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Krishna Pusuluri and Andrey Shilnikov

Phys. Rev. E **98**, 040202 — Published 30 October 2018

DOI: [10.1103/PhysRevE.98.040202](https://doi.org/10.1103/PhysRevE.98.040202)

Homoclinic chaos and its organization in a nonlinear optics model

Krishna Pusuluri*

Neuroscience Institute, Georgia State University,
Petit Science Center, 100 Piedmont Av., Atlanta, GA 30303, USA

Andrey Shilnikov†

Neuroscience Institute, and Department of Mathematics and Statistics,
Georgia State University, Petit Science Center, 100 Piedmont Av., Atlanta, GA 30303, USA
(Dated: October 9, 2018)

Abstract: We developed a powerful computational approach to elaborate on the onset mechanisms of deterministic chaos due to complex homoclinic bifurcations in diverse systems. Its core is the reduction of phase space dynamics to symbolic binary representations that lets one detect regions of simple and complex dynamics as well as fine organization structures of the latter in parameter space. Massively parallel simulations shorten the computational time to disclose highly detailed bifurcation diagrams to a few seconds.

New directions in science are launched by new tools much more often than by new concepts. Dyson F. [1]. Break-through discovery of deterministic chaos in [infrared gas] lasers in nonlinear optics was established and pioneered both theoretically and experimentally long time ago [2–6]. Recent developments in semiconductor lasers and nano-optics have stimulated newest advances in optical synchronization and photonic integrated circuits for the needs of cryptography [7–12]. Nowadays, a real advance in deterministic nonlinear science stimulating the progress of cutting-edge engineering is hardly possible without significantly deepening the knowledge and beneficial usage of complex elements borrowed from dynamical systems theory. This in turn is hardly possible without development and incorporation of new mathematical and computational tools, including for parallel Graphics Processing Unit(GPU) based platforms.

In this letter we demonstrate how our newly developed toolkit, called “Deterministic Chaos Prospector (DCP),” along with the bifurcation-parameter continuation technique, lets one quickly and fully disclose and elaborate on the origin of complex chaotic dynamics in a 6D model of a resonant 3-level optically-pumped laser (OPL) [13, 14]. In addition to simple dynamics associated with stable equilibria and periodic orbits, it reveals a broad range bifurcation structures that are typical for many ODE models from nonlinear optics and other diverse applications [15–21]. These include homoclinic orbits and heteroclinic connections between saddle equilibria that are the key building blocks of deterministic chaos in most systems. Their bifurcation curves with characteristic spirals around T(terminal)-points along with other codimension-2 points are the organizing centers that shape regions of complex and simple dynamics in the parameter space of such systems. The detection of these bifurcations has long remained the state-of-the-art involving a meticulous and time consuming parameter continuation technique to disclose a few sparse elements of the otherwise rich and fine organization of the bifurcation set. We note that while the brute-force approach

based on the evaluation of Lyapunov exponents can effectively locate stability windows within regions of chaos [22, 23], it fails to disclose these essential structures that are imperative for the exhaustive understanding of complex dynamics and their origin. We will demonstrate how our approach exploiting the sensitivity of deterministic chaos and its symbolic representation using binary sequences combined with Lempel-Ziv complexity algorithms [24], can effectively reveal regions of complex, structurally unstable and simple stable dynamics in this and other systems.

The 3-level optically pumped laser model [13, 14] is given by

$$\begin{aligned}\dot{\beta} &= -\sigma\beta + 50p_{23}, \\ \dot{p}_{21} &= -p_{21} - \beta p_{31} + aD_{21}, \\ \dot{p}_{23} &= -p_{23} + \beta D_{23} - ap_{31}, \\ \dot{p}_{31} &= -p_{31} + \beta p_{21} + ap_{23}, \\ \dot{D}_{21} &= -b(D_{21} - D_{21}^0) - 4ap_{21} - 2\beta p_{23}, \\ \dot{D}_{23} &= -b(D_{23} - D_{23}^0) - 2ap_{21} - 4\beta p_{23},\end{aligned}\tag{1}$$

