Rényi α entropies of quantum states in closed form: Gaussian states and a class of non-Gaussian states

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Phys. Rev. E **97**, 062141 — Published 21 June 2018

DOI: 10.1103/PhysRevE.97.062141

Rényi-α entropies of quantum states in closed form: Gaussian states and a class of non-Gaussian states

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(Dated: June 8, 2018)

In this work, we study the Rényi-α entropies $S_\alpha(\hat{\rho}) = (1-\alpha)^{-1}\,\ln\{\text{Tr}(\hat{\rho}^\alpha)\}$ of quantum states for N bosons in the phase-space representation. With the help of the Bopp rule, we derive the entropies of Gaussian states in closed form for positive integers $\alpha = 2, 3, 4, \cdots$ and then, with the help of the analytic continuation, acquire the closed form also for real-values of $\alpha > 0$. The quantity $S_2(\hat{\rho})$, primarily studied in the literature, will then be a special case of our finding. Subsequently we acquire the Rényi-α entropies, with positive integers α, in closed form also for a specific class of the non-Gaussian states (mixed states) for N bosons, which may be regarded as a generalization of the eigenstates $|n\rangle$ (pure states) of a single harmonic oscillator with $n \geq 1$, in which the Wigner functions have negative values indeed. Due to the fact that the dynamics of a system consisting of N oscillators is Gaussian, our result will contribute to a systematic study of the Rényi-α entropy dynamics when the current form of a non-Gaussian state is initially prepared.

PACS numbers: 03.65.Ta, 11.10.Lm, 05.45.-a

I. INTRODUCTION

The Rényi-α entropy defined as $S_\alpha(\hat{\rho}) = (1-\alpha)^{-1}\,\ln\{\text{Tr}(\hat{\rho}^\alpha)\}$ where $\alpha > 0$ is considered a generalization of the von-Neumann entropy $S_1(\hat{\rho})$ [1]. Its properties have recently been studied, e.g., in a generalized formulation of quantum thermodynamics, which is built upon the maximum entropy principle applied to $S_\alpha(\hat{\rho})$ [2], as well as in the discussion of its time derivative under the Lindblad dynamics, the result of which may be useful for exploring the dynamics of quantum entanglement in the Markovian regime [3]. However, their explicit expressions for $\alpha \neq 2$ have not been investigated extensively, even for relatively simple forms of states $\hat{\rho}$, such as the Gaussian states for N bosons. In fact, only the entropy $S_2(\hat{\rho}) = -\ln\{\text{Tr}(\hat{\rho}^2)\}$ has been the primary quantity for investigation thus far, where the purity measure $\text{Tr}(\hat{\rho}^2)$ is the first moment of probability $\overline{p} = \sum_j (p_j)^2$ with the eigenvalues p_j's of $\hat{\rho}$, whereas, e.g., the second moment $\overline{p^2} = \text{Tr}(\hat{\rho}^3)$ is needed for the entropy $S_3(\hat{\rho})$.

A Gaussian state is defined as a quantum state, the Wigner function of which is Gaussian, such as the canonical thermal equilibrium state of a single harmonic oscillator (including its ground state at $T = 0$), the coherent state and the squeezed state, etc. [4–9]. The Gaussian states have recently attracted considerable interest as the need for a better theoretical understanding increases in response to the novel experimental manipulation of such states in quantum optics, in particular for the quantum information processing with continuous variables. As is well-known, the statistical behaviors of an N-mode Gaussian state are fully characterized by its covariance matrix [cf. (17)], which can yield an evaluation of $S_2(\hat{\rho})$ straightforwardly. Besides, the so-called Wigner entropy defined as $S_W(\hat{\rho}) = -\int dqdp\; W_\rho(q,p)\,\ln\{W_\rho(q,p)\}$ has been in investigation for the Gaussian states in [10], where the Wigner function is explicitly given by [11–15]

$$W_\rho(q,p) \,=\, \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi\, \exp\left(-\frac{i}{\hbar}p\,\xi\right)\left\langle q+\frac{\xi}{2}\right|\hat{\rho}\left|q-\frac{\xi}{2}\right\rangle \tag{1}$$

for a single mode, for the simplicity of notation, as well as the Weyl-Wigner c-number representation of the operator $\hat{A}$ given by

$$A(q,p) \,=\, \int_{-\infty}^{\infty} d\xi\, \exp\left(-\frac{i}{\hbar}p\,\xi\right)\left\langle q+\frac{\xi}{2}\right|\hat{A}\left|q-\frac{\xi}{2}\right\rangle\,, \tag{2}$$

together giving rise to the expectation value

$$\langle\hat{A}\rangle \,=\, \int dq\int dp\; W_\rho(q,p)\; A(q,p)\,. \tag{3}$$

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If the operator $\hat{A} = \hat{\rho}$, then its expectation value is nothing else than the purity measure $\text{Tr}(\hat{\rho}^2) = \int dqdp\, W_\rho(q,p)\, A(q,p) = (2\pi\hbar) \int dqdp\, \{W_\rho(q,p)\}^2$, which is the case of the Rényi parameter $\alpha = 2$. Interestingly, the entropy $S_2(\hat{\rho})$ for N-mode Gaussian states has been shown to coincide with $S_W(\hat{\rho})$ up to a constant [16]. However, the moments $\overline{p^{\alpha-1}} = \text{Tr}(\hat{\rho}^\alpha)$ and the resulting entropies in closed form where $\alpha > 2$ have still been unknown even for the Gaussian states.

