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State dependent jump processes: Itô–Stratonovich interpretations, potential, and transient solutions

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Abstract

The abrupt changes that are ubiquitous in physical systems often are well characterized by shot noise with a state dependent recurrence frequency and jump amplitude. For such state dependent behavior, we derive the transition probability for both the Itô and Stratonovich jump interpretations, and subsequently use the transition probability to pose a master equation for the jump process. For exponentially distributed inputs, we present a novel class of transient solutions, as well as a generic steady state solution in terms of a potential function and the Pope-Ching formula. These new results allow us to describe state dependent jumps in a double well potential for steady state particle dynamics, as well as transient salinity dynamics forced by state dependent jumps. Both examples showcase a stochastic description that is more general than the limiting case of Brownian motion to which the jump process defaults in the limit of infinitely frequent and small jumps. Accordingly, our analysis may be used to explore a continuum of stochastic behavior from infrequent, large jumps to frequent, small jumps approaching a diffusion process.

Keywords: marked Poisson process, double well potential, anomalous jumps, diffusion processes, δ-pulse noise, two-sided exponential distribution, Fokker-Planck equation, Itô Stratonovich dilemma

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I. INTRODUCTION

The traditional tenet that denies sudden changes, an axiom in the works of Leibniz [1] and epitomized by the maxim 'Natura non facit saltus' – Nature does not make jumps [2], is clearly challenged by the abrupt transitions that are common in nature, from random bursts in gene expression [3], to the atomic transitions (quantum jumps) of electrons between energy levels [4]. Jumps are synonymous with delta-pulse trains and shot noise. References to shot noise first appeared in the study of vacuum tubes where it represents the random transfer of discrete charge units [5, 6]. Electron shot noise occurs in many solid state devices such as $p-n$ junctions [7–12]. In addition, shot noise occurs with optical devices where it represents the transfer of discrete packets of photons [13].

More generally, jumps are ubiquitous in a variety of fields such as queuing theory [14], stock market modeling [15–17], insurance risk [18], population dynamics [19], and of course, stochastic processes in general [15, 20]. Typically, these jumps punctuate a continuous time process [15], as in biology, where the jumps represent the sudden drop in voltage caused by nerve excitation [21–23], and in environmental science and engineering, where jumps may reasonably represent natural phenomena such as fires [24, 25], rainwater infiltration [26–28], extreme events [29], avalanches [30], runoff and streamflow [31–33], large earthquakes [34, 35], volcanic eruptions [36], and solar flares [37].

The jump process is defined by both the jump amplitudes and the frequency of jump events. In many models, the frequency and amplitudes of jumps are considered to be independent of the system state. In contrast, for many natural systems, both the jump frequency and amplitude depend on the system state. This state dependence may be critical. For instance, a state dependent frequency may create both persistent jump behaviors and preferential states [38, 39]. Similarly, a state dependent amplitude is essential for naturally limiting the system response to the jump [40, 41]. For example, a jump of rainfall infiltration is limited by the degree of soil saturation [42–44]. However, for white noise, this effect varies for different interpretations of the jump process— the well known Itô- Stratonovich dilemma [41, 45]. Although some work has begun to address this issue [41, 45], the effects of this state dependence in amplitude and frequency have yet to be examined together or in terms of transition probability density functions (PDFs).

Toward this goal, here we define the transition PDFs in terms of a state dependent frequency and jump amplitude for both the Itô and Stratonovich interpretations of the jump process. Unlike previous definitions, here the transition PDFs are defined in terms of a jump function for which the forcing input and state dependence are not necessarily separable. Furthermore, we discuss the generality of the limiting conditions under which a state dependent jump process converges to a diffusion process. A detailed derivation of the limits to the corresponding Fokker-Planck equations is provided in Appendix A for both Itô and Stratonovich interpretations. For the master equation in terms of the Stratonovich jump prescription, we then consider an exponential PDF of forcing inputs and present a novel general solution in terms of a potential function. We use this result to analyze particle dynamics in a double well potential based on the state dependence of both the jump frequency and amplitude. We also present a class of transient solutions and demonstrate the result by analyzing a transient solution for soil salinity dynamics.

II. JUMP PROCESS

Consider a system evolving in time because of a deterministic component and jump perturbations with random timing and amplitudes, as described by the stochastic differential equation (SDE), i.e., Langevin-type equation,

$$\frac{d\chi}{dt} = m(\chi, t) + \xi(\chi, t), \tag{1}$$

where $m(\chi, t)$ is a deterministic function, and $\xi(\chi, t)$ represents the jumps, which generally are a state dependent noise that perturbs the system. More specifically, these jumps are defined as

$$\xi(\chi, t) = \sum_{i=1}^{N(t)} b(\chi, z)\delta(t - t_i), \tag{2}$$

where, as indicated by Dirac delta function, $\delta(\cdot)$, the function $b(\chi, z)$ is instantaneous at the arrival times $\{t_i\}(i = 1, 2, ...)$. These arrival times are modeled as a non-homogeneous Poisson process with a (state dependent) rate of $\lambda(\chi, t)$. For each jump, the function $b(\chi, z)$ depends on the state variable, χ, and mutually independent random forcing inputs, z, with a probability distribution, $p_z(z)$ [46, 47]. Though not explicitly stated here, the function $b(\chi, z)$ generally also could be dependent on time.

Typically for Eq. (2), the literature [e.g., 48] considers the less general case of $b(\chi, z) = b(\chi)z$ [40, 41]. This implicitly assumes that any dependence on z has been factored out, i.e., $b(\chi, z) = b(\chi)b_z(z)$, and subsequently, $b_z(z)$, has been lumped into a new jump distribution $\check{p}_z(z)$ based on a change of variables, i.e.,

$$\check{p}_z(z) = \frac{p_z(b_z^{-1}(z))}{b_z'(b_z^{-1}(z))}, \tag{3}$$

where $b_z'(\cdot)$ is the derivative with respect to z, and $b_z^{-1}(\cdot)$ is the inverse of $b_z(z)$ [49].

Though equation (1) is the basis of many modeling approaches, there is one major caveat due to the white-noise character of the forcing. More specifically, for the function $b(\chi, z)$ the value of χ is undetermined at the arrival times $\{t_i\}(i = 1, 2, ...)$ of the delta function, and like the case of Gaussian noise [40, p. 230], it does not stipulate whether one assumes the value of χ before the jump, after the jump, or conceivably an intermediate value between both extremes [40]. The Stratonovich interpretation uses for χ in $b(\chi, z)$ an intermediate point between the states before and after a jump and preserves the rules of standard calculus. In Section II D, we show that for $b(\chi, z) = b(\chi)z$ this χ corresponds to an average point only in the limit of small jumps. While for the Itô interpretation, Itô's lemma performs the role of the standard calculus chain rule [50], and thus χ is interpreted as the value immediately before a jump. The Stratonovich approach corresponds to taking the zero limit of the correlation time of the jump [41] and accordingly represents the limit of a system that continuously evolves during the jump process. This Itô—Stratonovich dilemma has been explored for the specific case of $b(\chi, z) = b(\chi)z$, linear drift, and a homogeneous Poisson process [41], but thus far has not been examined for the more general case of $b(\chi, z)$, a nonhomogeneous Poisson process, and a generic drift function.

A. Master Equation

In both interpretations of the jump process, the PDF $p_\chi(\chi, t)$ evolves in time as

$$\partial_t p_\chi(\chi, t) = -\partial_\chi J(\chi, t), \tag{4}$$

where the current, $J(\chi, t) = J_m(\chi, t) + J_\xi(\chi, t)$, is the sum of the drift component

$$J_m(\chi, t) = m(\chi, t) p_\chi(\chi, t), \tag{5}$$

and the jump induced current

$$J_\xi(\chi, t) = J_{\chi u}(\chi, t) - J_{u\chi}(\chi, t). \tag{6}$$

The first component, $J_{\chi u}(\chi, t)$, is the current from jumping away from a prior state χ to any posterior state u, while the second component, $J_{u\chi}(\chi, t)$, is the current from jumping from a prior (antecedent) state u and arriving at a (posterior) state χ. These currents are

$$J_{\chi u}(\chi, t) = \int_{-\infty}^{\chi} p_\chi(x, t) \int_{-\infty}^{\infty} W(u|x, t) du dx \tag{7}$$

$$J_{u\chi}(\chi, t) = \int_{-\infty}^{\chi} \int_{-\infty}^{x} W_{(\cdot)}(x|u, t) p_\chi(u, t) du dx, \tag{8}$$

where $W(u|x, t)$ is the transition PDF of jumping from a state x and transitioning to any state u, while $W_{(\cdot)}(x|u, t)$ is the transition PDF of jumping away from a prior (antecedent) state u and transitioning to any (posterior) state x. For $W_{(\cdot)}(x|u, t)$, the subindex, $_{(\cdot)}$, indicates either $W_I(x|u, t)$ or $W_S(x|u, t)$ for the respective Itô or Stratonovich interpretations of the jump transition.

The transition PDF (per unit time) for jumping away from a state must equal the frequency of jumping. This frequency, $\lambda(\chi, t)$, is independent of the jump interpretation. Thus, integrating over all of the potential posterior (future) states u provides the overall rate $\lambda(\chi, t)$ of exiting the state χ [32], i.e.,

$$\int_{-\infty}^{\infty} W(u|\chi, t) du = \lambda(\chi, t). \tag{9}$$

The complementary transition PDF (per unit time) for jumping to any state is the frequency of exiting u with a transition amplitude of $\Delta\chi = \chi - u$ [32], i.e.,

$$W_{(\cdot)}(\chi|u, t) = \lambda(\chi, t) \int_{-\infty}^{\infty} p_{\Delta\chi|uz}(\Delta\chi|u, z) p_z(z) dz, \tag{10}$$

which is found by integrating the PDF of transition amplitudes, $p_{\Delta\chi|uz}(\Delta\chi|u,z)$, over the PDF of forcing inputs, $p_z(z)$.

Now, following the probability currents of Eqs. (5), (7), and (8), we express the master equation (4) as

$$\partial_t p_\chi(\chi,t) = -\partial_\chi \left[m(\chi,t)p_\chi(\chi,t)\right] - \lambda(\chi,t)p_\chi(\chi,t) + \int_{-\infty}^{\chi} W_{(\cdot)}(x|u,t)p_\chi(u,t)du, \qquad (11)$$

where on the r.h.s., the second term is based on Eq. (7) with the substitution of Eq. (9), and the transition PDF $W_{(\cdot)}(x|u,t)$ must be defined based on an interpretation of the jump transition amplitude.

The transition amplitude is derived from Eq. (1) at the instance of a jump, i.e.,

$$\frac{d\chi}{dt} = b(\chi,z)\delta(t-t_i), \qquad (12)$$

where as indicated by Dirac delta function at the times $\{t_i\}$ $(i=1,2,...)$, the infinite change of the jump overrides all other terms of Eq. (1). Within Eq. (12), $b(\chi,z)$ must be interpreted with either the Itô or Stratonovich conventions of stochastic calculus. For the two conventions, we construct two different versions of $p_{\Delta\chi|uz}(\Delta\chi|u,z)$ and $W_{(\cdot)}(\chi|u,t)$ (i.e., $W_I(\chi|u,t)$ and $W_S(\chi|u,t)$).

B. Itô prescription

Following the Itô convention, $b(u,z)$ depends on u, the state before (i.e., antecedent to) the jump. Accordingly, the jump transition is given as

$$\Delta\chi = \chi - u = b(u,z). \qquad (13)$$

This expression is retrieved by integrating both sides of Eq. (12) as $\int_u^{u+\Delta\chi} d\chi = \int_{t_i^-}^{t_i^+} b(\chi(t),z)\delta(t-t_i)dt$, where t_i^- and t_i^+ are the respective times immediately before and after a jump, and following the Itô convention, $\chi(t_i) = u$ is the state immediately prior to the jump. Consequently, $b(\chi(t_i),z)$ is interpreted as $b(u,z)$.

This expression of Eq. (13) is the basis of a conditional PDF, i.e.,

$$p_{\Delta\chi|zu}(\Delta\chi|z,u) = \delta(\chi - u - b(u,z)), \tag{14}$$

where Dirac delta function, $\delta(\cdot)$, indicates a deterministic relationship that may be posed as a function of the jump magnitude, z, i.e.,

$$g(z) = \chi - u - b(z,u). \tag{15}$$

Based on Eq. (15), $p_{\Delta\chi|zu}(\Delta\chi|z,u)$ also can be written as

$$p_{\Delta\chi|zu}(\Delta\chi|z,u) = \delta(g(z)) = \frac{\delta(z - z_n(\chi,u))}{|g'(z_n(\chi,u))|}, \tag{16}$$

where $g'(\cdot)$ is the derivative with respect to z, and $z_n(\chi,u)$ is the root for which $g(z_n) = 0$ (see Appendix A of [32]). Eq. (16) is useful in facilitating integration over z.