with parameters a , β , and $\sigma = \{1.5; 10\}$, being the Rabi flopping quantities representing the electric field amplitudes at pump and emission frequencies, and the cavity loss parameter, resp.; b is the ratio of population to polarization decay rates; p_{ij} is the normalized density matrix element corresponding to the transitions between levels i and j , while D_{ij} is the population difference between the i -th and j -th levels. Note that Eqs. (1) are \mathbb{Z}_2 -symmetric under the involution $(\beta, p_{21}, p_{23}, p_{31}, D_{21}, D_{23}) \leftrightarrow (-\beta, p_{21}, -p_{23}, -p_{31}, D_{21}, D_{23})$, which is typical for Lorenz-like systems [17, 25]. Depending on (a, b) -values, the laser model (1) has either a single non-lasing steady state, O , or an extra pair of equilibria, C^\pm (Fig. 1a), emerging as O loses stability through a pitch-fork (PF) bifurcation and becomes a saddle. All three steady states can independently undergo super-critical Andronov-Hopf (AH) bifurcations (curves labelled with AH_0 and $AH_{1,2}$ in the (a, b) -parameter plane in Fig.2) giving rise to stable periodic orbits (PO) in the phase space of the laser

model (1). Both structural and dynamical instability in

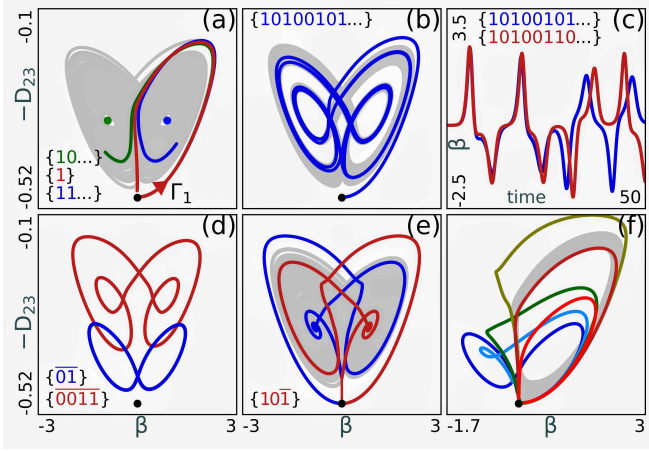


FIG. 1. (color online) (a) $(\beta, -D_{23})$ -phase space projection showing the primary homoclinic orbit (red, coded as $\{1\}$) splitting leftward/rightward (green/blue, $\{11\}$ or $\{1\bar{1}\}$) when the separatrix Γ_1 misses the saddle O (black dot) after completing a single turn around the saddle-focus C^+ , with the Lorenz attractor (in grey) in background (b) Chaotic transient of Γ_1 generating a binary sequence starting with $\{10100101\dots\}$ (c) Time-evolutions of the β -coordinate of Γ_1 (in (b)) and of a close trajectory (red), and their binary codes, before they diverge. (d) Two stable symmetric POs coded as $\{0\bar{1}\}$ and $\{001\bar{1}\}$ (e) Heteroclinic connections (red $\{10\bar{1}\}$, blue symmetric counterpart) at the T_1 -point (Fig. 2) (f) Samples (P_j) of the primary homoclinic orbit morphing to a double loop after the inclination-flip, IF_1 , on the curve H_0 in the (a, b) -parameter plane in Fig. 2; here $\sigma = 1.5$

the model are due to an abundance of homoclinic bifurcations (HB) of the saddle O , whose 1D unstable separatrix Γ_1 (and the symmetric counterpart Γ_2) densely fills out the two spatially-symmetric wings of the butterfly-shaped strange attractor (Fig. 1a,e) [25]. As parameters are varied Γ_1 constantly and unpredictably changes its flip-flop switching patterns within the Lorenz attractor. These patterns change whenever Γ_1 comes back to O to undergo a homoclinic bifurcation. This observation is the core for the proposed symbolic approach that converts chaotic and periodic patterns of Γ_1 around the equilibria C^\pm into binary sequences $\{k_n\}$ as follows:

$$k_n = \begin{cases} 1, & \text{when the separatrix } \Gamma_1 \text{ turns around } C^+; \\ 0, & \text{when the separatrix } \Gamma_1 \text{ turns around } C^-. \end{cases}$$