Therefore it will be interesting to study the Rényi-α entropies in arbitrary orders for arbitrary quantum states explicitly in the phase-space representation [cf. (1)-(3)], and then exactly evaluate them for some specific states such as N-mode Gaussian states, as well as a certain class of non-Gaussian states, where the Wigner function can possess negative values indeed and so the Wigner entropy $S_W(\hat{\rho})$ is not directly well-defined. In fact, the full knowledge of density matrix (p_j's) and its diagonalization is practically hardly possible to achieve for N-mode generic cases. Therefore, the phase-space representation may also be favorable for studying the moments $\text{Tr}(\hat{\rho}^\alpha)$, with no need to diagonalize the density matrix directly. Moreover, this phase-space approach will be useful for studying systematically the quantum-classical transition behaviors of the entropies. In fact, it is known that all Rényi-α entropies tend asymptotically to the von-Neumann one in the classical limit (e.g., [10, 17, 18]).

The general layout of this paper is as follows: In Sec. II we provide a generic framework for the moments of density operator in the phase-space representation for arbitrary quantum states. In Sec. III we apply this rigorous framework to N-mode Gaussian states and derive the Rényi-α entropies in closed form, as well as generalize the results available in the literature. In Sec. IV the same discussion will take place for a class of non-Gaussian states. Finally we give the concluding remarks of this paper in Sec. V.

II. HIGHER-ORDER MOMENTS OF DENSITY OPERATOR IN PHASE-SPACE REPRESENTATION

We begin with the case of the Rényi parameter $\alpha = 3$, in which $\text{Tr}(\hat{\rho}^3) = \text{Tr}(\hat{\rho}\,\hat{\rho}^2) = \overline{p^2}$ is in consideration. To do so, we apply the Bopp rule for the Weyl-Wigner representation of the operator product $\hat{B}_1\hat{B}_2$, explicitly given by [14]

$$(B_1B_2)(q,p) = B_1\left(q - \frac{\hbar}{2i}\frac{\partial}{\partial p}, p + \frac{\hbar}{2i}\frac{\partial}{\partial q}\right) B_2(q,p) \tag{4}$$

(for a single mode), in which $\hat{B}_1 = \hat{\rho}$ and $\hat{B}_2 = \hat{\rho}$ for our purpose. Then we have, with $\hat{A} = \hat{B}_1\hat{B}_2$, the second moment of probability

$$\begin{aligned}\overline{p^2} &= \int dqdp\, W_\rho(q,p)\, A(q,p) \\ &= (2\pi\hbar)^2 \int dqdp\, W_\rho(q,p)\, W_\rho\left(q - \frac{\hbar}{2i}\frac{\partial}{\partial p}, p + \frac{\hbar}{2i}\frac{\partial}{\partial q}\right) W_\rho(q,p)\,. \end{aligned}\tag{5}$$

With the help of the Taylor expansion given by $W_\rho(q + h_q, p + h_p) = \sum_{k=0}^\infty (1/k!)(h_q\,\partial_q + h_p\,\partial_p)^k\, W_\rho(q,p)$ with $h_q = -(\hbar/2i)\,\partial_p$ and $h_p = (\hbar/2i)\,\partial_x$, we can obtain, on the right-hand side of (5),

$$W_\rho\left(q - \frac{\hbar}{2i}\frac{\partial}{\partial p}, p + \frac{\hbar}{2i}\frac{\partial}{\partial q}\right) W_\rho(q,p) = \sum_{k=0} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k (\partial_{1q}\,\partial_{2p} - \partial_{1p}\,\partial_{2q})^k\; W_1(q,p)\, W_2(q,p)\,, \tag{6}$$

where the operators ∂_1 and ∂_2 affect $W_1 = W_\rho$ and $W_2 = W_\rho$ alone, respectively. Due to the symmetry between W_1 and W_2, all terms with k odd can be shown to vanish indeed. It is also easy to see that in the classical limit of $\hbar \to 0$, only the term of $k = 0$ is non-vanishing, and so Eq. (6) will reduce to its classical counterpart $\{W_{\text{cl}}(q,p)\}^2$.

