# Nonlinearity-induced localization in a periodically driven semidiscrete system

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# Nonlinearity-induced localization in a periodically-driven semi-discrete system

R. Driben[1], V. V. Konotop[2], B. A. Malomed[3,4,5], T. Meier[1], and A. V. Yulin[5]
[1]*Department of Physics and CeOPP, University of Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany*
[2] *Centro de Física Teórica e Computacional and Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Ed. C8, Lisboa 1749-016, Portugal*
[3] *Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering,Tel Aviv University, P.O.B. 39040, Tel Aviv, Israel*
[4] *Center for Light-Matter Interaction, Tel Aviv University, P.O.B. 39040, Tel Aviv, Israel*
[5] *ITMO University, 49 Kronverskii Ave., St. Petersburg 197101, Russian Federation*
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We demonstrate that nonlinearity plays a constructive role in supporting the robustness of dynamical localization in a system which is discrete in one dimension, and continuous in the orthogonal one. In the linear regime, time-periodic modulation of the gradient strength along the discrete axis leads to the usual rapid spread of an initially confined wave packet. Addition of the cubic nonlinearity makes the dynamics drastically different, inducing robust localization of moving wave packets. Similar nonlinearity-induced effects are also produced in the presence of a combination of static and oscillating linear potentials. The predicted dynamical localization in the nonlinear medium can be realized in photonic lattices and Bose-Einstein condensates.

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## I. INTRODUCTION

The possibility of Bloch oscillations (BO) [1], i.e., the occurrence of a temporally-oscillating (ac) electric current originating from spatial oscillations of the electron charge density in a crystal biased by a static uniform (dc) electric field, in the absence of scattering effects, was predicted by Bloch and Zener almost 90 years ago. Being initially far from feasible experimental realizations, this prediction caused debates regarding the actual existence of the BO, which lasted for several decades. The proof securing that the BO should be physically realizable, as predicted by effective Hamiltonians that include a finite number of bands, was theoretically provided in the early 1990s when rigorous upper limits for the interband tunneling rates had been established, see, e.g., Ref. [2]. Also in the 1990s, BO had been first observed experimentally in the temporal domain in electrically-biased semiconductor superlattices, using optical interband excitation by femtosecond laser pulses [3]. A few years later, BO has been realized with the use of ultracold atoms in optical lattices [4] measured in the momentum space. Very recently BO of a Bose-Einstein condensate (BEC) were observed by the direct measurement in the real space [5]. Also it was successfully emulated in optics, using arrayed waveguides [6]. These results prove that BO is a general physical effect relevant for a large class of systems, as also shown in detail by many further works were focused on the subject [7].

In Refs. [6] it was shown that, if the phase velocity of the waves varies linearly as a function of discrete coordinate $n$ of waveguides in the array, the position of the light beam is an oscillating function of propagation distance $z$, which is the optical counterpart of the electronic BO dynamics. In the latter context, the influence of the optical Kerr nonlinearity was considered too. However, in models of arrayed waveguides, which include solely the discrete diffraction, the nonlinearity was shown to produce a destructive effect on the BO dynamics. On the other hand, it was shown in a recent work [8] that adding another dimension, with continuous diffraction in that direction, may result in a constructive effect of the nonlinearity, *viz.*, localization of the wave packet in space and the emergence of a quasi-solitonic regime of the propagation. Thus, one may expect the existence of a new species of robust nonlinear hybrid wave packets, combining features of solitons and Bloch-oscillating waves. In photonic systems, the robustness of the hybrid wave packets in the presence of the anomalous group-velocity dispersion may lead to prediction of resonant radiation with nontrivial properties, similar to how it was predicted in the spatial domain under the action of diffraction [9]. This methodology is conceptually different from several proposed ways to guide nonlinear waves experiencing BO in a nonlinear regime that includes variation of the nonlinearity strength in the course of the evolution [10, 11], in the spirit if the "nonlinearity management" [12] techniques.