As such, the periodic sequence $\{111\dots\}$, or $\{\bar{1}\}$, corresponds to Γ_1 converging to the equilibrium state C^+ or a periodic orbit emerging from it through AH-bifurcation, while the sequence $\{100\dots\}$ or $\{1\bar{0}\}$ corresponds to Γ_1 converging to C^- and so forth. Wherever small parameter variations do not change Γ_1 -progressions and hence their binary representations, the system demonstrates structurally stable dynamics, which can be due to stable equilibria or periodic orbits (PO), such as the symmetric POs turning once [figure-8 shaped] or twice around C^-

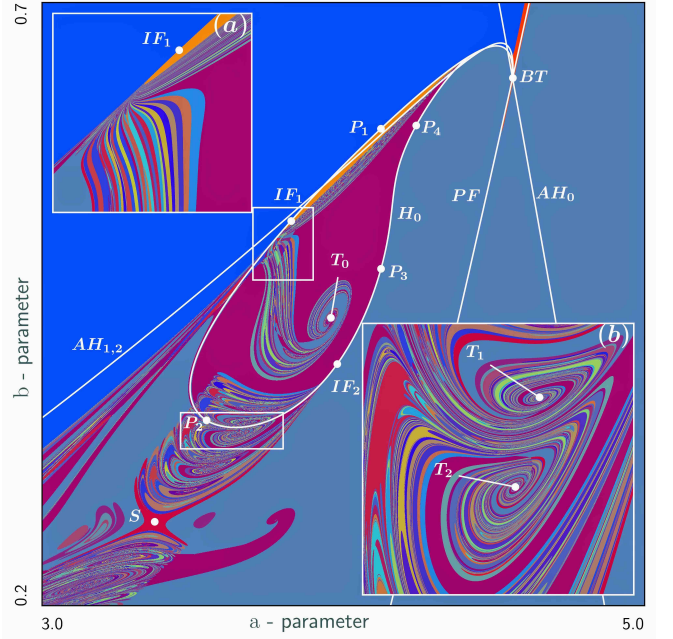


FIG. 2. (color online) (a, b) -parameter sweep of $[5-12]$ -length reveals an abundance of homoclinic bifurcations emerging from two cod-2 points, IF_1 & IF_2 , on H_0 , that corresponds to the primary homoclinic butterfly of saddle O , along with self-similar characteristic spirals around T-points, labelled $T_{0,1,2}$, corresponding to distinct heteroclinic cycles between O and saddle-foci C^\pm . Cod-2 Bogdanov-Takens, BT , unfolding includes Andronov-Hopf AH_0 , $AH_{1,2}$ and pitch-fork PF bifurcation curves for O and C^\pm , resp.; here $\sigma = 1.5$.

and C^+ in Fig. 1d, with corresponding binary sequences $\{0\bar{1}\}$ and $\{001\bar{1}\}$, resp. An aperiodic binary sequence is associated with chaotic dynamics that is characterized by the sensitive dependence on small parameter variations that change Γ_1 -progressions and corresponding symbolic sequences (Fig. 1c). Changes occur at homoclinic bifurcations when Γ_1 comes back to saddle O . The primary homoclinic orbit (shown in Fig. 1a,f) coded with a finite sequence $\{1\}$ separates periodic patterns coded as $\{\bar{1}\}$ and $\{1\bar{0}\}$. It occurs on the bifurcation curve H_0 in the (a, b) -parameter plane in Fig. 2. There are two special points labeled as IF_1 and IF_2 on H_0 that correspond to the so-called inclination-flip (IF) bifurcation of codimension-two [26]. Its feature is that it gives rise to instant homoclinic chaos in the phase space and complex bifurcation structures in the parameter space of the system. With our new computational-symbolic toolkit we can clearly and quickly identify such bifurcations and their fine organizations in the parameter space along with regions of chaotic and regular dynamics. First, we define a formal power series $P(N)$ for a finite binary sequence $\{k_n\}$ of length N , after omitting the first j symbols for initial transients of the separatrix Γ_1 or any other trajec-