Based on Eqs. (10) and (14), the transition PDF in the Itô sense for a state dependent marked Poisson process becomes

$$W_I(\chi|u,t) = \lambda(u,t) \int_{-\infty}^{\infty} \delta\left(\chi - u - b(u,z)\right) p_z(z) dz, \tag{17}$$

where the product of $\lambda(u,t)$ and $p_{\Delta\chi|u}(\Delta\chi|u)$ describes the transition to any χ. If $b(\chi,z) = b(\chi)z$, where $b_z(z)$ is absorbed into the jump distribution $\check{p}_z(z)$ of Eq. (3), which we now call $p_z(z)$, the transition PDF of Eq. (17) simplifies to

$$W_I(\chi|u,t) = \frac{\lambda(u,t)}{|b(u)|} p_z\left(\frac{\chi - u}{b(u)}\right), \tag{18}$$

which is derived from Eq. (16) where $g(z) = \chi - u - b(u)z$, $z_n(\chi,u) = \frac{\chi - u}{b(u)}$, and $g'(z_n(\chi,u)) = -b(u)$.

C. Stratonovich prescription

Following the Stratonovich convention, the state χ in $b(\chi,z)$ is interpreted as an intermediate value between the antecedent and posterior states. The Stratonovich jump transition is derived by posing Eq. (12) in terms of an integrated variable [40, p. 230], i.e.,

$$\frac{d\eta(\chi, z)}{dt} = \delta(t - t_i) \tag{19}$$

$$\eta(\chi, z) = \int \frac{1}{b(\chi, z)} d\chi, \tag{20}$$

where as will be shown in the next section, $\eta(\chi, z)$ indicates that the argument χ of $b(\chi, z)$ is evaluated for each infinitesimal increment of the overall transition, $\Delta\chi$. Eq. (19) can thus be formally integrated as

$$\eta(\chi, z) - \eta(u, z) = 1. \tag{21}$$

The jump transition $\Delta\chi = \chi - u$ is implicit in the difference between the function $\eta(\cdot, \cdot)$ after the jump $\eta(\chi, z)$ and before the jump, $\eta(u, z)$. In other words, we recover Eq. (21) by integrating both sides of Eq. (19) as $\int_u^{u+\Delta\eta} d\eta = \int_{t_i^-}^{t_i^+} \delta(t - t_i)dt$, where t_i^- and t_i^+ are the respective times immediately before and after a jump.

Eq. (21) is the basis of a conditional PDF for the jump transition, i.e.,

$$p_{\Delta\chi|u,z}(\Delta\chi|u, z) = \frac{1}{|b(\chi, z)|}\delta(\eta(\chi, z) - \eta(u, z) - 1), \tag{22}$$

where we have used a change of variables [e.g., 51], i.e., $p_{\Delta\chi|uz}(\Delta\chi|u, z) = p_{\Delta\eta|uz}(\Delta\eta|u, z) \left|\frac{d\eta}{d\chi}\right|$ for which $\frac{d\eta}{d\chi} = \frac{1}{b(\chi,z)}$ and $p_{\Delta\eta|uz}(\Delta\eta|u, z) = \delta(\eta(\chi, z) - \eta(u, z) - 1)$. Again similar to Eq. (14), the delta function in Eq. (22) indicates a deterministic relationship, i.e.,

$$g(z) = \eta(\chi, z) - \eta(u, z) - 1, \tag{23}$$

which we interpret as a function of z. With Eq. (23), the PDF $p_{\Delta\chi|u,z}(\Delta\chi|u, z)$ may be posed in the form of Eq. (16), which is useful for facilitating the integration of $p_{\Delta\chi|u,z}(\Delta\chi|u, z)$ over z.

From Eqs. (10) and (22) and the rate $\lambda(u, t)$, the transition PDF in the Stratonovich sense becomes

$$W_S(\chi|u, t) = \frac{\lambda(u, t)}{|b(\chi, z)|} \int_{-\infty}^{\infty} \delta\left(\eta(\chi, z) - \eta(u, z) - 1\right) p_z(z)dz. \tag{24}$$

If $b(z,\chi) = b(\chi)z$ and thus $\eta(\chi, z) = \frac{\eta(\chi)}{z}$, the conditional PDF of Eq. (22) may be simplified based on the scaling property of the delta function, i.e.,

$$p_{\Delta\chi|z,u}(\Delta\chi|z,u) = \frac{1}{|b(\chi)|}\delta(\eta(\chi) - \eta(u) - z). \tag{25}$$

Accordingly, the simplified transition PDF is given by

$$W_S(\chi|u,t) = \frac{\lambda(u,t)}{|b(\chi)|}p_z\left(\eta(\chi) - \eta(u)\right), \tag{26}$$

which follows from Eq. (16) where $g(z) = \eta(\chi) - \eta(u) - z$, $z_n(\chi, u) = \eta(\chi) - \eta(u)$, and $g'(z_n(\chi, u)) = 1$. Though not explicitly mentioned in previous works [41, 52], Eq. (26) is the transition probability density that is used to pose the master equation in terms of the Stratonovich jump prescription.

D. Jump Process Simulation

When numerically simulating the jump process, the jump transition at times $\{t_i\}$ $(i = 1, 2, ...)$ must be consistent with the jump interpretation adopted in the description. For the Itô interpretation, the jump transition is given by Eq. (13). For the Stratonovich interpretation, the jump transition amplitude derived from Eq. (21) is given by

$$\Delta\chi = \chi - u = \eta^{-1}\left(\eta(u,z) + 1, z\right) - u, \tag{27}$$

where u is the state prior to the jump, and $\eta^{-1}(\cdot,\cdot)$ is the inverse function, i.e., $\chi = \eta^{-1}(y,z)$ for which $y = \eta(\chi, z)$. These expressions not only are useful in comparing realizations for different jump prescriptions, but also highlight the differences between the different jump prescriptions.

For the common assumption of $b(\chi, z) = b(\chi)z$, these jump transitions of Eqs. (13) and (27) simplify to

$$\Delta\chi = zb(u) \tag{28}$$

$$\Delta\chi = \eta^{-1}\left(\eta(u) + z\right) - u, \tag{29}$$

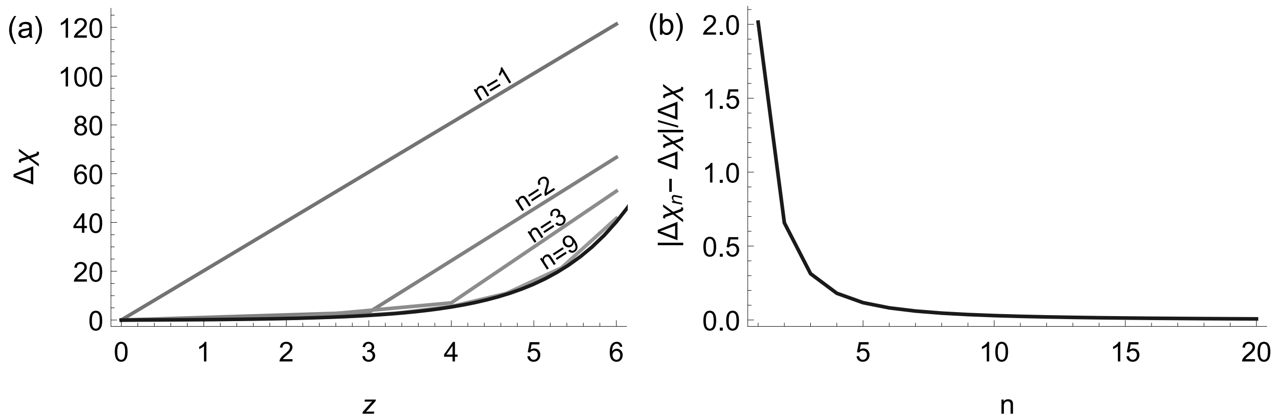

FIG. 1. For the Stratonovich jump interpretation, a) comparison of the Eq. (29) jump transition, $\Delta\chi$, (black line) and jump transition, $\Delta\chi_n$, for the summation within Eq. (30) based on different forcing input values z and different values of n, and b) for different values of n, the difference between $\Delta\chi$ of Eq. (29) and the summation within Eq. (30), $\Delta\chi_n$. In both cases, $b(\chi) = \chi$, $\eta(\chi) = \ln[|\chi|]$, and the antecedent state is $u = 0.1$.

for the respective Itô and Stratonovich interpretations.

Eq. (29) also is consistent with the Stratonovich interpretation of $b(\chi)$ for a Brownian motion. For a Brownian motion, the argument of $b(\chi)$ is correctly interpreted as the average of values immediately before and after an infinitesimally small random transition, i.e., $b\left(\frac{u+\chi}{2}\right)$ where u is the antecedent value and χ is posterior value (see Eq. (4.10) of [40]). This same interpretation also applies to the Stratonovich jump process when each jump is considered as a consecutive series of infinitesimal values, i.e.,

$$\Delta\chi = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{z}{n} b\left(\frac{u_j + \chi_j}{2}\right) = \eta^{-1}\left(\eta(u) + z\right) - u, \tag{30}$$

where the jump transition, $\Delta\chi$, is the sum of an n number of consecutive transitions each with an antecedent value u_j and a posterior value χ_j caused by the same consecutive forcing input, $\Delta z = z/n$. The values u_j and χ_j are defined recursively following the Stratonovich transition of Eq. (29). The posterior value is defined as

$$\chi_j = \eta^{-1}\left(\eta\left(u_j\right) + \frac{z}{n}\right), \tag{31}$$

which is based on the antecedent value u_j prior to the input of $\Delta z = z/n$. The antecedent value is initially u, but with each consecutive transition, the previous posterior, χ_{j-1}, value becomes the next antecedent value, u_j, i.e.,

$$u_j = \begin{cases} u & j \leq 1 \\ \chi_{j-1} & j > 1 \end{cases}, \tag{32}$$

where based on Eq. (31), $\chi_{j-1} = \eta^{-1}\left(\eta\left(u_{j-1}\right) + \frac{z}{n}\right)$ for which u_{j-1} is the previous antecedent value.

We examine the convergence of Eq. (30) based on the transition, $\Delta\chi_n$, as defined by the summation in Eq. (30). For the case of $n = 1$, the overall jump transition, $\Delta\chi_1$, greatly deviates from the actual Stratonovich transition of Eq. (29) (Fig. 1a). As n increases, $\Delta\chi_n$ rapidly converges to the Stratonovich transition (Fig. 1b). Because of this rapid convergence, $\Delta\chi_n$ may approximate the Stratonovich transition in cases where the jump is considered as a series of consecutive transitions. As shown by Eq. (30), the Stratonovich interpretation is ideal for representing jump transitions based on a continuous feedback from the concurrent increase (or decrease) in state variable as the forcing input, z, increases.

E. Jump Process Diffusive Limit

The forward master equation (11) provides a general description of a Markov process with state dependent transitions and thus acts as a framework for evaluating a stochastic process in terms of both coarser, larger jump transitions and finer, frequent transitions approaching a diffusion. For the limiting case of small, infinitely frequent jumps, the state dependence of both the jump amplitude and frequency directly translates to the state-dependence of the diffusion coefficient. In Appendix A for the case of $b(\chi, z) = b(\chi)z$, we show in detail how the master equation (11) converges to the Itô and Stratonovich versions of the Fokker-Planck equation, respectively, under the limiting scenario of infinitely small jumps occurring infinitely often. The explicit and detailed derivations of Appendix A also help clarify the conditions in which this convergence is possible, and in particular the condition that the mean forcing amplitude is zero.

