_i\right)m_j P_\infty}\right)\right]\left(1-e^{-\left(\sum_{i=j}^{l} k_i\right)m_j P_\infty}\right)^{n-1},$$
$$p_{co_j} < p < p_{co_{j-1}}. \quad (24)$$

We note that numerical simulations show excellent agreement with the theory of Eq. 24 (Fig. 4). We also note that in the case of interdependent networks, the final transition is now also abrupt [5, 6, 36]. The abruptness of this transition is caused by a long cascade process that takes place in interdependent networks and which has been previously found for different models [53].

B. Random Regular Network Formed of Hierarchical Networks

Lastly, we consider the case of an RR NON where each network depends on exactly z other networks. We will assume that for each pair of interdependent networks only a fraction q of the nodes are interdependent and we will allow feedback (in contrast to what was done for treelike NON). However, we will still restrict the dependencies such that they must be within the same community at the lowest level of the hierarchy. We will also carry out the attack on all the networks, as opposed to attacking only one of them in the treelike case.

The effects of the dependencies now imply that the likelihood of a node to survive all interdependencies is $p_{dep} = \left(1-q+qm_jP_\infty\right)^z$ [36, 54]. Again a node must survive in its own network as well, so combining this with Eq. (18) yields,

$$P_\infty = p_{co_{j-1}}\left[\frac{1}{m_j}e^{-k_j}(1-r_j)\left(1-e^{-\left(\sum_{i=j+1}^{n} k_i\right)m_j P_\infty}\right)\right.$$
$$\left. + r_j\left(1-e^{-\left(\sum_{i=j}^{l} k_i\right)m_j P_\infty}\right)\right]\left(1-q+qm_jP_\infty\right)^z,$$
$$p_{co_j} < p < p_{co_{j-1}}. \quad (25)$$

For the RR NON the last transition will be continuous (for low values of q) as for an RR NON formed of Erdős-Rényi networks the transition may be continuous [54]. We observe excellent agreement between the theory of Eq. (25) and simulations in Fig. 5.

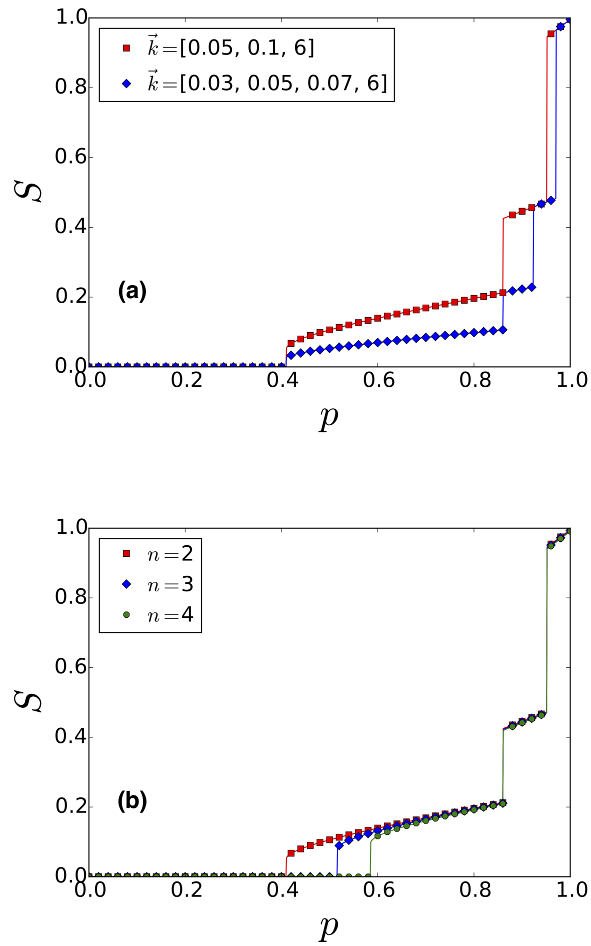

FIG. 4. **(a).** The case of two interdependent networks with the values of degree at each layer as given in the legend. Points are simulations averaged over 10 realizations of networks with $N = 10^6$ nodes and lines are theory fr om Eq. (24). **(b).** Varying the number of networks with the degree vector fixed to $\vec{k} = (0.05, 0.1, 6)$.

VI. REALISTIC EXTENSIONS

Here we will consider two basic extensions to the framework developed above. One type of extension involves considering degree distributions that are not Poisson and uncorrelated at each layer, while a second considers trade-offs between adding links at different layers of the hierarchy. We will also discuss several other possible extensions, however leaving them for future work.