A phenomenon somewhat related to the BO, which may be induced by time-modulated gradient potentials, is dynamical localization (DL). It was predicted in Ref. [13] within the framework of the tight-binding approximation for electrons in solids. Whereas usually an initially localized wave packet will spread out when driven by an oscillating bias, the DL implies that the wave packet remains localized. This is the case for a single-band tight-binding system, when the ratio of the amplitude of the bias and its modulation frequency takes resonant values, namely, roots of Bessel function $J_0$. Thus, the conductance, provided for by the delocalization of the wave packets, vanishes for such a special choice of the ac bias. Later, the DL was studied theoretically for ultracold atoms trapped in optical lattices [14], where it may be used for the coherent control of atoms and for the realization of the phase transition between the superfluid and Mott-insulator states [15]. Furthermore, spin-orbit coupled atoms allow for implementation of DL in a two-component spinor system [16]. The DL were also predicted to exist in nonlinear discrete systems [17, 18]. The same effect was also predicted to significantly alter the optical absorption in semiconductor superlattices and the effective dimensionality of excitons [19]. In another solid-state setting, a similar effect was theoretically studied under the action of combined dc and ac electric fields [20]. DL has been experimentally observed in atomic systems (see, e.g., Refs. [21] and [22]) and in transport properties of semiconductor superlattices (see Ref. [23]). DL effects are known too in photonic settings [25], where periodic corrugation of waveguides makes it possible to optically emulate the time-periodic linear force acting on a quantum particle, and thus realize localization of optical dynamical modes and Bloch oscillations, as was predicted theoretically in Ref. [24], and realized experimentally in various settings [25], see also a review of photonic realizations of the DL and related phenomena in Ref. [26].

The main aim of the present work is to demonstrate that the DL persists in the nonlinear propagation regime for hybrid soliton-BO wave packets in systems containing an extra continuous dimension, in addition to the discrete one. We also report results for a combination of static and oscillating gradient potentials. In the latter case, the electronic transport is typically supported by multi-photon-assisted tunneling [27–30]. As predicted in Ref. [27] for special relations between the amplitude and modulation frequency of the fields, one can selectively suppress particular single- or multiple-photon transitions, thus designing specific transport properties, as shown in Ref. [30] for semiconductor superlattices. We demonstrate that, in the semi-discrete system considered here, these effects may persist in the presence of significant nonlinearity (note, that exact DL in a nonlinear system can be observed only in integrable models [17]).

The rest of the paper is organized as follows. In Sect. II we introduce the model and present some considerations for it. Numerical and analytical results are reported and discussed in Sect. III. The paper is completed by Sect. IV.

## II. THE MODEL AND ANALYTICAL RESULTS FOR THE LINEAR CASE

Semi-discrete optical systems with intrinsic cubic and quadratic nonlinearities were theoretically introduced, respectively, in Refs. [31]-[34] and [35]. Here we start with the semi-discrete model proposed in Ref. [8] as the system of linearly-coupled Gross-Pitaevskii equations (GPEs) [36] for a BEC loaded into an array of waveguides/wires, written here in the normalized form:

$$i\frac{\partial u_n}{\partial t}+\frac{1}{2}\frac{\partial^2 u_n}{\partial x^2}+\kappa(u_{n-1}+u_{n+1}-2u_n) +\gamma(t)nu_n+g|u_n|^2u_n=0. \tag{1}$$

Here $u_n(x,t)$ is the mean-field wave function in the $n$-th waveguide, $\kappa$ is the array's coupling constant, $\gamma(t)$ is the strength of the time-dependent gradient potential acting in the discrete direction ($n$), and nonlinearity coefficient $g>0$ accounts for the intrinsic self-attraction of the BEC. By means of obvious rescaling, we fix values of $\kappa=2$ and $g=1$, except of setting $g=0$ in the consideration of the linear version of the system.

In terms of optics, Eq. (1), with $t$ and $x$ replaced, respectively, by the propagation distance, $z$, and reduced time, $\tau$, is the system of coupled nonlinear Schrödinger equations (NLSEs) modeling the light propagation in an array of coupled optical fibers in the presence of a transverse linear gradient of the waveguide's effective index [31], with the magnitude of the gradient that may be modulated along $z$. In the latter case, $u_n$ are scaled envelope amplitudes of the electromagnetic waves in the fibers, the group-velocity dispersion is anomalous, and the cubic term with $g>0$ represents the focusing Kerr nonlinearity. Alternatively, the same system (1), with $x$ being the transverse coordinate, models the spatial-domain light propagation in a stack of parallel planar waveguides, the second derivative representing the paraxial diffraction in the waveguides [34, 37].