This convergence is particularly interesting for the steady state condition. Specifically, for the case of $b(\chi, z) = b(\chi)z$, the jump process description can be linked to the well known steady state solutions for the Itô and Stratonovich Fokker-Planck equations as follows. Both solutions may be written in terms of a potential function, i.e.,

$$p_\chi(\chi) = N e^{-\Phi_{(\cdot)}(\chi)}, \tag{33}$$

where N is a normalization constant such that $\int_{-\infty}^{\infty} p_\chi(\chi) d\chi = 1$, and the potential function, $\Phi_{(\cdot)}(\chi)$, is specific to Eqs. (A14) and (A27) of Appendix A for the respective Itô and Stratonovich version of the Fokker-Planck equation, i.e.,

$$\Phi_I(\chi) = \int \left(-\frac{m(\chi)\lambda_o}{D_o \lambda(\chi) b(\chi)^2} + 2\frac{\partial_\chi b(\chi)}{b(\chi)} + \frac{\partial_\chi \lambda(\chi)}{\lambda(\chi)} \right) d\chi \tag{34}$$

$$\Phi_S(\chi) = \int \left(-\frac{m(\chi)\lambda_o}{D_o \lambda(\chi) b(\chi)^2} + \frac{\partial_\chi b(\chi)}{b(\chi)} + \frac{\partial_\chi \lambda(\chi)}{2\lambda(\chi)} \right) d\chi, \tag{35}$$

where $\Phi_I(\chi)$ is the Itô potential, $\Phi_S(\chi)$ is the Stratonovich potential, and $\int \frac{\partial_\chi b(\chi)}{b(\chi)} d\chi = \ln[|b(\chi)|]$. For both potentials, we have substituted for the state dependent diffusion coefficient, $D(\chi) = \frac{2D_o\lambda(\chi)}{\lambda_o}$; see Eq. (A12) of Appendix A. Thus, the potentials clearly identify the link with the state dependent jump frequency, $\lambda(\chi)$, and jump amplitude $b(\chi)z$, where z is implicit to the diffusion coefficient resulting from the Eq. (A11) limit of infinitely frequent and small jumps. Considering this limit in the Fokker-Planck equations (A14) and (A27) gives rise to a connection with the jump process that typically is not considered in presentations of the Fokker-Planck equation.

To illustrate how the Fokker-Planck steady state solution may provide a reasonable representation of high frequency jump processes, we first consider the simple case of a constant jump frequency, λ_o, with independent jump amplitudes, i.e., $b(\chi, z) = b(\chi)z$ and $b(\chi) = 1$. We consider both the jump and diffusion processes to share identical descriptions of a linear drift, $m(\chi) = -k\chi$, as well as a zero mean forcing amplitude, $\langle z \rangle = 0$. Accordingly, we assume the jump process is forced by a two-sided exponential PDF,

$$p_z(z) = \begin{cases} \frac{\gamma}{2} e^{-\gamma z} & z \geq 0 \\ \\ \frac{\gamma}{2} e^{\gamma z} & z < 0, \end{cases} \tag{36}$$

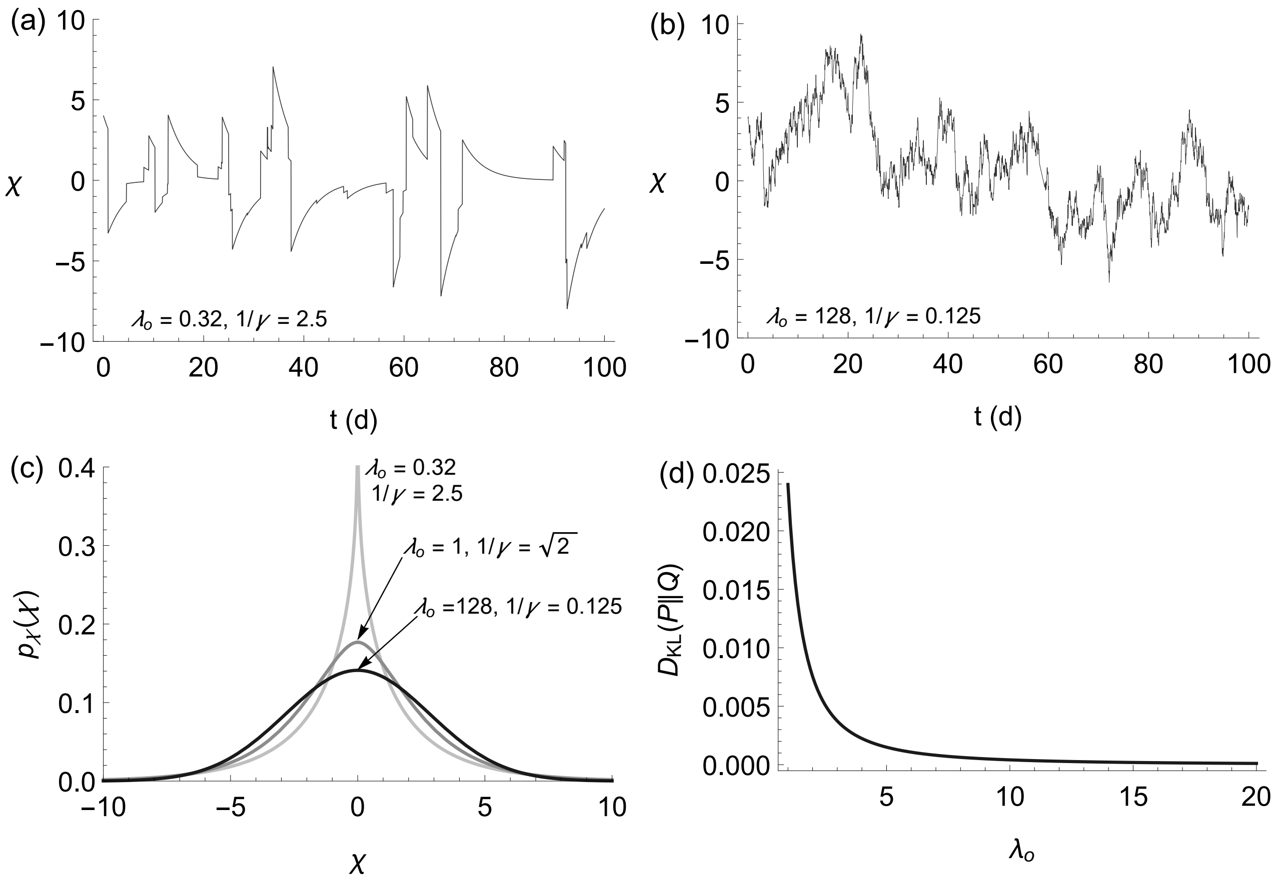

FIG. 2. Examples of a) a trajectory for infrequent and large jumps, b) a trajectory for frequent but tiny jumps, c) the steady state jump process PDF of Eq. (37) for different jump frequencies, and d) the Kullback-Leibler divergence, $D_{\mathrm{KL}}(P\|Q)$, between the jump process distribution, P, with the PDF of Eq. (37), and the diffusion process distribution, Q, with the Fokker-Planck steady state PDF of Eq. (33). In all cases, $D_o = 2$ and γ and D_o are related by Eq. (38). Here, $b(\chi, z) = b(\chi)z$, $b(\chi) = 1$, $p_z(z)$ is given by Eq. (36), the drift is given by $m(\chi) = -k\chi$, and $k = 0.25$.

where γ is the scale parameter. Thus, the jump process system trajectories fluctuate from both positive and negative jumps and are forced back to zero by the drift (Fig. 2a). For steady state conditions, the known solution to the master equation (11) is [46]

$$p_\chi(\chi) = \frac{2^{\frac{1}{2}\left(1-\frac{\lambda_o}{k}\right)}|\chi|^{-\frac{1}{2}\left(1-\frac{\lambda_o}{k}\right)}\gamma^{1-\frac{1}{2}\left(1-\frac{\lambda_o}{k}\right)}K_{\frac{1}{2}\left(1-\frac{\lambda_o}{k}\right)}(\gamma|\chi|)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\left(1-\frac{\lambda_o}{k}\right)\right)}, \tag{37}$$

where $\Gamma(\cdot)$ is the gamma function, and $K_n(\cdot)$ is the modified Bessel function of the second kind [53]. When $\lambda_o = 2k$, this jump process steady state PDF (37) is identical to the PDF of the forcing inputs (36) [54].

The resulting process provides for a continuum of stochastic behavior between a process with infrequent but large jump transitions (Fig. 2a) and a process with infinitely frequent but small transitions approaching a diffusion process (Fig. 2b). Since in this case there is no state dependence, the corresponding diffusion process is represented by either the Itô or Stratonovich version of the Fokker-Planck equation, for which the steady state solution is given by Eqs. (33), (34), and (35). The corresponding diffusion coefficient is calculated from the jump process parameters, i.e.,

$$D_o = \frac{\lambda_o}{\gamma^2}, \tag{38}$$

where λ_o is the jump frequency (independent of the state variable), and γ^{-1} is the average jump amplitude. For jump parameters related by a constant D_o in Eq. (38), the jump process PDF (37) rapidly converges to a Gaussian shape as the jump frequency increases (Fig. 2c). Accordingly, as shown by Fig. 2d, there is rapid decrease in the Kullback-Leibler divergence, i.e., the relative entropy, $D_{\mathrm{KL}}(P\|Q)$, between the jump process distribution, P, with the PDF of Eq. (37) and the diffusion process distribution, Q, with the PDF of Eq. (33). The relationship of Fig. 2d is the same for any assumed value of D_o in Eq. (38). At jump frequencies as small as $\lambda_o = 10$, one observes little difference between the steady state statistics of the jump and diffusion processes (Fig. 2d).

In the case of state dependence, the steady state solution of the diffusion process is based on functions for the jump frequency and amplitude, i.e., $\lambda(\chi, t)$ and $b(\chi)$. Thus, we can derive a diffusion process PDF that approximates the statistics of any high frequency jump process. Moreover, if the jumps are occurring extremely often, the state dependence of the frequency, $\lambda(\chi, t)$, approximately has the same effect as the state dependence of the jump amplitude. Under such conditions, we reasonably may assume a constant frequency, λ_o, and subsequently merge the state dependent component of the frequency into a new amplitude function, i.e.,

$$\hat{b}(\chi) = \sqrt{2D_o \frac{\lambda(\chi, t)}{\lambda_o}} b(\chi), \tag{39}$$

which is based on the Itô and Stratonovich versions of the Fokker-Plank equation and the corresponding Kramers-Moyal expansion of the jump process; see Appendix A. This approximation provides simplicity with little loss of fidelity in the simulation of high frequency jump processes with state dependence.

III. SOLUTIONS FOR THE STRATONOVICH INTERPRETATION

While analytical solutions to the Fokker-Planck equation are well known [e.g., 15, 55], little attention has been focused on analytical solutions to the more general jump process description of the master equation (4). Here, for the Stratonovich prescription of the jump process, we develop a general class of solutions for both transient and steady state conditions, for which the steady state solution is presented in terms of both a potential function and the Pope-Ching formula [56].

The solution to Eq. (11) starts with a change of variables based on the Stratonovich jump prescription, i.e.,

$$y = \eta(\chi) = \int \frac{1}{b(\chi)} d\chi \tag{40}$$

$$\chi = \eta^{-1}(y). \tag{41}$$

For this change of variables, the PDF $p_y(y,t)$ is given by

$$p_\chi(\chi,t) = p_y(y,t) \left| \frac{dy}{d\chi} \right|. \tag{42}$$

We then transform the master equation (11) by substituting for χ and $p_\chi(\chi,t)$ with Eqs. (41) and (42), multiplying both sides by $\frac{d\chi}{dy}$, and noting $\frac{d\chi}{dy} = b(\chi)$, i.e.,

$$\frac{\partial}{\partial t} p_y(y,t) = -\frac{\partial}{\partial y}\left[\frac{m\left(\eta^{-1}(y)\right)}{b(\eta^{-1}(y))} p_y(y,t)\right] - \lambda\left(\eta^{-1}(y),t\right) p_y(y,t) + \int_{-\infty}^{y} \lambda(\eta^{-1}(u),t) p_z\left(y-u\right) p_y(u,t) du, \tag{43}$$

where on the r.h.s. the first term is the derivative of the current $J_m(\chi,t)$ of Eq. (5), the second term is the derivative of the current $J_{\chi u}(\chi,t)$ of Eq. (7) and the last term represents the derivative of the current $J_{u\chi}(\chi,t)$ of Eq. (8) based on the Stratonovich transition PDF $W_S(\chi|u,t)$ of Eq. (26).

This master equation (43) is not solved readily, but we find a few general results for an assumed exponential distribution of the forcing inputs, i.e.,

$$p_z(z) = \Theta(z)\gamma e^{-\gamma z}, \tag{44}$$

where γ^{-1} is the average input, and $\Theta(z)$ is the Heaviside step function. Exponential inputs have been central to studying physical and environmental processes, in particular for the simpler case of $b(\chi, z) = z$ [e.g., 26]; however only specific solutions have been derived for state dependent jumps [e.g., 41, 52].