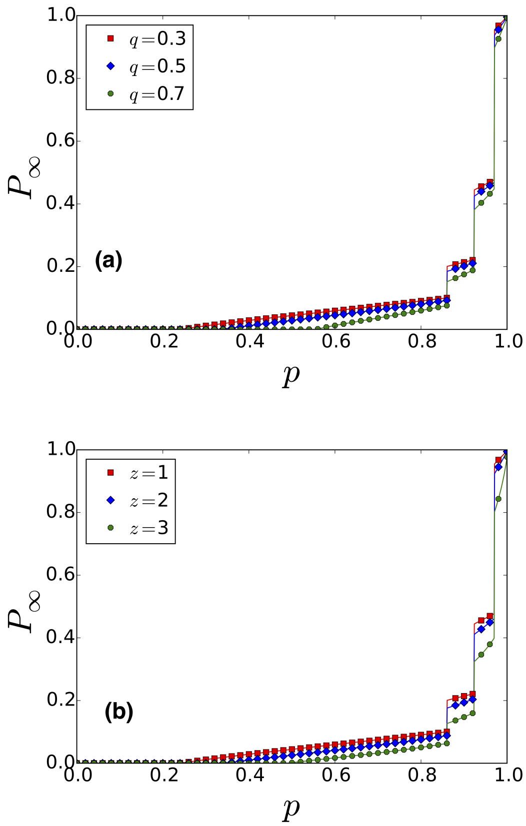

FIG. 5. A random regular network of networks where each network depends on z other networks such that they form loops. We vary both **(a)** q the level of interdependence between the networks (with $z = 1$) and **(b)** z the number of networks each network depends on (with $q = 0.3$). Symbols are simulations averaged over 10 realizations on networks with $N = 10^6$ nodes and lines are theory from Eq. (25).

A. Varying Degree Distributions

The first extension we will consider is a degree distribution that is not Poisson at each layer of the hierarchy. For this purpose we will use a power-law distribution with the likelihood of a node having k links given by $P(k) \sim k^{-\lambda}$. This type of degree distribution is common to many networks such as social networks, biological networks, and others [55].

We consider the case where the lowest layer of our hierarchy has a power-law distribution while the links at the higher layers have a Poisson distribution as in the above versions of our model. In Fig. 6a we present results for two different hierarchical networks with different degree distributions at higher layers and a scale-free distribution at the lowest layer. In both cases we still observe the characteristic multiple jumps as in our earlier models. In fact, in the case of scale-free networks, since $p_c \rightarrow 0$ for an isolated scale-free network with $\lambda < 3$, we expect that the number of jumps will nearly always approach its upper bound of one less than the number of layers $(l-1)$, which will be followed by a final transition at $p_c \rightarrow 0$. Relating back to Eq. (19) a scale-free distribution with $\lambda < 3$ would imply that the RHS of the equation is always near 0 and thus for every layer the LHS will be greater. Our results in Fig. 6a indeed confirm this as we find that for the case shown of $l = 4$ there are always three jumps.

Furthermore, if we consider the case where the scale-free distribution is not at the lowest layer, we expect that all nodes at the scale-free layer would have to be removed before the network would segregate and thus there would be no jumps passed the layer where the scale-free distribution existed. Alternatively, one could restrict *a priori* a specific set of nodes to have interlinks between them at the given layer according to a scale-free distribution. In this case once all such nodes were removed, the network would presumably segregate into distinct communities so long as the chosen set was not overly large.

A second extension involving degree distributions could relate to having the same nodes be interconnected at each layer. In this sense, one would first choose some set of nodes to be interconnected at the next to lowest layer and then choose a subset of those nodes to be interconnected at the above layer, followed by a subset of those nodes interconnected at the next higher layer, etc. If all the layers remained Poisson distributed, the main effect of these correlations between interconnections would be that the cutoff values of p in Eq. (14) would change. In this case, rather than having the cutoff be given by the sum of the degrees at all the layers, it would be given by $p_{co_i} = e^{-k_i}$, as only nodes that are interconnected at the given layer have been removed since those with interlinks at higher layers also must have interlinks at the lower layers (note that we assume that the k_i are decreasing as one moves up the layers). The remainder of our original derivation would remain virtually unchanged. Interlinks between layers could also simply be correlated rather than the absolute nature suggested above and then the calculation of the cutoff values would be more complicated.