Now we generalize this single-mode expression into that of N modes. Let $\vec{q} = (q_1, q_2, \cdots, q_N)^T$ and $\vec{p} = (p_1, p_2, \cdots, p_N)^T$. Then, Eq. (6) will easily be transformed into

$$\begin{aligned} & W_\rho\left(\sum_{n=1}^N \left(q_n - \frac{\hbar}{2i}\frac{\partial}{\partial p_n}\right), \sum_{n=1}^N \left(p_n + \frac{\hbar}{2i}\frac{\partial}{\partial q_n}\right)\right) \cdot W_\rho(\vec{q},\vec{p}) \\ &= \sum_{k=0} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \sum_{n=1}^N (\partial_{1q_n}\partial_{2p_n} - \partial_{1p_n}\partial_{2q_n})^k\; W_1(\vec{q},\vec{p})\, W_2(\vec{q},\vec{p})\,. \end{aligned}\tag{7}$$

We substitute into (7) the expression

$$W_\rho(\vec{q},\vec{p}) = \int \frac{d^N\vec{\mathfrak{q}}\, d^N\vec{\mathfrak{p}}}{(2\pi\hbar)^N}\, \widetilde{W}_\rho(\vec{\mathfrak{q}},\vec{\mathfrak{p}})\, \exp\left\{-\frac{i}{\hbar}\sum_n (q_n\mathfrak{p}_n + p_n\mathfrak{q}_n)\right\} \tag{8}$$

where the symbol $\widetilde{W}_\rho(\vec{\mathfrak{q}},\vec{\mathfrak{p}})$ denotes the Fourier transform of $W_\rho(\vec{q},\vec{p})$. After making some algebraic manipulations, we can finally transform (7) into the compact form

$$\int \frac{d^{2N}\vec{x}_1}{(2\pi\hbar)^N}\frac{d^{2N}\vec{x}_2}{(2\pi\hbar)^N}\,\widetilde{W}_\rho(\vec{x}_1)\,\widetilde{W}_\rho(\vec{x}_2)\,\exp\left\{-\frac{i}{\hbar}\,(\vec{x})^T\,\hat{\Lambda}\,(\vec{x}_1+\vec{x}_2)\right\}\,\exp\left\{\frac{i}{2\hbar}(\vec{x}_1)^T\,\hat{\Omega}\,\vec{x}_2\right\}\,, \tag{9}$$

in which the vector $\vec{x} = (q_1,p_1,\cdots,q_N,p_N)^T \in \mathbb{R}^{2N}$, and $\hat{\Lambda} = \oplus_{n=1}^{N}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, as well as $\hat{\Omega} = \oplus_{n=1}^{N}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$; for a single mode, we have the reduced expressions here, $(\vec{x})^T\,\hat{\Lambda}\,(\vec{x}_1+\vec{x}_2) \to q\,(p_1+p_2)+p\,(q_1+q_2)$ and $(\vec{x}_1)^T\,\hat{\Omega}\,\vec{x}_2 \to q_1\,p_2 - p_1\,q_2$. It is also easy to note that this integral form is real-valued, by exchanging the variables $(\vec{x}_1 \leftrightarrow \vec{x}_2)$ with the relation $(\vec{x}_1)^T\,\hat{\Omega}\,\vec{x}_2 = -(\vec{x}_2)^T\,\hat{\Omega}\,\vec{x}_1$. Then the second moment is given by [cf. (5)-(7) and (9)]

$$\overline{p^2} = (2\pi\hbar)^N \int d^{2N}\vec{x}\;W_\rho(\vec{x})\,W_{\rho^2}(\vec{x})\,, \tag{10}$$

where $W_{\rho^2}(\vec{x})$ denotes the integral form (9) multiplied by $(2\pi\hbar)^N$.

Along the lines similar to the case of $\alpha = 3$, we can study the next case of $\alpha = 4$. By applying the Bopp rule (4) for $A(q,p) = (2\pi\hbar)^3\,(B_1B_2B_3)(q,p)$ with $\hat{B}_1 = \hat{B}_2 = \hat{B}_3 = \hat{\rho}$, we can have the third moment of probability

$$\begin{aligned}\overline{p^3} &= \int dqdp\,W_\rho(q,p)\,A(q,p) \qquad (11)\\ &= (2\pi\hbar)^3 \int dqdp\,W_\rho(q,p)\,W_\rho\left(q-\frac{\hbar}{2i}\frac{\partial}{\partial p},p+\frac{\hbar}{2i}\frac{\partial}{\partial q}\right)\,W_\rho\left(q-\frac{\hbar}{2i}\frac{\partial}{\partial p},p+\frac{\hbar}{2i}\frac{\partial}{\partial q}\right)\,W_\rho(q,p)\end{aligned}$$

for a single mode. Following the steps provided in (6)-(9) for the second moment, it is straightforward to derive an expression of the third moment for N modes, which will exactly be the counterpart to (10). We easily observe that the same techniques will be employed also for $\alpha = 5,6,7,\cdots$. In fact, we can finally derive an expression of the jth moment, with $j = \alpha - 1$, for N modes