A. General Steady State Solution and Potential Function

For the exponential distribution of forcing inputs (44), the solution to the master equation (43) under steady state conditions is given by

$$p_y(y) = N\frac{|b(\eta^{-1}(y))|}{|m\,(\eta^{-1}(y))\,|} e^{-\gamma\, y - \int \frac{\lambda(\eta^{-1}(u))b(\eta^{-1}(u))}{m\,(\eta^{-1}(u))} du}, \tag{45}$$

where N is an integration constant such that $\int_{-\infty}^{\infty} p_y(y)dy = 1$. This solution is found from an ordinary differential equation (ODE) that is retrieved by multiplying Eq. (43) by an integrating function $e^{\gamma y}$ and differentiating [e.g., 26, 57, 58]. After applying a change of variables, we may pose the solution of Eq. (45) in terms of χ, i.e.,

$$p_\chi(\chi) = \frac{N}{|m(\chi)|} e^{-\gamma \int \frac{d\chi}{b(\chi)} - \int \frac{\lambda(\chi)}{m(\chi)} d\chi}, \tag{46}$$

where $\lambda(\chi)$ is the state dependent arrival frequency of inputs, and N is the normalization constant such that $\int_{-\infty}^{\infty} p_\chi(\chi)d\chi = 1$. This solution unifies and extends previous results of [41] and [52], both of which were limited to specific forms of functions for $m(\chi)$ and $b(\chi)$.

Rather surprisingly, in cases where $b(\chi)$ is a rectangular hyperbola, the solution of Eq. (46) also represents processes forced by a two-sided exponential distribution of z. For such cases, the jump transition of Eq. (29) then is modeled as

$$\Delta\chi = \eta^{-1}\left(\eta(u) + |z|, \mathrm{sgn}[zb(\chi)]\right) - u, \tag{47}$$

which differs from the typical approach because of the functional dependence on $\mathrm{sgn}[zb(\chi)]$. The transition now is forced by the absolute value $|z|$ because the direction of the transition is governed by the inverse function, $\eta^{-1}(\cdot,\cdot)$, that now depends on a sign function, i.e., $\mathrm{sgn}[zb(\chi)]$. This sign function determines the direction of the transition and generally represents the two real roots of $\eta(\cdot,\cdot)$ in cases where $b(\chi)$ is a rectangular hyperbola, which will be used later in describing double well potentials.

The steady state solution (46) also may be written in terms of a potential function, i.e.,

$$p_\chi(\chi) = Ne^{-\Phi(\chi)}, \tag{48}$$

where N is a normalizing constant, and the effective potential is given by

$$\Phi(\chi) = \int \left(\frac{\gamma}{b(\chi)} + \frac{\lambda(\chi)}{m(\chi)} + \frac{\partial_\chi m(\chi)}{m(\chi)}\right) d\chi. \tag{49}$$

where $\int \frac{\partial_\chi m(\chi)}{m(\chi)} d\chi = \ln[|m(\chi)|]$.

Furthermore, note that the ensemble average of the velocity squared and the acceleration conditional on χ respectively are given by

$$\langle\dot{\chi}^2|\chi\rangle = m(\chi)^2 \tag{50}$$

$$\langle\ddot{\chi}|\chi\rangle = m(\chi)\left(-\gamma\frac{m(\chi)}{b(\chi)} - \lambda(\chi) + \partial_\chi m(\chi)\right). \tag{51}$$

Following Eqs. (50) and (51), we may pose Eq. (46) in terms of the Pope and Ching formula [56], i.e.,

$$p_\chi(\chi) = \frac{N}{\langle\dot{\chi}^2|\chi\rangle} e^{\int \frac{\langle\ddot{\chi}|\chi\rangle}{\langle\dot{\chi}^2|\chi\rangle} d\chi}, \tag{52}$$

which shows that this general solution of Eq. (46) also satisfies the differential equation $-\frac{d}{d\chi}\left(\langle\ddot{\chi}|\chi\rangle p\right) + \frac{d^2}{d\chi^2}\left(\langle\dot{\chi}^2|\chi\rangle p\right) = 0$ [55, 59].

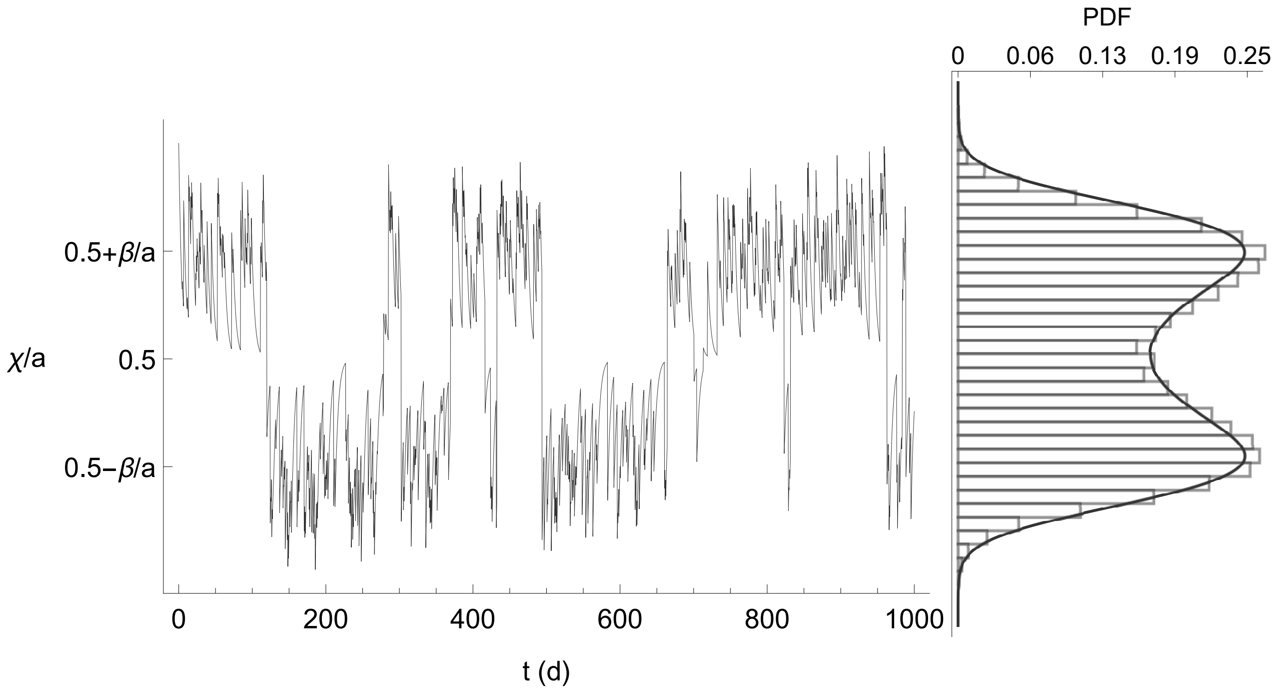

FIG. 3. For the symmetric double well potential of Eq. (59), a simulated trajectory (line) and a comparison of the simulated distribution (histogram bars) to the PDF of Eq. (48) (black line). The parameter values for the constitutive functions of $m(\chi)$, $\lambda_2(\chi)$, $b_2(\chi)$ and $\tilde{p}_z(z,\chi)$ are $a = 10$, $\beta = 1.5$, $k = 0.25$, $\lambda_o = 0.25$, and $\gamma = 0.04$, $f(\chi - a/2) = \alpha$, and $\alpha = 49/50$.

IV. DOUBLE WELL POTENTIALS

The general potential solution (48) now can be applied to the interesting case of a jump process within a double well potential (Fig. 3). Such a process may be of interest in a variety of fields, from preferential states and bistability in natural sciences [60, 61] to quantum mechanics, where the double well potential conveys the idea of a superposition of classical states [62]. The double well potential also may represent bistable physical and chemical systems such as second order phase transitions [63], nuclear fission and fusion [64, 65], chemical reaction rates [66, 67], and isomerization processes [68]. While in the literature the noise within a double well potential is typically represented by Brownian motion [69], here we extend the double well potential processes to include the case where both the jump amplitude and frequency are state dependent. This may be especially useful in describing anomalous jumps between two states [70], as well as in describing natural processes such as abrupt changes between two climatic states [71].

We consider a family of double well potential functions based on a linear drift function, i.e.,

$$m(\chi) = k\left(\frac{a}{2} - \chi\right), \tag{53}$$

where k [1/T] is the rate constant that controls the intensity of the drift, which is symmetric about the position $a/2$ [L] (Fig. 4b). The frequency of jump events may be given by either a first or second order expression, i.e.,

$$\lambda_1(\chi) = \lambda_o \frac{2\gamma a}{\beta}\left|\chi - \frac{a}{2}\right| + \lambda_o \tag{54}$$

$$\lambda_2(\chi) = \lambda_o \frac{4\gamma a}{\beta^2}\left(\chi - \frac{a}{2}\right)^2 + \lambda_o, \tag{55}$$

where λ_o [1/T] is a minimum frequency, γ [1/L] is the inverse of the average jump amplitude, and β [L] controls the positioning of the local minima of the double potential wells (Fig. 4a). Because these expressions are symmetric about $a/2$, both result in a symmetric double well potential. The corresponding expressions for the state dependence of the jump respectively are based on 1st and 3rd order polynomials of χ, i.e.,

$$b_1(\chi) = \frac{\beta^2 k}{2\lambda_o a\left(\chi - \frac{a}{2}\right)} \tag{56}$$

$$b_2(\chi) = \frac{\beta^4 k}{4\lambda_o a\left(\chi - \frac{a}{2}\right)^3}, \tag{57}$$

where both are negative valued functions for $x < a/2$, positive valued functions for $x > a/2$, with a discontinuity at $x = a/2$ (Fig. 4b).

Specific examples of double well potentials are retrieved from Eq. (49) by substituting for $m(\chi)$ with Eq. (53) and substituting for $\lambda(\chi)$ and $b(\chi)$ with either Eqs. (54) and (56) or Eqs. (55) and (57), respectively, i.e.,

$$\phi_1(\chi) = \frac{\lambda_o \gamma a}{k\beta^2}\left(\left|x - \frac{a}{2}\right| - \beta\right)^2 + \frac{k - \lambda_o}{k}\ln[|a - 2\chi|] \tag{58}$$

$$\phi_2(\chi) = \frac{\lambda_o \gamma a}{k\beta^4}\left(\left(x - \frac{a}{2}\right)^2 - \beta^2\right)^2 + \frac{k - \lambda_o}{k}\ln[|a - 2\chi|], \tag{59}$$

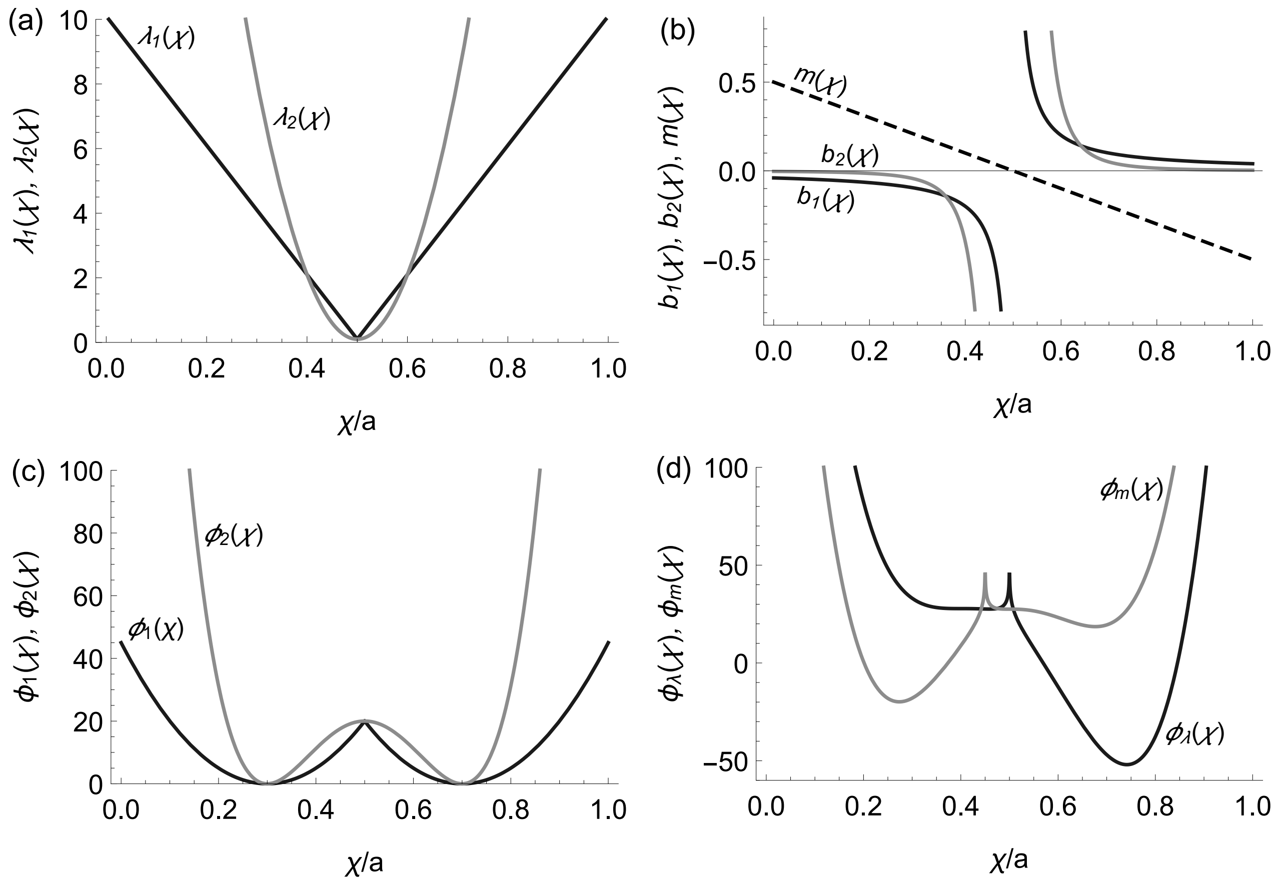

FIG. 4. Comparison of a) the frequency functions of Eqs. (54) and (55), b) the jump dependence of Eqs. (56) and (57) and the drift of Eq. (53), c) the symmetric double well potentials of Eqs. (58) and (59), and d) the asymmetric double well potentials of Eqs. (64) and (67). Parameter values are $k = 0.1$, $a = 10$, $\lambda_o = 0.1$, $\beta = 2$, $\gamma = 2$, $\epsilon = 0.5$.