As concerns estimates of physical parameters, most relevant are ones for the above-mentioned spatial-domain realization in the stacked waveguides. For a typical experimentally relevant value of the transverse waist of the probe optical beam, $\sim 10$ $\mu$m, and its per-layer power, $\sim 300$ kW [38], the unit of $\Delta t=1$ corresponds to $\sim 0.5$ mm in physical units, hence the modulation frequency $\omega\sim 0.1$ (relevant to results reported below) implies the propagation distance $\sim 3$ cm (the actual length of the waveguide may be up to $\sim 10$ times larger). Further, the gradient's strength $\gamma\sim 0.1$ (which is also relevant to the actual results) implies the difference in the refractive index between adjacent cores of the waveguiding system $\Delta n\sim 10^{-3}$, which is a technologically feasible value.

First, we address the transport mechanism in the linear system, with $g=0$ in Eq. (1), keeping solely the discrete direction in it. For the harmonic format of the time modulation of the transverse gradient, with

$$\gamma(t)=\gamma_0\cos(\omega t). \tag{2}$$

The DL is realized if condition

$$J_0\left(\gamma_0/\omega\right)=0, \tag{3}$$

with Bessel function $J_0$, is satisfied [13, 14, 17, 18]. The combined ac-dc-driven regime was analyzed in Ref. [27], and realization of this effect was elaborated for the photoexcited electronic transport in semiconductor superlattices in Ref. [30]. Based on those studies, one can predict

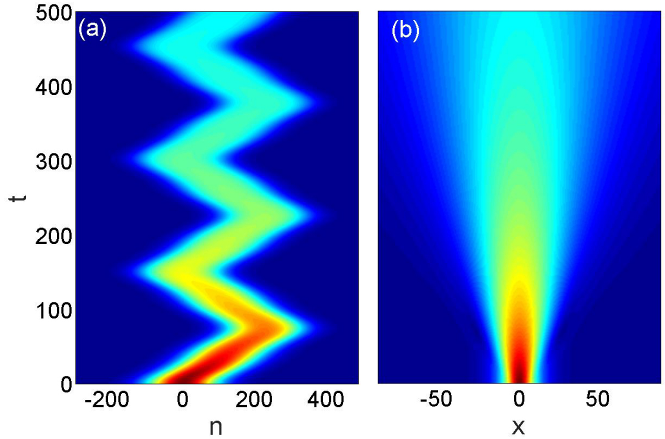


FIG. 1: (Color online) Evolution of the wave packet in the linear regime, i.e., with $g = 0$ in Eq. (1), is displayed in the $(n, t)$ (a) and $(x, t)$ (b) planes. The gradient strength is $\gamma_0 = 0.1$ and the modulation frequency is $\omega = 0.0416$, i.e., the ratio $\gamma_0/\omega \approx 2.4$ is very close to the first root of the zeroth-order Bessel function $J_0$. Other parameters are $\kappa = 2$ and $g = 0$. The initial condition is taken as per Eqs. (7) and (8) with $a_0 = 0.15$ and $w = 100$ (i.e., the initial Gaussian is quite broad).

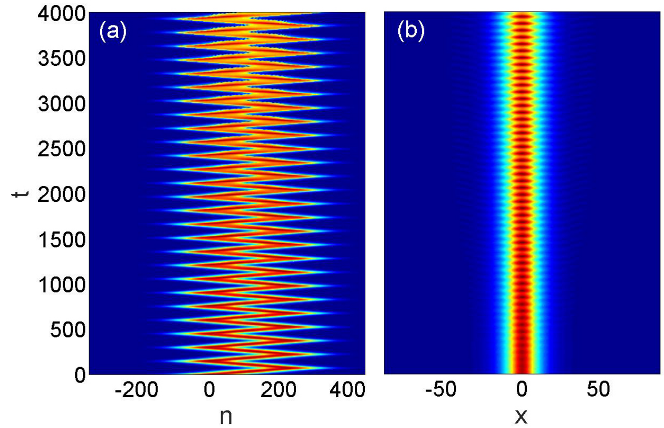


FIG. 2: (Color online) Long-time evolution of the wave packet in the nonlinear system, displayed in the $(n, t)$ (a) and $(x, t)$ (b) planes. The figure deminstrates dynamical localization of the wave packet driven by the time modulation of the gradient potential, with the strength and frequency corresponding to the first root of Bessel function $J_0$. The parameters are $\gamma_0 = 0.1$ and $\omega = 0.0416$. Other parameters are fixed as $\kappa = 2$ and $g = 1$, as in other numerical simulations.The initial condition is given by Eqs. (7) and (8) with $a_0 = 0.15$ and $w = 100$.

that, for the time modulation of the gradient's strength of the form

$$\gamma(t) = \gamma_{\rm dc} + \gamma_0 \cos(\omega t), \tag{4}$$

the dynamical transport is suppressed if the following two conditions are met simultaneously, for integer $m$:

$$\gamma_{\rm dc} = m\omega, \;\; J_m\left(\gamma_0/\omega\right) = 0, \tag{5}$$

since in this case the $m$-photon-assisted tunneling vanishes in the tight-binding limit.