$$\overline{p^j} = \mathrm{Tr}(\hat{\rho}^{j+1}) = (2\pi\hbar)^N \int d^{2N}\vec{x}\,W_\rho(\vec{x})\,W_{\rho^j}(\vec{x})\,, \tag{12}$$

which is valid for an arbitrary state $\hat{\rho}$ with $j = 1,2,3,\cdots$ [cf. (10)]. Here, a factor of the integrand

$$\begin{aligned}W_{\rho^j}(\vec{x}) &= (2\pi\hbar)^{(j-1)N}\left[W_\rho\left(\sum_{n=1}^{N}\left(q_n-\frac{\hbar}{2i}\frac{\partial}{\partial p_n}\right),\sum_{n=1}^{N}\left(p_n+\frac{\hbar}{2i}\frac{\partial}{\partial q_n}\right)\right)\right]^{j-1} W_\rho(\vec{x}) \qquad (13)\\ &= \int \frac{d^{2N}\vec{x}_1\cdots d^{2N}\vec{x}_j}{(2\pi\hbar)^N}\left\{\prod_{\nu=1}^{j}\widetilde{W}_\rho(\vec{x}_\nu)\right\}\,\exp\left\{-\frac{i}{\hbar}\,(\vec{x})^T\,\hat{\Lambda}\,\sum_{\nu}^{j}\vec{x}_\nu\right\}\,\exp\left\{\frac{i}{2\hbar}\sum_{\nu<\mu}^{j}(\vec{x}_\nu)^T\,\hat{\Omega}\,\vec{x}_\mu\right\}\end{aligned}$$

[cf. (9) for $j = 2$]. Consequently, we can explicitly discuss the Rényi-α entropies $S_\alpha(\hat{\rho})$ where $\alpha = j+1 = 2,3,4,\cdots$. In the next sections, we will evaluate those moments of probability and the corresponding entropies in closed form for some specific states.

For comparison, we briefly consider two additional entropies now. The first one is the von-Neumann entropy $S_{\mathrm{vN}}(\hat{\rho})$. It is easy to rewrite this entropy as

$$\begin{aligned}S_1(\hat{\rho}) &= -\sum_\nu p_\nu\,\ln\{\overline{p}\,(p_\nu/\overline{p})\} = -\ln(\overline{p}) + \sum_\nu p_\nu \sum_{\mu=2}\frac{1}{\mu}\,\{1-p_\nu\,(\overline{p})^{-1}\}^\mu\\ &= S_2(\hat{\rho}) + \sum_{\mu=2}\frac{(-1)^\mu\,\mathcal{M}_\mu}{\mu\,(\overline{p})^\mu}\,, \qquad (14)\end{aligned}$$

where the central moments $\mathcal{M}_\mu := \overline{(p-\overline{p})^\mu} = \mathrm{Tr}(\hat{\rho}\,\{\hat{\rho}-\mathrm{Tr}(\hat{\rho}^2)\}^\mu)$. Therefore, the difference between $S_1(\hat{\rho})$ and $S_2(\hat{\rho})$ is explicitly given by that sum of all higher-order moments, each of which can be evaluated with the help of the Bopp rule, as shown. The second entropy is the Wigner entropy (for an N-mode state), explicitly given by

$$S_W(\hat{\rho}) = -\int d^{2N}\vec{x}\;W_\rho(\vec{x})\;\ln\{\hbar^N\,W_\rho(\vec{x})\}\,, \tag{15}$$

which is well-defined so long as $W_\rho(\vec{x}) \geq 0$ over the phase space. This also can be expanded as

$$S_W(\hat{\rho}) = \sum_{\nu=1} \frac{(-1)^\nu}{\nu} \int d^{2N}\vec{x}\; W_\rho(\vec{x})\, \{\hbar^N\, W_\rho(\vec{x}) - 1\}^\nu\,. \tag{16}$$

As seen, the Bopp rule was not at all employed here for each ν such that $\int \{W_\rho(\vec{x})\}^\nu \not\propto \mathrm{Tr}(\hat{\rho}^\nu)$ where $\nu \geq 3$. Therefore, the Wigner entropy, albeit well-defined, cannot directly be related to the von-Neumann entropy.