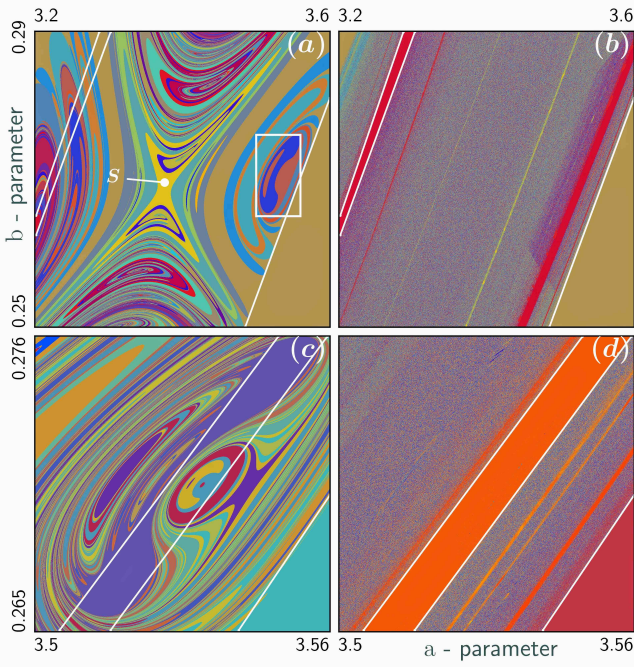


FIG. 3. (color online) (a) Short [8–15] and (b,d) long [100–123]-length (a, b)-sweeps reveal fine self-similar organization of homo- and heteroclinic bifurcations underlying the regions of chaotic and regular dynamics of model (1) for $\sigma = 1.5$. A small area (white box) in (a) is magnified with a longer [15–22]-sweep in (c). (b,d) reveal stability windows (solid colors) within “noisy” regions of structurally unstable chaos; white lines demarcate boundaries of some stability windows.

tory, as follows:

$$P(N) = \sum_{n=j+1}^{j+N} \frac{k_n}{2^{(N+j+1)-n}}. \quad (2)$$

By construction, the range of $P(N)$ is $[0, 1]$, including the sequences $\{0\}$ and $\{1\}$ (in the limit as $N \rightarrow \infty$). For example, $P(8)$ for the aperiodic sequence $\{10100101\}$ generated by Γ_1 in Fig. 1b, with $j = 0$ and $N = 8$, is given by: $P(8) = 1/2^8 + 0/2^7 + 1/2^6 + 0/2^5 + 0/2^4 + 1/2^3 + 0/2^2 + 1/2^1 = 0.64453125$. The P -quantities are used as invariants to discriminate or conjugate finite progressions of the separatrix Γ_1 of the saddle against each other to identify and trace down corresponding bifurcation curves in the parameter space. Moreover, the quantities generated from long periodic and aperiodic binary sequences let us efficiently detect regions of regular and chaotic dynamics, resp. Keeping σ fixed at 1.5 or 10, we 1) vary a and b to get a bi-parametric sweep on a 2000x2000 grid 2) to follow Γ_1 -progressions 3) generating binary sequences $\{k_n\}$ that 4) result in $P(N)$ -quantities. Next 5) we colormap all found $P(N)$ values onto the parameter plane, where regions are identified by their equivalent colors, and the borderlines between adjacent regions correspond to homoclinic bifurcation curves. The colormap differen-