For an initial pulse that is smooth with respect to $n$, one can approximate the finite difference in Eq. (1) by the continuum limit [39], which leads to the two-dimensional (2D) equation with two continuous coordinates, $x$ and $n$:

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \kappa\frac{\partial^2 u}{\partial n^2} + \gamma(t) n u + g|u|^2 u = 0. \tag{6}$$

Bearing this in mind, in the numerical simulations used below we consider the initial condition of the form

$$u_n^{(0)}(x) = \frac{A\left(n\right)}{\cosh\left[A\left(n\right) x\right]}. \tag{7}$$

with the Gaussian envelope:

$$A(n) = a_0 \exp\left(-n^2/w^2\right). \tag{8}$$

Thus, we will consider the evolution of the input localized as sech in the continuous coordinate, and as the Gaussian of width $w$ along the discrete coordinate.

## III. RESULTS

First we examine the evolution of the semi-discrete wave packet in the linear regime by setting $g = 0$ in Eq. (1). Figure 1(a) clearly shows that the wave packet performs regular oscillations in the discrete $(n)$ direction with period $2\pi/\omega$ of the modulation of the gradient potential. As we chose the values of the gradient strength $\gamma_0 = 0.1$ and frequency $\omega_0 = 0.0416$, which correspond to the first root of $J_0$ in Eq. (3), the wave packet remains localized in the $n$-domain in the course of the evolution, which is typical for the DL; however, the localization degree is decaying with the increase of $t$. Concomitantly, the diffraction along the continuous $x$-axis leads to rapid spreading of the wave packet, as seen in Fig. 1(b). Thus, in the linear regime the semi-discrete wave packet suffers delocalization in the course of its evolution, even when the DL condition (3) holds.

Next we launch the input into the full nonlinear system, corresponding to Eq. (1) with $g = 1$. The other parameters are same as in Fig. 1, i.e., the ratio of the gradient's strength and modulation frequency corresponds to the first root of $J_0$ in Eq. (3). Figures 2(a) and (b) display the nonlinear evolution in the $(n, t)$ and $(x, t)$ planes, respectively. Now, instead of the gradual decay of the linear wave packet displayed in Fig. 1, robust permanent localization of the wave packet is observed, in both the $n$- and $x$-directions. This and other examples demonstrate that the nonlinear propagation creates stable 2D dynamical semi-discrete solitons, which do not exist in the linearized system.