where for Eqs. (58) and Eq. (59), we have assumed integration constants of $c = (4\beta^2 + a^2)\frac{\lambda_o \gamma a}{4\beta^2 k}$ and $c = (2\beta^2 - a^2)\frac{\lambda_o \gamma a}{2\beta^2 k}$, respectively (Fig. 4c). These respective constants allow one to complete the square of the first term of the r.h.s. of Eqs. (58) and (59).

In quantum mechanics, these two potential functions have been used as simple models for systems (such as the ammonia molecule) that may reside in a superposition of nearly degenerate states [62]. For both potential functions, the corresponding PDF is given by Eq. (48), and the PDF shows two local maxima where the potential shows two local minima, which are at $\chi = \frac{a}{2} \pm \frac{\beta}{2}\left(1 + \sqrt{\frac{\lambda_o\gamma a + 2(\lambda_o - k)}{\lambda_o \gamma a}}\right)$ and $\chi = \frac{a}{2} \pm \frac{\beta}{2}\sqrt{2 + 2\sqrt{\frac{\lambda_o\gamma a + \lambda_o - k}{\lambda_o\gamma a}}}$ for Eqs. (58) and (59), respectively (Fig. 4c). When $k < \lambda_o$ the potentials wells are separated by a barrier of infinite strength (Fig. 4c). If $k = \lambda_o$ this barrier has a finite value of $\phi_{\max} = \frac{\lambda_o}{k}\gamma a$, and the local minima are located at $a/2 \pm \beta$. Conversely, when $k > \lambda_o$, small differences between k and λ_o result in the double well potential becoming a triple well potential with an additional potential well centered at $a/2$. As k continues to increase to values much greater than λ_o, the strength of the last term of the r.h.s. of Eqs. (58) and (59) may cause the system to converge to a single well potential.

For the positive jump amplitudes represented by the PDF of Eq. (44) and $b(\chi)$ of either Eqs. (56) or (57), the trajectories are repulsed from $a/2$ because of the jumps (Fig. 3). These trajectories then are attracted back to $a/2$ because of the drift (Figs. 3 and 4b). This drift is zero at $a/2$, and consequently, the drift never pushes a trajectory over the barrier to the neighboring potential well. Nevertheless, both potential functions (and PDFs) describe trajectories over the two potential wells (Figs. 3 and 4c). Hence, the trajectories must jump between neighboring potential wells, and accordingly, the jump amplitudes must be both positive and negative (Fig. 3). Thus, because both $b_1(\chi)$ and $b_2(\chi)$ represent rectangular hyperbolas, the distribution of forcing inputs is a state-dependent, two-sided exponential distribution, i.e.,

$$\tilde{p}_z(z,\chi) = \begin{cases} f(\chi - a/2)\gamma e^{-\gamma z} & z \geq 0 \\ (1 - f(\chi - a/2))\gamma e^{\gamma z} & z < 0, \end{cases} \tag{60}$$

where the fractional weight $f(\chi - a/2)$ controls the relative probability density for a positive and negative jump. This function $f(\chi - a/2)$ must be symmetric about $a/2$ to maintain the symmetry indicated by the potential functions of Eqs. (58) and (59).

For this two-sided exponential distribution, the jump transition is described by Eq. (47). Accordingly, the jump transition is simulated based on the absolute value of the forcing input, $|z|$, because the direction of the transition is determined by the respective inverse functions, i.e.,

$$\eta_1^{-1}(y) = \frac{a}{2} + \mathrm{sgn}[b_1(\chi)z]\frac{1}{2}\sqrt{\frac{4\beta^2 ky}{\lambda_o} + a^2} \tag{61}$$

$$\eta_2^{-1}(y) = \frac{a}{2} + \mathrm{sgn}[b_2(\chi)z]\beta\left(\frac{ky}{\lambda_o}\right)^{1/4}, \tag{62}$$

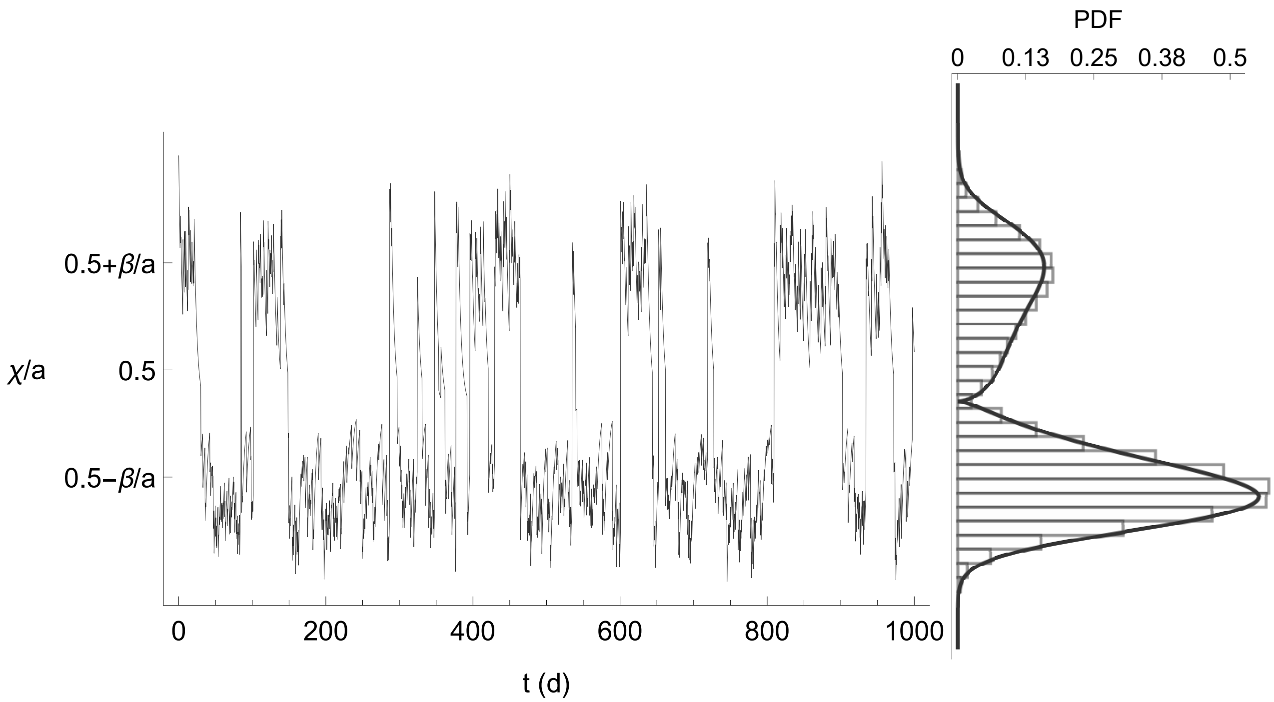

FIG. 5. For the asymmetric double well potential of Eq. (67), a simulated trajectory (line) and a comparison of the simulated distribution (histogram bars) to the PDF of Eq. (48) (black line). The parameter values for the constitutive functions of $m(\chi)$, $\lambda_2(\chi)$, and $\tilde{p}_z(z,\chi)$ are $a = 10$, $\beta = 1.5$, $k = 0.25$, $\lambda_o = 0.35$, and $\gamma = 0.04$, $f(\chi - a/2) = \alpha$, $\alpha = 49/50$, and $\epsilon = 0.5$.

where following Eq. (40) $\eta_1^{-1}(y)$ and $\eta_1^{-1}(y)$ are derived from $b_1(\chi)$ and $b_2(\chi)$. As indicated by Eqs. (61) and (62) if either $z\, b_1(\chi)$ or $z\, b_2(\chi)$ is negative (positive), then the jump creates a decrease (increase) in the state variable χ. This underlying process is more generic (and complex) than one may initially perceive from a cursory inspection of $b_1(\chi)$ and $b_2(\chi)$ of Eqs. (56) and (57) and the jump distribution $p_z(z)$ of Eq. (44), and these potential functions represent a steady state solution with $f(\chi - a/2)$ mediating the random transition (i.e., anomalous jumping) between the two states (i.e., potential wells).

The double well potential becomes asymmetric for a small perturbation, ϵ, in the location of either the frequency of the jump $\lambda(\chi)$ or the drift, $m(\chi)$. We examine such an asymmetry for the second double well potential $\phi_2(\chi)$. For a small perturbation, ϵ, in the frequency location, i.e.,

$$\lambda_\epsilon(\chi) = \lambda_o \frac{4\gamma a}{\beta^2}\left(\chi - \frac{a}{2} + \epsilon\right)^2 + \lambda_o, \tag{63}$$

the frequency function is centered around $\frac{a}{2}+\epsilon$. With Eq. (63), we then retrieve the potential function from Eq. (49) with substitutions for $m(\chi)$ of Eq. (53) and $b(\chi)$ of Eq. (57), i.e.,

$$\phi_\lambda(\chi,\epsilon) = \phi_2(\chi) - \epsilon\frac{8\lambda_o\gamma a}{\beta^2 k}\left(\chi - \frac{a}{2}\right) - \epsilon\frac{4\lambda_o\gamma a}{\beta^2 k}\ln[|a-2\chi|], \tag{64}$$

where the potential asymmetry is controlled by either a positive or negative value of ϵ (Fig. 4d). For $k < \lambda_o + \frac{4\epsilon^2\lambda_o\gamma a}{\beta^2}$ the potentials wells are separated by a barrier of infinite strength. When $k = \lambda_o + \frac{4\epsilon^2\lambda_o\gamma a}{\beta^2}$ this barrier has a finite value of $\phi_{\max} = \frac{\lambda_o}{k}\gamma a$. Similar to the symmetric version, the potential well of Eq. (64) also is centered at $a/2$. This asymmetric potential, $\phi_\lambda(\chi)$, not only corresponds to the perturbed frequency of Eq. (63), but also to a different version of the state-dependent, two-sided exponential distribution of forcing inputs, i.e.,

$$\tilde{p}_z(z,\chi) = \begin{cases} f(\chi - a/2)\gamma e^{-\gamma z} & z \geq 0 \\ (1 - P_\chi(a/2))(1 - f(\chi - a/2))\gamma e^{\gamma z} & z < 0 \ \& \ x \leq a/2 \\ P_\chi(a/2)(1 - f(\chi - a/2))\gamma e^{\gamma z} & z < 0 \ \& \ x > a/2, \end{cases} \tag{65}$$

where the frequency of these transitions now is weighted by the probability or each potential well, as described by the CDF $P_\chi(a/2)$ where $P_\chi(\chi) = \int_{-\infty}^{\chi} p_\chi(\chi)d\chi$. These CDF weights provide consistency between the jump probability and the asymmetry of the probability density about $a/2$ (e.g., Fig. 4d).