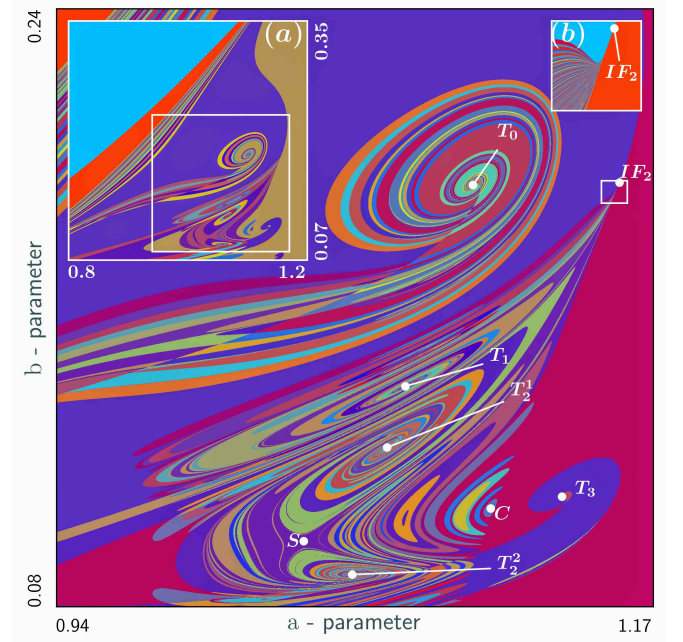


FIG. 4. (color online) [2–9]-length sweep discloses organization of homo/heteroclinic bifurcations originating from cod-2 inclination-flip IF_2 and multiple T-points: primary T_0 coded as $\{10\}$, secondary T_1 as $\{10\bar{1}\}$, and a pair $T_2^1 - T_2^2$ with code $\{11\bar{0}\}$ separated by a saddle (S) in the (a, b)-parameter plane; here $\sigma = 10$. Inset (a) shows a larger (a, b)-sweep of [1–7]-length; (b) [16–23]-long sweep depicts dense loci of homoclinic bifurcation curves originating from IF_2 .

tiates between $P(N)$ -values grouped into 2^{24} bins with preset RGB-color values. Such sweeps can be massively parallelized by running separate threads on a GPU. For example, the sweep of [5–12]-length, i.e. with the first five symbols omitted, shown in Fig. 2 takes about 8 seconds to run on a Tesla K40 GPU by Nvidia. It is superimposed with the curves, obtained by parametric continuation, corresponding to pitch-fork (PF), Andronov-Hopf (AH_0 and $AH_{1,2}$ for O and C^\pm) and the primary homoclinic (H_0) bifurcations all originating from the codimension-2 Bogdanov-Takens point (BT) [26]. Fig. 1f shows how the primary homoclinic loop transmutes into a double loop along the curve H_0 . The sweep reveals the way the inclination-flip IF_1 and IF_2 points give rise to jets of homoclinic bifurcation curves spiraling to various self-similar cod-2 Bykov terminal T-points, including T_0 and T_1 , that correspond to heteroclinic connections linking the saddle O with saddle-foci C^+ , C^- and generating periodic sequences $\{10\}$, $\{10\bar{1}\}$ (Fig. 1e), resp.

Figure 3a shows that with longer sequences we can obtain more detailed sweeps disclosing multiple T-points of smaller scales near the saddle point, S , that are not seen in Fig. 2. These spiral structures around T-points (identical to T_2^1 and T_2^2 in Fig. 4) morph into closed loops (like those shown in Fig. 3c) after collapsing into the saddle

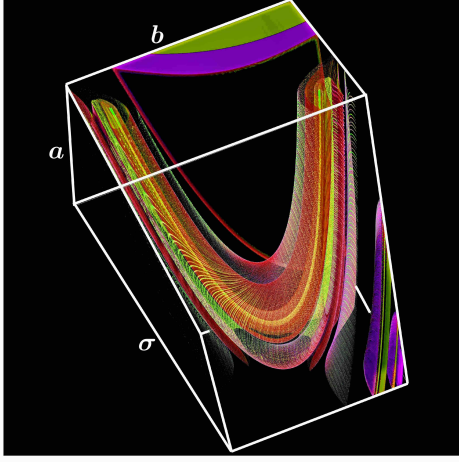


FIG. 5. (color online) Fragment of the 3D (a, b, σ) -parameter space (based on a $2000 \times 2000 \times 2000$ grid) depicting various nested elliptic and hyperbolic paraboloids whose contour curves appear as spirals around T-points, and/or concentric circles near saddles in the bi-parametric sweeps in Figs. 2-4.

as σ -parameter is varied (shown in Suppl. Movie 1 [27].) Figures 3b,d present the sweep of [100–123]-length, i.e., after skipping the first 100 transient symbols. Here regions with solid colors of constant $P(23)$ -values represent the stability windows corresponding to simple (periodic) Morse-Smale dynamics, whereas multi-colored noisy regions refer to structurally unstable chaotic dynamics, as described later.