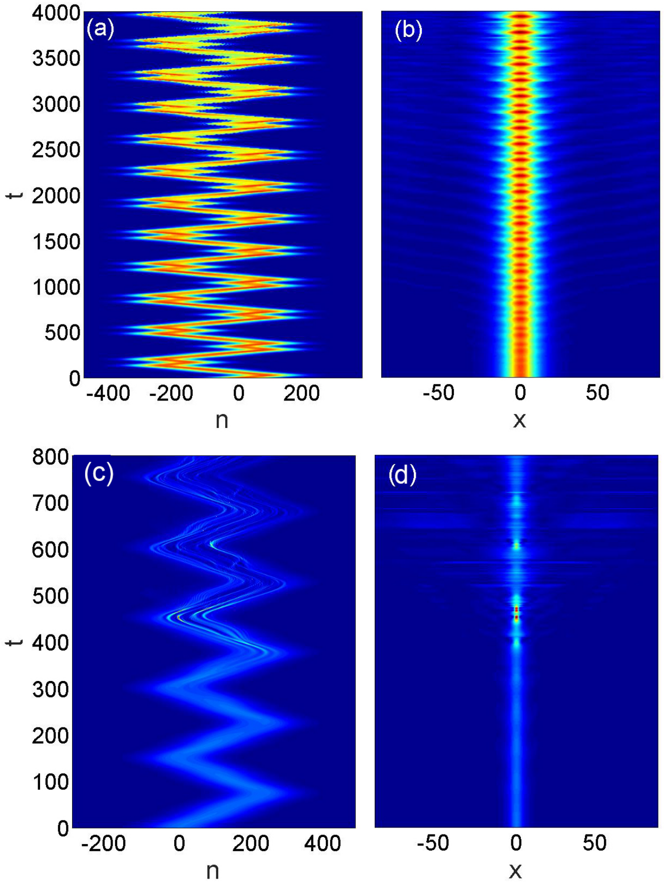


FIG. 3: (Color online) The long-time evolution of the wave packet in the nonlinear system in the $(n,t)$ (a) and $(x,t)$ (b) planes, respectively. Panels (a) and (b) display the dynamical localization of the wave packet driven with the modulation frequency corresponding to the second root of the Bessel function $J_0$ [see Eq. (3)]. Accordingly, the parameters are chosen as $\gamma_0 = 0.1$, $\omega_0 = 0.0181$, and $a_0 = 0.15$. Panels (c) and (d) display the evolution for the same parameters as in (a) and (b), but with a larger input amplitude, $a_0 = 0.3$.

The dynamical localization is also found when modulation frequency $\omega$ is reduced so that the ratio of $\gamma_0$ and $\omega$ corresponds to the second root of $J_0$ in Eq. (3), as shown in Figs. 3(a) and (b). However, in a still stronger nonlinear regime, corresponding to amplitude $a_0$ which is twice as large, the robustness of the established wave packets drops significantly, as we reach the quasi-collapse-driven dynamical regime [31], with the wave packet splittings, similar to what was reported in Ref. [8]. Figures 3(c) and (d) illustrate this situation. Thus, similar to the case of the BO, cf. Ref. [8], the well-pronounced DL regime occurs at optimal strengths of nonlinearity, which can be identified by systematic simulations with varying $a_0$ and/or $g$.

To characterize the dynamics of the wave packet in both the linear and nonlinear systems, we define the average of a semi-discrete function, $f_n(x,t)$, carried by the wave packet $u_n(x,t)$, as

$$\langle f \rangle = \frac{1}{P}\int_{-\infty}^{+\infty}\sum_n f_n(x,t)|u_n(x,t)|^2 dx,$$

with norm $P \equiv \sum_n \int_{-\infty}^{+\infty} |u_n(x,t)|^2 dx$. This definition allows one to explore average positions of the wave packet along the $x$ and $n$ directions, i.e., $\langle x \rangle$ and $\langle n \rangle$, respectively. Furthermore, we define a deformation parameter characterizing "combined" changes of the wave-packet's widths in the course of the evolution [8]:

$$\Delta(t) = \sqrt{[N(t) - N(0)]^2 + [X(t) - X(0)]^2}\,, \qquad (9)$$

where average widths of the wave packet in the $n$ and $x$ directions are

$$N(t) = \sqrt{\langle n^2 \rangle - \langle n \rangle^2},\ X(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \qquad (10)$$

If deformations with respect to $n$ and $x$ are strongly anisotropic, $\Delta$ estimates the largest one. For the ideal case of a totally robust DL, $\Delta(t)$ would remain time independent, while growing or decreasing $\Delta(t)$ corresponds to ongoing deformation of the wave packet. These indicators, pertaining to the dynamical regimes displayed in Figs. 2(a,b) and 3(a,b), are presented, severally, in three top and three bottom panels of Fig. 4. It is clearly observed that the growth of the soliton's width in the course of the long-time evolution is strongly suppressed. Furthermore, the top panel in Fig. 4 demonstrates that the soliton may even slowly shrink in the $x$ direction. Thus, we conclude that the optimal nonlinear regime, supporting the robust wave packets in the semi-discrete 2D model, system was introduced in Ref. [8], supports the stable DL as well. In Fig. 4, the characteristics of the linear-propagation regimes are plotted by dashed curves along with solid curves representing their nonlinear counterparts for the same values of the parameters, except for $g = 0$ in the linear system. It is clearly seen that, in latter system, the packets spread fast in the $x$ direction, leading to fast growth of $\Delta(t)$, in contrast with the robust self-trapping of the quasi-solitons in the full nonlinear model. For the comparison all the parameters besides the switched off nonlinear interaction were taken similar. After a very long evolution, such as corresponding to $t = 12000$ for the case presented in Fig. 2 and in three upper panels of Fig. 4, the wave packets undergo splitting, which makes the description in terms of Fig. 4 irrelevant. However, the estimate of the values of physical parameters for the realization of the present system in optics, given above, implies that so large values of $t$ correspond to the propagation distance which is $\sim 100$ times larger than the limit achievable in the experimental setting.