III. RÉNYI ENTROPIES OF N-MODE GAUSSIAN STATES

We first consider the case of an N-mode Gaussian state, which is explicitly given by [8, 9]

$$W_\rho(\vec{x}) = \frac{\exp\{-(\vec{x}-\vec{d})^T\, (\hat{\sigma})^{-1}\, (\vec{x}-\vec{d})\}}{\pi^N\, \{\det(\hat{\sigma})\}^{1/2}}\,, \tag{17}$$

in which the vector $\vec{x} \in \mathbb{R}^{2N}$ as defined above, and the first moments $d_j := \overline{x}_j = \mathrm{Tr}(\hat{\rho}\, \hat{\mathcal{X}}_j)$ where the vector of operators $\vec{\mathcal{X}} = (\hat{Q}_1, \hat{P}_1, \cdots, \hat{Q}_N, \hat{P}_N)^T$, as well as the $2N$-by-$2N$ matrix $\hat{\sigma} := 2\,\hat{\underset{\sim}{\sigma}}$ where the covariance matrix $\hat{\underset{\sim}{\sigma}}$ denotes the corresponding second moments, given by $\underset{\sim}{\sigma}_{jk} = \mathrm{Tr}[\hat{\rho}\,(\hat{\mathcal{X}}_j\, \hat{\mathcal{X}}_k + \hat{\mathcal{X}}_k\, \hat{\mathcal{X}}_j)/2] - d_j\, d_k$. Then, $\mathrm{Tr}(\hat{\rho}^2) = \hbar^N\, \{\det(\sigma)\}^{-1/2}$. As is well-known, the Fourier transform of (17) is Gaussian, too, and explicitly given by

$$\widetilde{W}_\rho(\vec{x}) = \frac{\exp\{-(4\hbar^2)^{-1}\, (\vec{x}-\vec{d})^T\, \hat{\Lambda}\, \hat{\sigma}\, \hat{\Lambda}\, (\vec{x}-\vec{d})\}}{(2\pi\hbar)^N}\,, \tag{18}$$

which can be acquired by applying the L-dimensional Gaussian integral for $L = 2N$ [19]

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_L\, \exp\left\{-\frac{1}{2}\,(\vec{y})^T\, \hat{\Upsilon}\, \vec{y} + (\vec{v})^T\, \vec{y}\right\} = \left\{\frac{(2\pi)^L}{\det(\hat{\Upsilon})}\right\}^{1/2} \exp\left\{\frac{1}{2}\,(\vec{v})^T\, \hat{\Upsilon}^{-1}\, \vec{v}\right\} \tag{19}$$

with $\vec{y}, \vec{v} \in \mathbb{R}^L$. Here the symbol $\hat{\Upsilon}$ denotes a symmetric L-by-L matrix. Since the first moments d_j's give rise to the displacement of Wigner function (17) but do not change its shape, we will set $\vec{d} = 0$ from now on, without loss of generality for our discussion of the probability moments and Rényi-α entropies which remain unchanged with respect to this displacement.

Now we substitute (17)-(18) into the framework (12)-(13) and then apply (19) for $L = 2N(j+1)$ with $\vec{v} = 0$ and $dy_1 \cdots dy_L = d^{2N}\vec{x} \prod_{\nu=1}^{j} d^{2N}\vec{x}_\nu$ in order to acquire the probability moments $\overline{p^j}$ in closed form. This will easily result in

$$\overline{p^j} = \mathrm{Tr}(\hat{\rho}^{j+1}) = (2\hbar)^{-jN}\, \{\det(\sigma)\}^{-1/2}\, \{\det(\hat{D}_{j+1})\}^{-1/2}\,, \tag{20}$$

where $\hat{\Upsilon} \to \hat{D}_{j+1}$ in (19) can be expressed as a $(j+1)$-by-$(j+1)$ block symmetric matrix, each block of which is a $2N$-by-$2N$ matrix; for the simplest case of $(N = 1, j = 1)$, we explicitly have the 4-by-4 symmetric matrix

$$\hat{D}_2 = \begin{pmatrix} \frac{\sigma_{22}}{4\hbar^2} & \frac{\sigma_{12}}{4\hbar^2} & 0 & \frac{i}{2\hbar} \\ \frac{\sigma_{12}}{4\hbar^2} & \frac{\sigma_{11}}{4\hbar^2} & \frac{i}{2\hbar} & 0 \\ 0 & \frac{i}{2\hbar} & \frac{\sigma_{22}}{\det(\hat{\sigma})} & \frac{-\sigma_{12}}{\det(\hat{\sigma})} \\ \frac{i}{2\hbar} & 0 & \frac{-\sigma_{12}}{\det(\hat{\sigma})} & \frac{\sigma_{11}}{\det(\hat{\sigma})} \end{pmatrix}\,, \tag{21}$$

where $\det(\hat{\sigma}) = \sigma_{11}\, \sigma_{22} - (\sigma_{12})^2$. The matrices $\hat{D}_{j+1}$'s for N modes are formally provided in Tab. I.