The (a, b) -sweep of [2-9]-length in Fig. 4 demonstrates the intrinsic re-arrangement of the bifurcation constituents of complexity for a different cut at $\sigma = 10$. Here, the secondary inclination-flip point, (IF_2), gives rise to loci of outgoing homoclinic curves that are re-directed by a saddle point (S), and spiral onto multiple T-points. The heteroclinic connections at the T-points, T_0 - T_3 , are given by $\{10\}$, $\{10\bar{1}\}$, $\{11\bar{0}\}$, and $\{\bar{1}\}$, respectively. The T-points T_2^1 and T_2^2 , separated by the saddle S , correspond to the same heteroclinic connection $\{11\bar{0}\}$. Note that here the primary homoclinic curve spirals onto the primary T-point T_0 . The T-point T_3 belongs to the stability window dominated by the symmetric figure-8 periodic orbit (Fig. 1d) in the long run. The semi-annular structures around C are, in fact, the remnants of the spirals around T_3 , where the other halves of the spirals are disintegrated by the stable periodic orbit existing near T_3 . With small σ -variations, T_3 crosses over the stability boundary near C , so that both ends of the semi-annular structures merge to complete spirals around T_3 (as demonstrated in Suppl. Movie 2 [27].) Meanwhile, T-points T_2^1 and T_2^2 merge with the saddle S to transform into concentric cycles. These structures in the 2D sweeps are the contour curves of the corresponding surfaces in the 3D (a, b, σ) -parameter space of

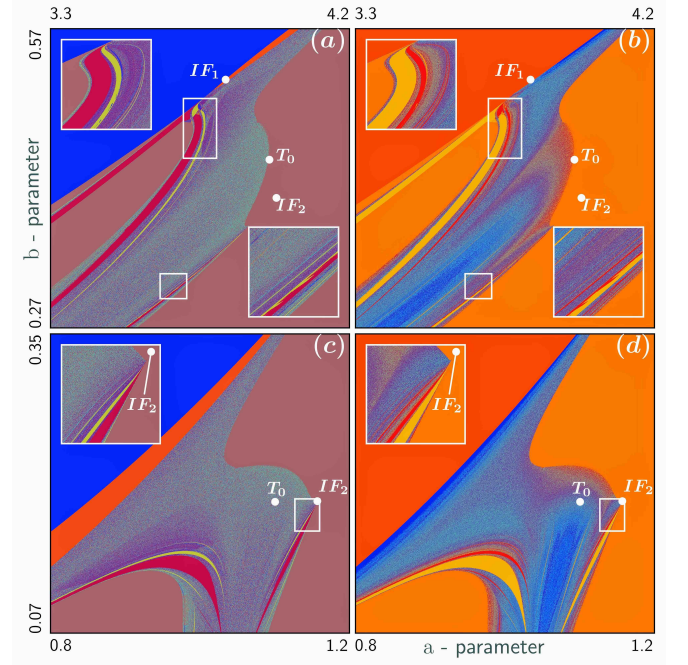


FIG. 6. (color online) Long [1000–1999]-length sweeps to detect a multiplicity of stability windows (solid colors; dark red due to stable PO $\{0011\}$ in Fig. 1d) within noisy/multi-color regions of chaos adjacent to IF_1 and IF_2 points in the (a, b) -parameter plane using PC algorithm in (a) and (c), and LZ-complexity in (b) and (d). Sweeps at $\sigma = 1.5$ (a,b) and $\sigma = 10$ (c,d) to compare with Fig. 2 and 4.