For a small perturbation in the location of the drift, i.e.,

$$m_\epsilon(\chi) = k\left(\frac{a}{2} - \chi + \epsilon\right), \tag{66}$$

the double well potential again becomes asymmetric (Figs. 4d and 5). The corresponding potential function is found from Eq. (49) with substitutions for $m_\epsilon(\chi)$ of Eq. (66), $\lambda_2(\chi)$ of Eq. (55), $b_2(\chi)$ of Eq. (57), i.e.,

$$\phi_m(\chi,\epsilon) = \phi_2(\chi) + \epsilon\frac{2\lambda_o\gamma a}{\beta^2 k}\left(2\left(\chi - \frac{a}{2}\right) + 3\epsilon\right) - \epsilon\frac{4\lambda_o\gamma a}{\beta^2 k}\ln[|a - 2(\chi+\epsilon)|] + \frac{k-\lambda_o}{k}\ln\left[\left|\frac{a-2(\chi+\epsilon)}{a-2\chi}\right|\right], \tag{67}$$

where as indicated by the term $\ln[\cdot]$, the double well potential is no longer centered at $a/2$ and potential barrier is only of a finite value when $k = \lambda_o + \frac{4\epsilon^2\lambda_o\gamma a}{\beta^2}$ (Fig. 5). This asymmetric potential, $\phi_m(\chi)$, not only corresponds to the perturbed drift of Eq. (66), but also to a different state-dependent, two-sided exponential distribution of forcing inputs, i.e.,

$$\tilde{p}_z(z,\chi) = \begin{cases} f(\chi - a/2)\gamma e^{-\gamma z} & z \geq 0 \\ \left(\frac{\langle\lambda_m\rangle}{\langle\lambda\rangle}\Theta[\epsilon] + (1 - P_\chi(a/2))(1 - f(\chi - a/2))\right)\gamma e^{\gamma z} & z < 0 \ \& \ x \leq a/2 \\ \left(\frac{\langle\lambda_m\rangle}{\langle\lambda\rangle}\Theta[-\epsilon] + P_\chi(a/2)(1 - f(\chi - a/2))\right)\gamma e^{\gamma z} & z < 0 \ \& \ x > a/2, \end{cases} \tag{68}$$

where the Heaviside step function $\Theta(\cdot)$ is right continuous, i.e., $\Theta(0) = 1$, $\langle\lambda_m\rangle$ is the frequency at which a trajectory crosses the location $a/2$ where the jump direction changes, and $\langle\lambda\rangle$ is the average frequency of jumping from the larger potential well. These average frequencies are given respectively by

$$\langle\lambda_m\rangle = |m_\epsilon(a/2)|p_\chi(a/2) \tag{69}$$

$$\langle\lambda\rangle = \int_{-\infty}^{a/2} \lambda_2(\chi) N e^{-\phi_m(\chi,|\epsilon|)} d\chi, \tag{70}$$

where N is the normalization constant of Eq. (48). The first expression describes the average rate at which the drift causes a trajectory to cross $a/2$, while the second expression is the average rate of jumping from the larger potential and crossing back over $a/2$ (Fig. 5). The expression $\langle\lambda\rangle$ is for the larger potential well as indicated by the absolute value $|\epsilon|$ within the potential function.

Assuming the trajectories (e.g., Figs. 3 and 5) represent particle movement, we may use the formula of Pope and Ching of Eq. (52) to examine the particle dynamics in terms of the ensemble average velocity squared and acceleration of Eqs. (50) and (51). The ensemble average of the velocity squared may describe the average kinetic energy of the particle, i.e., $E_k = \frac{1}{2}m_p\langle\dot{\chi}^2|\chi\rangle$ for which m_p is the mass. Accordingly, the kinetic energy increases with the distance from $a/2$. The ensemble average acceleration then describes the power applied to the particle, i.e., $P_w = m_p\langle\ddot{\chi}|\chi\rangle m(\chi)$, where m_p again represents the particle mass. This repels the particle away from $a/2$, and reaches a local maximum right before the minima of each double well potential, as shown by the ensemble average acceleration (Fig. 6a). The symmetry of this acceleration mostly is controlled by the symmetry of the frequency function. A small perturbation in the frequency produces large changes in the symmetry of the acceleration (Fig. 6b, black line). Conversely, a small perturbation in the drift, while

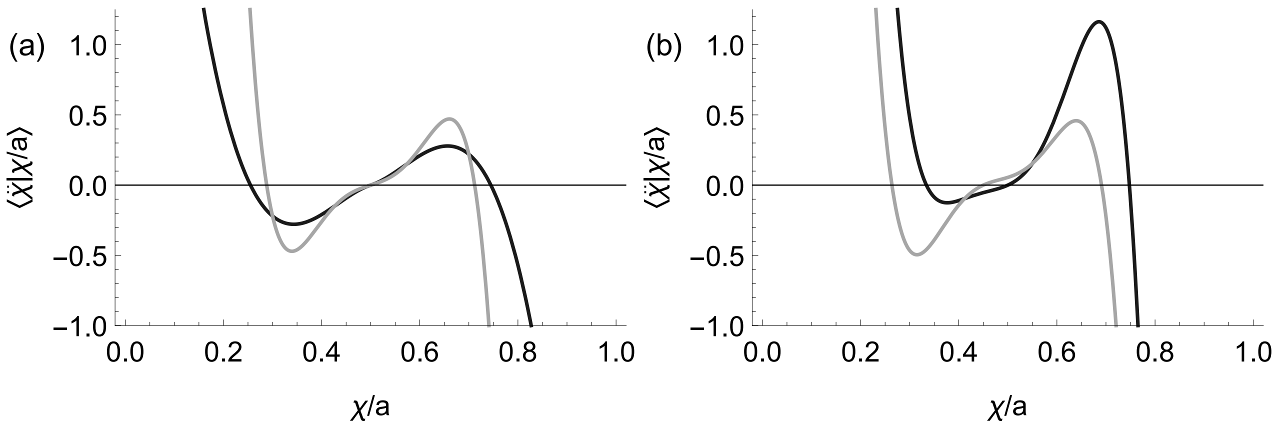

FIG. 6. The ensemble average acceleration of Eq. (51) for a) the symmetric double potential $\Phi_1(\chi)$ of Eq. (58) (black line) and $\Phi_2(\chi)$ of Eq. (59) (gray line) and b) for the asymmetric double potentials $\Phi_\lambda(\chi)$ of Eq. (64) (black line) and $\Phi_m(\chi)$ of Eq. (67) (gray line). Parameter values are $k = 0.1$, $a = 10$, $\lambda_o = 0.1$, $\beta = 2$, $\gamma = 2$, $\epsilon = 0.5$.

altering the symmetry of the potential function (Fig. 5), does not significantly change the acceleration (Fig. 6b, gray line).

V. A CLASS OF TRANSIENT SOLUTIONS FOR THE STRATONOVICH INTERPRETATION

In the case of the Stratonovich jump interpretation, it also is possible to solve Eq. (4) for a class of transient solutions. The solutions are derived by starting with the transformed master equation (43) and assuming a (y dependent) linear drift, i.e.,

$$m_y(y) = \frac{m\left(\eta^{-1}(y)\right)}{b(\eta^{-1}(y))} = \kappa y, \tag{71}$$

where κ [1/T] is a generic constant that adjusts the drift. Note that the drift, $m_y(y)$, accommodates a variety of χ dependent drift functions, $m(\chi)$, and jump functions, $b(\chi)$, that satisfy the following relationship, i.e.,

$$m(\chi) = \kappa b(\chi) \int \frac{1}{b(\chi)} d\chi, \tag{72}$$

where examples of the constant κ are given in Table I. In addition to Eq. (71), we assume a homogeneous Poisson process, i.e., $\lambda\left(\eta^{-1}(y), t\right) = \lambda$, an exponential PDF of forcing inputs

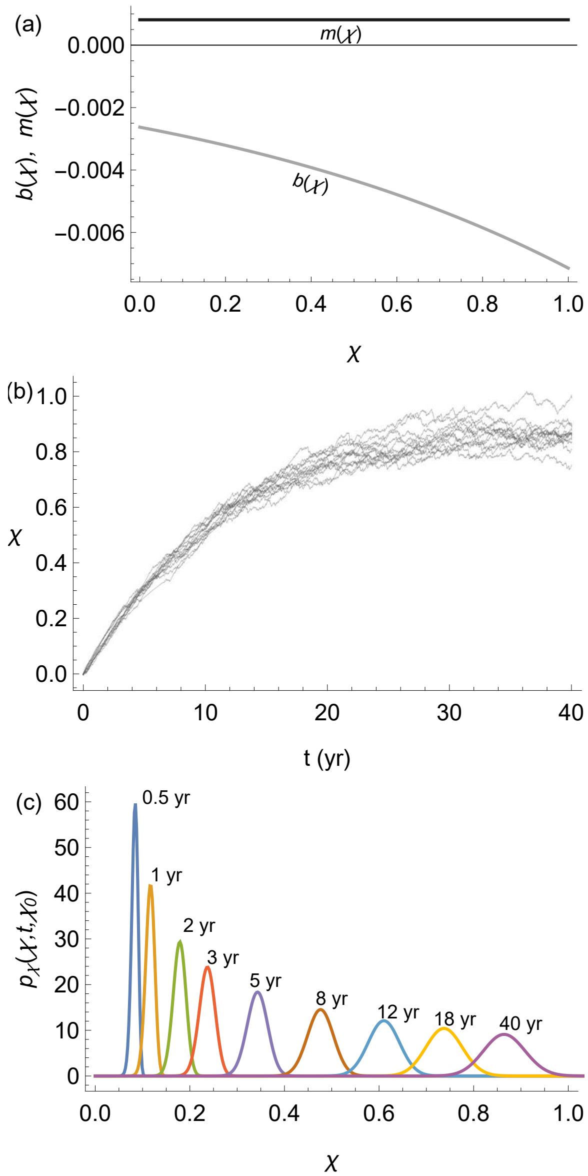

FIG. 7. For the variable $\chi = X/w$, a) the constant drift, $m(\chi) = k$, and state dependent function, $b(\chi) = \beta e^{-n\chi}$, b) realizations of the transient dynamics, and c) the continuous part of the transient PDF $p_\chi(\chi, t, \chi_0)$ of Eq. (74). Though not shown, the PDF includes an atom of probability of strength $e^{-\lambda t}$ located at $\chi = \frac{1}{n} \ln \left[e^{\kappa t + n\chi_0} \right]$. Parameter values are $\lambda = 0.17$ d^{-1}, $w = 90$ g, $\alpha = 0.2$ cm, $\gamma = 1/\alpha$ cm^{-1}, $\beta = \beta_s/w$ cm^{-1}, $\beta_s = -1/e^n$ g cm^{-1}, $n = 1$ [-], $k = k_s/w$ d^{-1}, $k_s = 0.03$ g d^{-1} and $\chi_0 = 0.05$ [-]. (Color version available online).

given by Eq. (44), and an initial condition of $p_y(y,0,y_0)=\delta(y-y_0)$. We find a transient solution by converting the master equation (43) with a laplace transform, solving the resulting equation with the method of characteristics, and subsequently inverting the Laplace transform solution [72, 73], i.e.,

$$\begin{aligned} p_y(y,t,y_0) = {} & e^{-\lambda t}\delta\left(y-y_0e^{\kappa t}\right)-\frac{\lambda\gamma}{\kappa}e^{-\lambda t-\gamma\left(y-y_o e^{\kappa t}\right)} \\ & \cdot\left(e^{-\kappa t}-1\right)\, {}_1F_1\left(1+\frac{\lambda}{\kappa};2;\gamma\left(y-y_0e^{\kappa t}\right)\left(1-e^{-\kappa t}\right)\right)\Theta\left(y-y_0e^{\kappa t}\right), \end{aligned} \tag{73}$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function of the 1st kind, and $\Theta(\cdot)$ is the Heaviside step function. The solution in terms of the original state variable is $p_\chi(\chi,t,\chi_0)=p_y(\eta(\chi),t,\eta(\chi_0))\left|\frac{dy}{d\chi}\right|_{y=\eta(\chi)}$, i.e.,

$$\begin{aligned} p_\chi(\chi,t,\chi_0) = {} & \frac{e^{-\lambda t}}{|b(\chi)|}\delta\left(\eta(\chi)-\eta(\chi_0)e^{\kappa t}\right)-\frac{\lambda\gamma}{|b(\chi)|\,\kappa}e^{-\lambda t-\gamma\left(\eta(\chi)-\eta(\chi_o)e^{\kappa t}\right)} \\ & \cdot\left(e^{-\kappa t}-1\right)\, {}_1F_1\left(1+\frac{\lambda}{\kappa};2;\gamma\left(\eta(\chi)-\eta(\chi_0)e^{\kappa t}\right)\left(1-e^{-\kappa t}\right)\right)\Theta\left(\eta(\chi)-\eta(\chi_0)e^{\kappa t}\right), \end{aligned} \tag{74}$$

where the expression is a mixed distribution consisting a continuous part and an atom of probability, which moves along a trajectory as described by the argument of the delta function, i.e., $\delta(\eta(\chi)-\eta(\chi_o)e^{-\kappa t})$. Following the property of Appendix A of [32], this delta function may be posed as $\frac{\delta(\chi-\chi_n)}{g'(\chi_n)}$ where $g(\chi)=\eta(\chi)-\eta(\chi_o)e^{-\kappa t}$ and χ_n is the root for $g(\chi_n)=0$. Examples of various transient solution functions are given in Table I.