Now we will evaluate $\det(\hat{D}_{j+1})$ with the help of the recurrence relation [20]

$$\det(\hat{D}_{j+1}) = \det(\hat{G}_j)\, \det(\hat{D}_j) = \det(\hat{G}_j)\, \det(\hat{G}_{j-1}) \cdots \det(\hat{G}_1)\, \{\det(\hat{\sigma})\}^{-1}, \tag{22}$$

TABLE I: The symmetric $2N(j+1)$-by-$2N(j+1)$ matrix $\hat{D}_{j+1}$, used for an evaluation of $\overline{p^j}$ in (20), is now expressed as a $(j+1)$-by-$(j+1)$ block matrix, where $j = 1,2,3,4,5,6$. Here $\vec{x} \in \mathbb{R}^{2N}$ and $\vec{x}_j \in \mathbb{R}^{2N}$. Consequently, $\hat{D}_7$ is a 7-by-7 block matrix covering $(\vec{x}, \vec{x}_1, \cdots, \vec{x}_6)$; $\hat{D}_6$ is a 6-by-6 block matrix covering $(\vec{x}, \vec{x}_1, \cdots, \vec{x}_5)$; $\cdots$; $\hat{D}_2$ is a 2-by-2 block matrix covering $(\vec{x}, \vec{x}_1)$. Besides, $\hat{D}_1$ is a 1-by-1 block matrix covering $\vec{x}$ only. In fact, $\hat{D}_1 = (\sigma)^{-1}$ as seen, and $\mathrm{Tr}(\hat{\rho}) = 1$ in (20), thus representing the trivial case. The explicit expression of $\hat{D}_{j+1}$ for $j = 7,8,9,\cdots$ will easily follow in the same way.

$$\hat{D}_7 = \begin{array}{|c|c|c|c|c|c|c||c}
\vec{x}_6 & \vec{x}_5 & \vec{x}_4 & \vec{x}_3 & \vec{x}_2 & \vec{x}_1 & \vec{x} & \\ \hline\hline
\hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & i\hat\Lambda/(2\hbar) & \vec{x}_6 \\ \hline
i\hat\Omega/(4\hbar) & \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & i\hat\Lambda/(2\hbar) & \vec{x}_5 \\ \hline
i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & i\hat\Lambda/(2\hbar) & \vec{x}_4 \\ \hline
i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & \hat\Omega/(4i\hbar) & \hat\Omega/(4i\hbar) & i\hat\Lambda/(2\hbar) & \vec{x}_3 \\ \hline
i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & \hat\Omega/(4i\hbar) & i\hat\Lambda/(2\hbar) & \vec{x}_2 \\ \hline
i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & i\hat\Omega/(4\hbar) & \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2) & i\hat\Lambda/(2\hbar) & \vec{x}_1 \\ \hline
i\hat\Lambda/(2\hbar) & i\hat\Lambda/(2\hbar) & i\hat\Lambda/(2\hbar) & i\hat\Lambda/(2\hbar) & i\hat\Lambda/(2\hbar) & i\hat\Lambda/(2\hbar) & (\hat\sigma)^{-1} & \vec{x} \\ \hline\hline
\end{array}$$

in which the generator $\det(\hat{G}_j) = \det\{\hat{A} - \hat{B}_j\,(\hat{D}_j)^{-1}\,\hat{C}_j\}$ (cf. Fig. 1). Here $\hat{A} = \hat\Lambda\hat\sigma\hat\Lambda/(4\hbar^2)$ is a 1-by-1 block matrix, as shown in Tab. I; $\hat{B}_j$ is a 1-by-j block matrix; $\hat{C}_j$ is a j-by-1 block matrix. Eq. (20) then reduces to a simple recursive form

$$\overline{p^j} = (2\hbar)^{-jN}\,\{\det(\hat{G}_j)\,\det(\hat{G}_{j-1})\cdots\det(\hat{G}_1)\}^{-1/2} = (2\hbar)^{-N}\,\{\det(\hat{G}_j)\}^{-1/2}\,\overline{p^{j-1}}\,, \tag{23}$$

where $\overline{p^0} = 1$. Now it is easy to see that the key ingredient is the generator $\det(\hat{G}_j)$. After some algebraic manipulations, every single step of which is provided in detail in the Appendix, we can finally arrive at the closed expression

$$\begin{aligned}
\det(\hat{G}_j) &= \frac{\mathcal{D}_\sigma}{(2\hbar)^{2N}}\left[\frac{1}{2} + (\mathcal{D}_\sigma)^{-1/2}\left\{\left[1 - \left\{\frac{(\mathcal{D}_\sigma)^{1/2}-1}{(\mathcal{D}_\sigma)^{1/2}+1}\right\}^j\right]^{-1} - \frac{1}{2}\right\}\right]^2 \\
&= \frac{1}{(2\hbar)^{2N}}\left\{\frac{F(j+1)}{2\,F(j)}\right\}^2\,,
\end{aligned} \tag{24}$$

in which the dimensionless quantities $\mathcal{D}_\sigma := (\det\hat\sigma)/\hbar^{2N}$ and $F_\sigma(j) := \{(\mathcal{D}_\sigma)^{1/2}+1\}^j - \{(\mathcal{D}_\sigma)^{1/2}-1\}^j$. Plugging this into (23), we can then obtain the jth moments of probability in closed form for an arbitrary N-mode Gaussian state, given by

$$\overline{p^j} = 2^{j+1}\,\{F_\sigma(j+1)\}^{-1}\,. \tag{25}$$

We easily observe that if $\mathcal{D}_\sigma = 1$, equivalent to $\hat\rho$ denoting a pure state and thus leading to $F_\sigma(j+1) = 2^{j+1}$, then $\overline{p^j} = 1$ for all j's. Some of the probability moments are explicitly provided in Tab. II. As expected, all higher-order

TABLE II: Explicit expressions of the probability moments for an arbitrary N-mode Gaussian state.