the model (1). Figure 5 demonstrates this saddle as the critical point of the 3D surface shaped as a hyperbolic paraboloid, constructed using 8 billion trajectories in the parameter space. Depending on the particular σ -cuts, the contour lines of the bent scroll-shaped surfaces may look like spirals or closed concentric circles in the projections in Figs. 2-4. While a detailed sweep for short-term transient dynamics lets us reveal the underlying homoclinic bifurcations, longer sweeps omitting initial transients are designed to localize stability windows corresponding to regular dynamics and regions of chaotic dynamics in the parameter space. We implemented two algorithms into our computational DCP toolkit to classify such regions depending on whether the corresponding binary sequences of solutions are periodic or not for the given parameter values. The first algorithm based on Eq. (2) includes periodicity correction (PC) to identify periodic structure within a binary sequence and to normalize it to its smallest valued circular permutation. For example, the symmetric figure-8 periodic orbit in Fig. 1d is coded with $\{0\bar{1}\}$ not with $\{\bar{1}0\}$. The second algorithm utilizes the Lempel-Ziv-76 (LZ) compression [24], to determine the normalized complexity (the number of words in vocabulary per sequence length) of the binary sequence. The LZ compression algorithm scans a sequence from left to right and adds a new word to

the vocabulary every time a previously unencountered substring is detected. Since all circular permutations of a periodic orbit have the same complexity, with this approach we can also detect stability windows amidst structurally unstable chaotic regions. This approach requiring only one solution per parameter set complements more expensive computational approaches based on the evaluations of the largest or several Lyapunov exponents.

Figure 6 represents the bi-parametric [1000-1999]-long sweeps to identify regions of simple and complex dynamics in model (1); here insets a/c and b/d represent the PC- and LZ-algorithm based sweeps, respectively. Regions of solid monotone colors correspond to the stability windows with stable equilibrium states and periodic orbits, while multi-colored noisy regions indicate that the dynamics are structurally unstable and chaotic. The sweeps in Figs. 5a-b (at $\sigma = 1.5$) are superimposed with the primary and secondary inclination-flip points, IF_1 and IF_2 , along with the primary T-point T_0 located next to the boundary between the regions of chaotic and stable periodic dynamics. They reveal multiple stability windows adjacent to IF_1 and IF_2 (magnified insets), including the wide one (in dark red) corresponding to a stable periodic orbit $\{0011\}$ (shown in Fig. 1d). This approach can clearly identify distinct periodic orbits and their stability windows mapped by different colors. Note that same stability windows (indicated with same colors) emerge near both IF_1 and IF_2 in the reversed order. The sweeps in Figs. 5c-d (at $\sigma = 10$) depict the primary T-point T_0 located inside the region of chaotic dynamics, and the stability windows accumulating to IF_2 . We note that the PC-algorithm lets one detect and identify a variety of stable periodic orbits efficiently even with short symbolic sequences (see Figs. 2b,d) compared to quite long sequences required by the LZ-algorithm that suits better for the detection of chaotic regions. This observation suggests the order to analyze the given sequence is to run it first through the PC-algorithm to detect periodic orbits, and next through the LZ-algorithm to detect complexity of aperiodic strings on a GPU. Other future enhancements for the DCP toolkit are to include the search algorithms for bifurcations of equilibrium states and periodic orbits such as period-doubling.

In conclusion, we have demonstrated the proficiency of the new symbolic toolkit for computational studies of both transient and long-term solutions to analyze the bifurcation mechanisms underlying the onset of chaotic and regular dynamics in the phase and parameter spaces of the given OPL model and similar deterministic systems. This work was in part funded by NSF grant IOS-1455527, RSF grant 14-41-00044 at the Lobachevsky University of Nizhny Novgorod, and MESRF project 14.740.11.0919. We thank GSU's Brains and Behavior initiative for fellowship and pilot grant support, NVIDIA corporation for the donation of Tesla K40 GPU, and the members of

Shilnikov's NeurDS lab for helpful discussions.

* pusuluri.krishna@gmail.com

† ashilnikov@gsu.edu

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- [27] See Supplemental Material at [URL to be inserted by publisher] for numerical details and movies of 3D sweeps.