B. Trade-offs between links at different layers

Another extension we consider is trade-offs among adding links at various layers. Ideally to consider such trade-offs one should specify costs for failures in connectivity at each layer of the hierarchy and also specify the cost of adding a link at each layer. A framework for

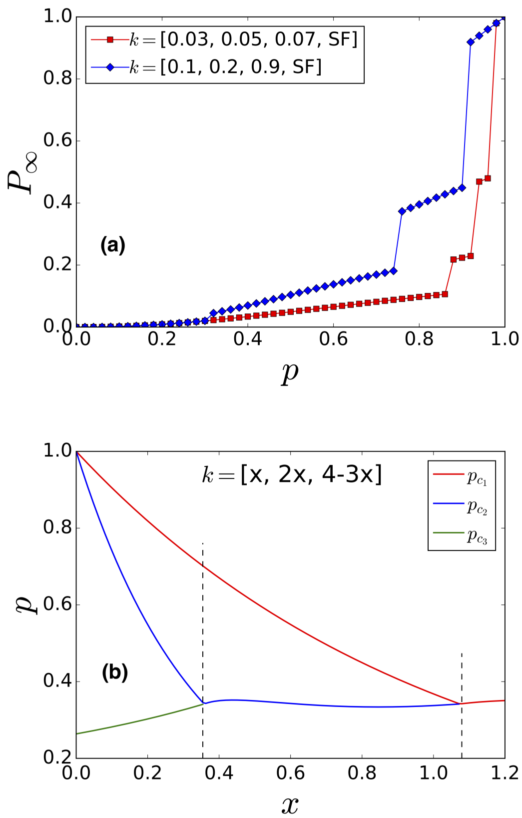

FIG. 6. **(a).** We consider two distinct hierarchical networks ($n = 1$) with a scale-free (SF) distribution at the lowest layer of the network. At the higher layers, the degree distribution is Poisson with the average given by the entries in $\vec{k}$. For the scale-free distribution we set $\lambda = 2.5$, $k_{min} = 2$, and $k_{max} = 1000$. The results are averaged over 10 realizations with $N = 10^6$ and we note that the lines here are not theory, but rather only a visual guide as all results are from simulations. **(b)** Here we consider a single network with a hierarchical structure composed of 3 layers, each with a Poisson distribution. However we vary the degree at each layer according to a fixed equation and fixed maximum degree. We assume that at the highest layer the degree is given by x, the next layer is given by $2x$, and the final layer is given by $4 - 3x$. We then plot the locations of p_{c_i} vs. x. We highlight with vertical dashed lines the two critical values of x where we move from having three transition to two ($x \approx 0.35$) and from having two transitions to a single one ($x \approx 1.05$). All the solid lines are based on the theory from Eq. (18).

considering costs similar to these can be found in the recent work of [56].

To give an example of a more simple version, we consider a hierarchical network of 3 layers with a fixed total degree. Furthermore, we place a condition stating that the number of links at the second layer must be twice (or more generally any factor of) the number of links at the first layer. Using Eqs. (19) and (21) we then find the critical point(s) of transition for the given network. In Fig. 6b we show how the critical points vary with the average degree at the first layer (defined as x). We find that there exist several possible critical values of x where the system moves from having three transitions to two and where it moves from having two transitions to a single one. These critical values of x represent what could be considered optimal trade-offs in preserving connectivity at specific layers. If we consider for example the first critical point in x, x_1, we note that this is the lowest value of x for which the middle layer does not segregate into communities and instead its connectivity is preserved so long as the overall network remains connected. Likewise the second critical point in x represents the first point where the network does not separate at the highest layer and instead the entire hierarchical network collapses all at once. This simple analysis shows that indeed trade-offs do exist between adding links at the different layers and provides some basic intuition into the more general case where costs are assigned to links at the different layers.

VII. DISCUSSION

In this work, we have studied the robustness of networks and networks of interdependent networks with a hierarchical structure. This structure is very common for many infrastructure networks, biological networks and others. We have found analytical solutions and confirmed these solutions through simulations for isolated hierarchical networks and for two different structures of interdependent hierarchical networks. The resilience of the network depends on the number of communities at each layer of the hierarchy, the degree at each layer of the hierarchy, the fraction of nodes removed, and also the parameters governing the interdependence (if present). We also extended our framework to consider the more realistic case of a scale-free distribution and different trade-offs between adding links at the various layers.

Our results show that hierarchical networks can undergo multiple abrupt transitions depending on the above parameters and that these transitions represent the separation of the network at different layers of the hierarchy. These results have potential applications in optimizing the resilience of networks in infrastructure and other fields.

We acknowledge the Israel Science Foundation, ONR, the Israel Ministry of Science and Technology (MOST) with the Italy Ministry of Foreign Affairs, BSF-NSF, MOST with the Japan Science and Technology Agency, the BIU Center for Research in Applied Cryptography and Cyber Security, and DTRA (Grant no. HDTRA-1-10-1- 0014) for financial support. We thank an anonymous referee for comments that significantly improved the manuscript.

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