$\overline{p^0} = \mathrm{Tr}(\hat\rho^1) = 1$
$\overline{p^1} = \mathrm{Tr}(\hat\rho^2) = (\mathcal{D}_\sigma)^{-1/2}$
$\overline{p^2} = \mathrm{Tr}(\hat\rho^3) = 4\,(3\,\mathcal{D}_\sigma + 1)^{-1}$
$\overline{p^3} = \mathrm{Tr}(\hat\rho^4) = 2\,\{(\mathcal{D}_\sigma)^{1/2}\,(\mathcal{D}_\sigma + 1)\}^{-1}$
$\overline{p^4} = \mathrm{Tr}(\hat\rho^5) = 16\,\{1 + 10\,\mathcal{D}_\sigma + 5\,(\mathcal{D}_\sigma)^2\}^{-1}$
$\overline{p^5} = \mathrm{Tr}(\hat\rho^6) = 16\,\{(\mathcal{D}_\sigma)^{1/2}\,(\mathcal{D}_\sigma + 3)(3\,\mathcal{D}_\sigma + 1)\}^{-1}$
$\overline{p^6} = \mathrm{Tr}(\hat\rho^7) = 64\,\{1 + 21\,\mathcal{D}_\sigma + 35\,(\mathcal{D}_\sigma)^2 + 7\,(\mathcal{D}_\sigma)^3\}^{-1}$

moments are expressed in terms of the first moment $\overline{p}$. The resulting Rényi-α entropies are plotted in Fig. 2. For comparison, it is instructive to note that for a (classical) Gaussian distribution $f(y)$, the moments $M_l = \overline{(y-\overline{y})^l} = 0$

for l odd while $M_l = (l-1)!!\,(M_2)^{l/2}$ for l even such that $M_4 = 3\,(M_2)^2, M_6 = 15\,(M_2)^3, \cdots$, as is well-known [21]. A comment is deserved here. Alternatively, we can derive Eq. (24) for a single mode only and then generalize this into its N-mode counterpart, thanks to the Williamson decomposition [8, 9, 22].

In comparison with the Williamson decomposition and for a later purpose, we consider a simple system in a single-mode Gaussian state now. This system is a linear oscillator coupled at an arbitrary strength to a heat bath consisting of N_b uncoupled oscillators where $N_b \gg 1$ (the Brownian oscillator) [23]. The *total* system of $N = N_b + 1$ modes, composed of oscillator $(\hat{Q}, \hat{P})$ and bath $\{(\hat{Q}_n, \hat{P}_n)|n = 1, 2, \cdots, N_b\}$, is assumed to be in the canonical thermal equilibrium state $\hat{\rho}_\beta \propto \exp(-\beta\hat{H})$ where $\beta = 1/(k_\mathrm{B}T)$ and the total-system Hamiltonian $\hat{H}$. Then the reduced state of the coupled oscillator only is given by [24]

$$\langle q|\hat{\mathsf{R}}|q'\rangle = \frac{1}{\sqrt{2\pi\,\langle\hat{Q}^2\rangle_\beta}}\,\exp\left\{-\frac{(q+q')^2}{8\,\langle\hat{Q}^2\rangle_\beta} - \frac{\langle\hat{P}^2\rangle_\beta\,(q-q')^2}{2\hbar^2}\right\}\,, \tag{26}$$

in which $\langle\hat{Q}^2\rangle_\beta = \mathrm{Tr}(\hat{\rho}_\beta\hat{Q}^2)$ and $\langle\hat{P}^2\rangle_\beta = \mathrm{Tr}(\hat{\rho}_\beta\hat{P}^2)$. This density matrix is, in general, *not* in diagonal form in the energy eigenbasis [25]. In the limit of vanishingly weak coupling, this state reduces to the canonical equilibrium state, $\hat{\mathsf{R}} \to \hat{\mathsf{R}}_\beta$ indeed, as required, thus yielding $\langle\hat{Q}^2\rangle_\beta \to (2\kappa^2)^{-1}\coth(\beta\hbar\omega/2)$ and $\langle\hat{P}^2\rangle_\beta \to (\hbar^2\kappa^2/2)\coth(\beta\hbar\omega/2)$ where $\kappa := (M\omega/\hbar)^{1/2}$, as is well-known [26].