Our model also proves its effectiveness in the case of the combined ac-dc modulation of the gradient's strength, taken as per Eq. (4). As mentioned above, in this case the stabilization occurs at frequencies obeying Eq. (5), i.e., they pertain to zeros of the Bessel functions

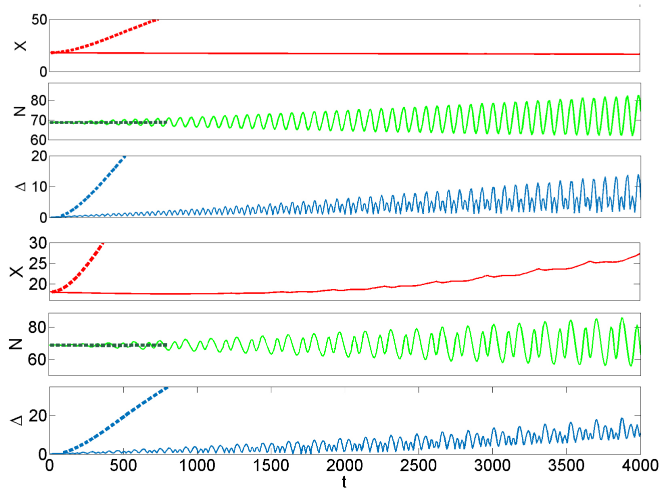


FIG. 4: (Color online) The temporal evolution of the overall spread of the wave packet $\Delta(t)$, together with its $N$- and $X$-components [see Eqs. (9) and (10)]. Examples of the DL, displayed in Figs. 2(a,b) and 3(a,b), are shown here in the three top and three bottom panels, respectively. Dashed curves show the same characteristics for the linear propagation regimes, with the same values of the control parameters, except for $g = 0$.

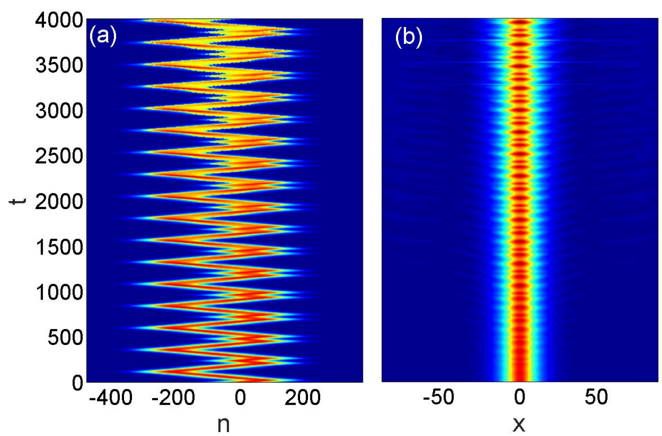


FIG. 5: (Color online) The evolution of the wave packet in case of the combined ac-dc modulation of the gradient's strength, defined as per Eq. (4), with the parameters corresponding to the first nontrivial zero of $J_1$, which is $\gamma_0/\omega \approx 3.8317$. The parameters are $\gamma_{\rm dc} = \omega = 0.0261$, which corresponds to $m = 1$ in Eq. (5), and $\gamma_0 = 0.1$.

of the order higher than zero. In particular, Fig. 5 demonstrates robust long-time evolution of the wave packet under the action of the combined ac-dc drive with $\gamma_{\rm dc} = \omega = 0.0261$, which corresponds to $m = 1$ in Eq. (5), $\gamma_0 = 0.1$ being the same as in Fig. 2. At other values of parameters satisfying Eq. (5) the combined dc-ac modulation pattern (4) produced very similar results.

## IV. CONCLUSIONS

To study the effect of the DL (dynamical localization) in nonlinear settings, we have introduced a semi-discrete 2D system, driven by the gradient potential with the time-periodic (ac) strength, applied in the discrete direction. In the absence of the nonlinearity, direct simulations demonstrate straightforward spreading of localized inputs, even if they satisfy the specific DL condition known from previous studies of linear models. The situation is found to be altogether different in the system with the cubic nonlinearity: in a certain range of the nonlinearity strength, the system features robust self-trapping of well-localized wave packets. In the presence of the nonlinearity, similar DL effects are also produced by the application of the gradient potential subject to the combined dc-ac temporal modulation. The DL effects predicted by the present analysis may be implemented in an effectively 2D BEC, loaded into a quasi-1D lattice potential, as well as in optics, in the temporal and spatial domains alike, using, respectively, an array of nonlinear fibers or a stack of planar waveguides.

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