We also consider the limiting case where the y dependent drift simply is constant, i.e.,

TABLE I. Examples of Transient Solution Functions[a]

	$b(\chi)$	$m(\chi)$[b]	$y=\eta(\chi)$	$\eta^{-1}(y)$	κ
Ex. 1	$\beta\chi^n$	$k\chi$	$\frac{\chi^{1-n}}{\beta(1-n)}$	$((1-n)y\beta)^{\frac{1}{1-n}}$	$k(1-n)$
Ex. 2	$\varrho+\beta\chi$	$k(\varrho+\beta\chi)\ln[\lvert\varrho+\beta\chi\rvert]$	$\frac{\ln[\lvert\varrho+\chi\beta\rvert]}{\beta}$	$\frac{e^{y\beta}-\varrho}{\beta}$	$k\beta$
Ex. 3	$\beta e^{n\chi}$	k	$-\frac{e^{-n\chi}}{n\beta}$	$\frac{1}{n}\ln\left[\left\lvert-\frac{1}{n\beta y}\right\rvert\right]$	$-kn$
Ex. 4	$\varrho+\beta e^{n\chi}$	$k(\varrho+\beta e^{n\chi})\left(n\chi-\ln\left[\lvert\varrho+\beta e^{n\chi}\rvert\right]\right)$	$\frac{n\chi-\ln[\lvert\varrho+\beta e^{n\chi}\rvert]}{n\varrho}$	$-\frac{1}{n}\ln\left[\left\lvert\frac{e^{-n\varrho y}-\beta}{\varrho}\right\rvert\right]$	$k\varrho n$

[a] Note that β, ϱ, and n are generic parameters of $b(\chi)$, and k is a generic parameter of the drift, $m(\chi)$.

[b] Note that the drift function is derived from Eq. (72).

$$m_y(y) = \frac{m\left(\eta^{-1}(y)\right)}{b(\eta^{-1}(y))} = \kappa, \tag{75}$$

in which case $m(\chi)/b(\chi) = \kappa$ [L/T], and thus $m(\chi)$ and $b(\chi)$ share the same functional dependency on χ. Similar to the previous case, we also assume an initial condition of $p_y(y, 0, y_0) = \delta(y - y_0)$, an exponential PDF of forcing inputs, and a homogeneous Poisson process, i.e., $\lambda\left(\eta^{-1}(y), t\right) = \lambda$. We find the corresponding solution by posing the mater equation (43) in terms of Laplace transforms, solving the resulting equation, and then transforming the solution with an inverse Laplace transform [74], i.e.,

$$\begin{aligned} p_y(y, t, y_0) &= e^{-\lambda t}\delta\left(y_0 - y - \kappa t\right) \\ &+ \sqrt{\frac{\gamma\lambda t}{y_0 - y - \kappa t}} \cdot I_1\left[2\sqrt{\gamma\lambda(y_0 - y - \kappa t)t}\right] e^{-y_0 + y + \gamma\kappa t - \lambda t}\Theta(y_0 - y - \kappa t), \end{aligned} \tag{76}$$

where $I_1[\cdot]$ is the modified Bessel function of the first kind [53]. With a change of variables, i.e., $p_\chi(\chi, t, \chi_0) = p_y(\eta(\chi), t, \eta(\chi_0)) \left|\frac{dy}{d\chi}\right|_{y=\eta(\chi)}$, we retrieve the solution in terms of the original state variable, i.e.,

$$\begin{aligned} p_\chi(\chi, t, \chi_0) = \frac{e^{-\lambda t}}{|b(\chi)|}\delta\left(\eta(\chi_0) - \eta(\chi) - \kappa t\right) + \frac{1}{|b(\chi)|}\sqrt{\frac{\gamma\lambda t}{\eta(\chi_0) - \eta(\chi) - \kappa t}} \\ \cdot I_1\left[2\sqrt{\gamma\lambda(\eta(\chi_0) - \eta(\chi) - \kappa t)t}\right] e^{-\eta(\chi_0) + \eta(\chi) + \gamma\kappa t - \lambda t}\Theta(\eta(\chi_0) - \eta(\chi) - \kappa t), \end{aligned} \tag{77}$$

and this solution describes a mixed distribution that consists of a continuous part and an atom of probability (represented by the Dirac delta function).

A. Soil Salinity Dynamics

The transient solutions just presented find use in modeling the dynamics of soil salinity based on the assumptions of previous models that only examined the steady state condition [e.g., 41, 52]. We consider salt is deposited into the soil layer at a constant rate k_s and subsequently leaches in proportion to the rainfall amount per storm event (Fig. 7a). Over a range of salt content w for which the normalized salt content is $\chi = X/w$, the proportional loss of salt may be captured by the function $b(\chi) = \beta e^{n\chi}$ for which $\beta = \beta_s/w$. Hence, the normalized deposition of salt is $k = k_s/w$, and the representation follows the functions of Example 3 of Table I. The probabilistic dynamics of salt content, which may be appreciated from looking at the ensemble of trajectories (Fig. 7b), is described by the transient solution of Eq. (74), as shown by Fig. 7c.

Initially, over the first few years, the salt concentration is tightly centered near the value of $\chi = \frac{1}{n} \ln [e^{\kappa t + n\chi_0}]$, which is the initial salt concentration relocated by the governing dynamics. At around a decade, the salt concentration (per unit area) shows significantly more variability in the range of about ± 5.6 g (Fig. 7c for which $5.6 = 0.062 \cdot 90$ g). This variability will affect the time at which the soil requires remediation to remove salt. From a decade onward, the variability increases while the median value of the PDF increases. Such behavior continues until approximate steady state conditions occur at around year 40. Thus, the transient PDF provides a basis for assessing the risk, costs, and benefits of remediating the soil at different junctures in time between the initial time and steady state conditions (Fig. 7c).

VI. CONCLUSION

For systems forced by random jumps, i.e., shot noise, we have provided a general theory for defining the jump transition for both the Itô and Stratonovich interpretations of the jump process. For the Stratonovich jump interpretation and an exponential PDF of forcing inputs, we have presented a steady state solution for the state variable PDF that is general to functions for the deterministic drift, state dependent recurrence frequency of jumps, and state dependent jump amplitudes. This solution allows us to provide a novel description of a jump process within a double well potential, where particle dynamics are forced by an input with a two-sided exponential distribution that then allows for anomalous jumps between the two potential wells. We have shown that small perturbations in the deterministic drift and the frequency of jumps create asymmetry between the strength of the two potential wells. We also have derived a class of transient solutions that are general to functions for the deterministic drift and state dependent jump amplitudes. As demonstrated with soil salinity dynamics, the transient solution provides a faster approach to assessing long term behavior versus the typical approach involving more onerous numerical simulations. In general, the processes investigated here provide a framework for moving stochastic process descriptions beyond the typical paradigms that assume noise driven diffusion represented by

Brownian motion.

It will be interesting to analyze the possibility of moving beyond the typical Itô and Stratonovich jump interpretations. For example, the jump process could be defined by directly imposing two distributions that respectively describe the variability of the state variable before and after the jump. Such a description naturally may be suited to representing stochastic renewal and control processes. Work along these lines will be presented elsewhere. Furthermore, even in steady state, the jump process represents a system that never reaches equilibrium, i.e., there is an asymmetry in the timescale of drifting to a state and jumping from a state. Because of this asymmetry, the system does not balance (in detail) the frequency of entering and exiting a particular state. Such a lack of a detail balance and the associated non-equilibrium state are of particular interest in statistical mechanics. Future work thus will consider the typical Brownian forcing in conjunction with a jump process description that could reveal new paradigms for a non-equilibrium steady state in stochastic thermodynamics, which primarily assumes a Brownian motion [75].

Appendix A: Jump Process Convergence to a Diffusion Process

For the case of $b(\chi, z) = b(\chi)z$, we show how the jump process converges to a diffusion process that is described by a Fokker-Planck equation with a state dependent diffusion coefficient. To show this convergence we expand the master equation (11) components representing the jump forcing, i.e.,

$$\partial_\chi J_\xi(\chi, t) = -p_\chi(\chi, t) \int_{-\infty}^{\infty} W(u|\chi, t) du + \int_{-\infty}^{\chi} W_{(\cdot)}(\chi|u, t) p_\chi(u, t) du, \tag{A1}$$

where $W_{(\cdot)}(\chi|u)$ represents either the Itô transition PDF, $W_I(\chi|u)$, of Eq. (18) or the Stratonovich transition PDF, $W_S(\chi|u)$, of Eq. (26). We link both cases to a diffusion process with a Taylor series expansion of the second term of Eq. (A1).

1. Itô description

For the Itô jump prescription, we introduce the jump transition by substituting for the antecedent state (before) a jump event, i.e.,

$$u = \chi - \upsilon, \tag{A2}$$

where $\upsilon = \Delta\chi$ is the jump transition. Upon substituting Eq. (A2) into Eq. (A1) and accounting for a change of variables for a probability distribution [e.g., 51], the jump component is posed as an integration over υ, i.e.,

$$\partial_\chi J_\xi(\chi, t) = -p_\chi(\chi, t) \int_\infty^{-\infty} W(\chi - \upsilon|\chi, t) \left|\frac{du}{d\upsilon}\right| d\upsilon + \int_\infty^0 W_I(\upsilon|\chi - \upsilon, t) p_\chi(\chi - \upsilon, t) \left|\frac{du}{d\upsilon}\right| d\upsilon, \tag{A3}$$

where $\left|\frac{du}{d\upsilon}\right| = 1$ and $W_I(\chi|u, t)$ has become a PDF of υ conditional on $\chi - \upsilon$, i.e.,

$$W_I(\upsilon|\chi - \upsilon, t) = \lambda(\chi - \upsilon, t) \int_{-\infty}^{\infty} \delta(\upsilon - b(\chi - \upsilon) z) p_z(z) dz, \tag{A4}$$

which is specific to the Itô transition PDF, $W_I(\chi|u, t)$, of Eq. (18). The term $W(\chi - \upsilon|\chi, t)$ often is given with the notation $W(\chi, -\upsilon, t)$, i.e., conditional on being in the present state χ there is a prior state at a distance $-\upsilon$. The second term $W_I(\upsilon|\chi - \upsilon, t)$ often is written as $W_I(\chi - \upsilon, \upsilon, t)$, i.e., conditional on begin at the prior state $\chi - \upsilon$ there is a transition of size υ [40].

Recognizing the second term of Eq. (A3) is a function of u (see Eq. (A2)), we pose $W_I(\upsilon|\chi - \upsilon, t) p_\chi(\chi - \upsilon, t)$ as a Taylor-series expansion around a transition to χ, i.e.,

$$\partial_\chi J_\xi(\chi, t) = -p_\chi(\chi, t) \int_\infty^{-\infty} W(\chi - \upsilon|\chi, t) d\upsilon + \int_\infty^0 \sum_{n=0}^{\infty} \frac{(-1)^n \upsilon^n}{n!} \frac{\partial^n}{\partial \chi^n} W_I(\upsilon|\chi, t) p_\chi(\chi, t) d\upsilon, \tag{A5}$$

where the distance from u is simply the negative jump distance; accordingly, $(-1)^n \upsilon^n = (u - \chi)^n$. Integrating $W_I(\upsilon|\chi, t)$ over υ defines the jump moments given by

$$M_n(\chi) = \int_\infty^0 \upsilon^n W_I(\upsilon|\chi, t) d\upsilon = \lambda(\chi, t) b(\chi)^n \langle z^n \rangle, \tag{A6}$$

which follows from the sifting property of the delta function within $W_I(\upsilon|\chi,t) = \lambda(\chi,t)\int_0^\infty \delta(\upsilon - b(\chi)z)p_z(z)dz$. Note that $\langle z^n\rangle = \int_0^\infty z^n p_z(z)dz$, and $W_I(\upsilon|\chi,t)$ is Eq. (A4) with $\chi-\upsilon$ replaced by χ based on the Taylor-series expansion. In addition, the first term on the r.h.s. of Eq. (A5), i.e.,

$$-\lambda(\chi,t)p_\chi(\chi,t) = -p_\chi(\chi,t)\int_\infty^{-\infty} W(\chi-\upsilon|\chi,t)d\upsilon, \tag{A7}$$

cancels with the zero order term of the expansion of Eq. (A5), i.e.,

$$\lambda(\chi,t)p_\chi(\chi,t) = \frac{(-1)^0}{0!}\frac{\partial^0}{\partial\chi^0}\left[M_0(\chi)p_\chi(\chi,t)\right]. \tag{A8}$$

Based on Eqs. (A6), (A7) and (A8), we may compactly pose Eq. (A1) as

$$\partial_\chi J_\xi(\chi,t) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\chi^n}\left[M_n(\chi)p_\chi(\chi,t)\right], \tag{A9}$$

and this is the so-called Kramers-Moyal expansion that is the basis of past derivations of the Fokker-Planck equation [66, 76]. Upon substitution of the jumps moments, $M_n(\chi)$, the Kramers-Moyal expansion for the Itô prescription of a marked Poisson process is given by

$$\partial_\chi J_\xi(\chi,t) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\chi^n}\left[\langle z^n\rangle\lambda(\chi,t)b(\chi)^n p_\chi(\chi,t)\right]. \tag{A10}$$

This jump description converges to a diffusion process under the limiting scenario of the jump weights approaching zero, i.e., $z\to 0$, while the density of jump events increases, i.e., $\lambda(\chi,t)\to\infty$, such that

$$\lim_{(\lambda,z)\to(\infty,0)}\langle z^2\rangle\lambda(\chi,t) = D(\chi,t), \tag{A11}$$

where the state dependent diffusion coefficient is given as

$$D(\chi,t) = 2D_o\frac{\lambda(\chi,t)}{\lambda_o}. \tag{A12}$$

This diffusion results from noting that 1) the jump frequency consists of a component independent of the state variable, i.e., $\lambda_o = 1/t_o$, 2) the jump frequency is equivalent to $\lambda(\chi, t) = \frac{\lambda(\chi,t)}{\lambda_o t_o}$, and 3) in the limit of Eq. (A11), the mean squared displacement, $\langle z^2 \rangle$, converges to $2D_o t_o$, where D_o is a diffusion coefficient. For Eq. (A11), convergence to $D(\chi, t)$ implies that $n \geq 3$ terms are zero, i.e., $\langle z^n \rangle \lambda(\chi, t) \to 0$, because $\langle z^n \rangle$ tends to zero faster than $\lambda(\chi, t)$ tends to infinity, while if $\langle z \rangle \neq 0$, the $n = 1$ term is infinite, i.e., $\langle z \rangle \lambda(\chi, t) \to \infty$, because $\lambda(\chi, t)$ tends to infinity faster than $\langle z \rangle$ tends to zero.

Thus, unless the jump magnitude PDF $p_z(z)$ is symmetric about the origin ($z = 0$), convergence only occurs if the $n = 1$ term of Eq. (A10) is balanced by the drift, i.e.,

$$m(\chi, t) = m_o(\chi, t) - \langle z \rangle \lambda(\chi, t) b(\chi), \tag{A13}$$

where $m_o(\chi, t)$ is a generic function and $\langle z \rangle \lambda(\chi, t) b(\chi)$ compensates for the average rate of increase from the jump process. For the drift of Eq. (A13) and $\partial_\chi J_\xi(\chi, t)$ of Eq. (A10), the master equation (11) converges to a diffusion process under the limit of Eq. (A11), i.e.,

$$\partial_t p_\chi(\chi, t) = -\frac{\partial}{\partial \chi} \left[m_o(\chi, t) p_\chi(\chi, t) \right] + \frac{1}{2} \frac{\partial^2}{\partial \chi^2} \left[D(\chi, t) b(\chi)^2 p_\chi(\chi, t) \right], \tag{A14}$$

and this is the Itô version of the Fokker-Planck for which the first term on the r.h.s. represents the deterministic drift and the second term represents the diffusion process. Note that the Fokker-Planck drift $m_o(\chi, t)$ is different than the jump process drift of Eq. (A13) unless the PDF $p_z(z)$ is symmetric about $z = 0$. The state dependent diffusion coefficient $D(\chi, t)$ differs from previous derivations in which the Poisson rate of jumping and thus the diffusion coefficient are constants [e.g., 41, 48].

2. Stratonovich Description

Here we also show the jump process convergence to a diffusion for the Stratonovich prescription of the jumps. For the Stratonovich jump prescription of Eq. (26), we consider Eq. (A1) under the change of variables given by Eqs. (40) - (42). Following this change of variables, Eq. (A1) becomes

$$\partial_y J_{\xi_y}(y,t) = -p_y(y,t)\int_{\eta(-\infty)}^{\eta(\infty)} W(u|y,t)du + \int_{\eta(-\infty)}^{y} W_S(y|u,t)p_y(u,t)du, \tag{A15}$$

where the transformed transition probabilities are given by

$$W(u|y,t) = W(u|\eta^{-1}(y),t) \tag{A16}$$

$$W_S(y|u,t) = \lambda(\eta^{-1}(u),t)\int_{-\infty}^{\infty} \delta(y-u-z)p_z\,(z)\,dz, \tag{A17}$$

for which the last expression is derived from the Stratonovich transition PDF of Eq. (26), $W_S(\chi|u,t)$. Equation (A15) is derived from (A1) by substituting for $p_\chi(\chi,t)$ based on Eq. (42), substituting for χ with Eq. (41), substituting $\partial_\chi J_\xi(\chi,t) = \frac{dy}{d\chi}\partial_y J_{\xi_y}(y,t)$, and then multiplying both sides by $\frac{d\chi}{dy}$. This derivative is given by

$$\frac{d\chi}{dy} = \frac{d\eta^{-1}(y)}{dy} = b(\eta^{-1}(y)), \tag{A18}$$

which is based on the property for the derivative of an inverse function, i.e., $\frac{d}{dy}\eta^{-1}(y) = \frac{d}{dx}\eta\,(\chi)\,\big|_{\chi=\eta^{-1}(y)}$. Similar to Itô prescription, we then introduce the jump transition into Eq. (A15) by substituting for the antecedent value given by

$$u = y - \upsilon. \tag{A19}$$

Subsequently, we expand Eq. (A15) around a transition to y, as was done for the Itô prescription of the previous section.

The methodology for expanding Eq. (A15) is the same as in previous Itô case, and the resulting expansion is the Kramers-Moyal expansion of Eq. (A9), but in terms of the variable y. For this Kramers-Moyal expansion, the jump moments are given by

$$M_n(y) = \int_{\infty}^{0} \upsilon^n W_S(\upsilon|y,t)d\upsilon = \lambda(\eta^{-1}(y),t)\langle z^n\rangle, \tag{A20}$$

where $\langle z^n\rangle = \int_0^\infty z^n p_z(z)dz$ and $W_S(\upsilon|y,t) = \lambda(\eta^{-1}(y),t)\int_0^\infty \delta(\upsilon-z)p_z(z)dz$. This term $W_S(\upsilon|y,t)$ is Eq. (A17) with a substitution for u based on Eq. (A19), after which $\lambda(\eta^{-1}(y-$

$\upsilon), t)$ is replaced with $\lambda(\eta^{-1}(y), t)$ because of the Taylor series expansion around a transition to y. Accordingly, based on Eq. (A20) and Eq. (A9) in terms of y, the expansion for the transformed jump process is given by

$$\partial_y J_{\xi_y}(y,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial y^n} \left[\langle z^n \rangle \lambda(\eta^{-1}(y), t) p_y(y,t) \right]. \tag{A21}$$

However, in Eq. (A21), the frequency, $\lambda(\eta^{-1}(y), t)$, represents a multiplicative function. Consequently, for consistency with the Stratonovich jump interpretation, this frequency must be merged into a new variable, i.e,.

$$\hat{y} = \hat{\eta}(\chi) = \int \frac{1}{b(\chi)\sqrt{\lambda(\chi,t)}} d\chi, \tag{A22}$$

where accordingly $\chi = \hat{\eta}^{-1}(\hat{y}, t)$ and $p_\chi(\chi, t) = p_{\hat{y}}(\hat{y}, t) \left| \frac{d\hat{y}}{d\chi} \right|$, and now

$$\frac{d\chi}{d\hat{y}} = \frac{d\hat{\eta}^{-1}(\hat{y})}{d\hat{y}} = \sqrt{\lambda(\hat{\eta}^{-1}(\hat{y}), t)}\, b(\hat{\eta}^{-1}(\hat{y})). \tag{A23}$$

Based on this change of variables, Eq. (A21) is posed as

$$\partial_{\hat{y}} J_{\xi_{\hat{y}}}(\hat{y},t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \hat{y}^n} \left[\langle z^n \rangle p_{\hat{y}}(\hat{y},t) \right], \tag{A24}$$

for which the corresponding $\hat{y}$ dependent drift is given as $m_{\hat{y}}(\hat{y}, t) = \frac{m(\hat{\eta}^{-1}(\hat{y}))}{b(\hat{\eta}^{-1}(\hat{y}))\sqrt{\lambda(\hat{\eta}^{-1}(\hat{y}),t)}}$. After transforming Eq. (A24) with a change of variables following Eqs. (A22) - (A23), we recover the Kramers-Moyal expansion for the Stratonovich jump prescription, i.e.,

$$\partial_\chi J_\xi(\chi,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\sqrt{\lambda(\chi,t)}b(\chi)} \left(\sqrt{\lambda(\chi,t)}b(\chi)\frac{\partial}{\partial\chi} \right)^n \left[\langle z^n \rangle \sqrt{\lambda(\chi,t)} b(\chi) p_\chi(\chi,t) \right], \tag{A25}$$

where $p_{\hat{y}}(\hat{y}, t) = p_\chi(\chi, t)\frac{d\chi}{d\hat{y}}$, $\sqrt{\lambda(\chi,t)}b(\chi) = \frac{d\chi}{d\hat{y}}$, $\left(\sqrt{\lambda(\chi,t)}b(\chi)\frac{\partial}{\partial\chi}\right)^n = \frac{\partial^n}{\partial\hat{y}^n}$, and Eq. (A25) is based on recognizing $\partial_{\hat{y}} J_{\xi_{\hat{y}}}(\hat{y}, t) = \frac{d\chi}{d\hat{y}} \partial_\chi J_\xi(\chi, t)$ and multiplying both sides of Eq. (A24) by $\frac{d\hat{y}}{d\chi}$. When $\lambda(\chi, t)$ is a constant, the terms of Eq. (A25) may be rearranged so the expression is equivalent to the form given by Eq. (D5) of [41].

We now consider the convergence of Eq. (A25) under the limit of Eq. (A11), i.e., infinite jump events as the forcing weights approach zero, $z \to 0$. Similar to the Itô case, unless the forcing input PDF, $p_z(z)$, is symmetric about the origin ($z = 0$), convergence only occurs if the $n = 1$ term of Eq. (A25) is balanced by the drift, i.e.,

$$m(\chi, t) = m_o(\chi, t) - \langle z \rangle \sqrt{\lambda(\chi, t)} b(\chi), \tag{A26}$$

where in comparison to Itô drift of Eq. (A13), the drift now must balance based on $\sqrt{\lambda(\chi, t)}$ instead of $\lambda(\chi, t)$. For the Stratonovich version of $\partial_\chi J_\xi(\chi, t)$ of Eq. (A25) and the drift term of Eq. (A26), the master equation (11) under the limit of Eq. (A11) converges to a diffusion process description, i.e.,

$$\partial_t p_\chi(\chi, t) = -\frac{\partial}{\partial \chi} \left[m_o(\chi, t) p_\chi(\chi, t) \right] + \frac{1}{2} \frac{\partial}{\partial \chi} \left[\sqrt{D(\chi, t)} b(\chi) \frac{\partial}{\partial \chi} \left[\sqrt{D(\chi, t)} b(\chi) p_\chi(\chi, t) \right] \right], \tag{A27}$$

which is the Stratonovich version of the Fokker-Planck equation where $D(\chi, t)$ is given by Eq. (A12) and follows from the Eq. (A11) limit of $\langle z^2 \rangle \lambda(\chi, t)$. Note that the drift for the corresponding jump process is given by Eq. (A26) and is different than the Fokker-Planck drift term unless $p_z(z)$ is symmetric about $z = 0$. In both Fokker-Planck equations (A14) and (A27), the diffusion coefficient $D(\chi, t)$ has the same dependence on the jump frequency, $\lambda(\chi, t)$.

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