The Wigner representation of (26) is then given by a Gaussian form

$$W_{\mathsf{R}}(q,p) = \frac{1}{2\pi\sqrt{\langle\hat{Q}^2\rangle_\beta\,\langle\hat{P}^2\rangle_\beta}}\,\exp\left\{-\frac{1}{2}\left(\frac{q^2}{\langle\hat{Q}^2\rangle_\beta} + \frac{p^2}{\langle\hat{P}^2\rangle_\beta}\right)\right\}\,, \tag{27}$$

where the (2-by-2) covariance matrix is in diagonal form, $(\hat{\underset{\sim}{\sigma}})_{jk} = \underset{\sim}{\sigma}_j\,\delta_{jk}$ with $\underset{\sim}{\sigma}_1 = \langle\hat{Q}^2\rangle_\beta$ and $\underset{\sim}{\sigma}_2 = \langle\hat{P}^2\rangle_\beta$. It follows that $\mathcal{D}_\sigma = (2v)^2$ where $v = (\underset{\sim}{\sigma}_1\,\underset{\sim}{\sigma}_2)^{1/2}/\hbar$. Only in the limit of vanishingly weak coupling (i.e., the canonical thermal equilibrium), we have $v = \overline{n} + (1/2)$ where the average number of quanta $\overline{n} = \{\exp(\beta\hbar\omega) - 1\}^{-1}$, as well as the two components become identical, $\underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_2$ with $\hbar = 1$ and $\kappa = 1$, which formally corresponds to a 1-mode component $(\underset{\sim}{\sigma}_1)$ of the symplectic spectrum for the $2N$-by-$2N$ covariance matrix of a general Gaussian state in N modes, as in the Williamson decomposition [8, 9].

Two additional comments are deserved here: First, if a density matrix $\hat{\rho}$ is restricted to a Gaussian form only, one can easily relate the eigenvalues $(\underset{\sim}{\sigma}_j)$ of the covariance matrix to the eigenvalues of $\hat{\rho}$ (in the Hilbert space) and use those eigenvalues $\underset{\sim}{\sigma}_j$ to compute $\mathrm{Tr}(\hat{\rho}^\alpha)$ directly, without resort to our general framework in the phase space: We now show this, for simplicity, for the single-mode case only (note that its N-mode generalization is straightforward). The most general single-mode Gaussian state can be expressed as a displaced squeezed thermal state (e.g., [9]) such that $\hat{\rho} = \hat{D}(\alpha)\,\hat{S}(z)\,\hat{\rho}_\beta\,\hat{S}^\dagger(z)\,\hat{D}^\dagger(\alpha)$ where the displacement operator $\hat{D}(\alpha)$ and the squeezing operator $\hat{S}(z)$, easily leading to $\mathrm{Tr}(\hat{\rho}^\alpha) = \mathrm{Tr}\{(\hat{\rho}_\beta)^\alpha\}$. It is also true that $\hat{\sigma} = \nu\,\hat{\mathbb{1}}$ for $\hat{\rho}_\beta$, where $\nu = 2\bar{n} + 1$, and $\hat{\rho}_\beta = \sum_k\{(\bar{n})^k/(\bar{n}+1)^{k+1}\}\,|k\rangle\langle k|$, yielding $\mathrm{Tr}(\hat{\rho}^\alpha) = \{(\bar{n}+1)^\alpha - \bar{n}^\alpha\}^{-1}$. This confirms the validity of Eq. (25) with $\mathcal{D}_\sigma = \nu^2$.

Second, it is easy to see from a harmonic oscillator in the thermal equilibrium $\hat{\rho}_\beta$ [cf. (27)] that $\mathcal{D}_\sigma \to 4\,(\beta\hbar\omega)^{-2} + 2/3 + \mathcal{O}(\hbar^2)$ in the limit of $\hbar \to 0$, where we used the series, $\coth(x) = x^{-1} + 3^{-1}\,x + \mathcal{O}(x^3)$ [27]. Therefore the quantity $\mathcal{D}_\sigma$ and the resulting Rényi-α entropies obviously diverge within this limit, as is the case with the von-Neumann entropy in the same limit for a (classical) harmonic oscillator, as is well-known. This divergence also confirms the uniform behavior of all Rényi-α entropies in the classical limit tending asymptotically to the von-Neumann entropy (also note the remark in the second last paragraph of Sec. I).

Now we consider the analytic continuation of the probability moment given in (25) by extending its domain from $\{j|j = 1, 2, 3, \cdots\}$ to $\{\alpha|\alpha \in \mathbb{R} > 0\}$. This enables us to introduce the Rényi-α entropy of an arbitrary N-mode Gaussian state such that

$$S_\alpha(\hat{\rho}) = \frac{\alpha\ln(2) - \ln\{F_\sigma(\alpha)\}}{1-\alpha} \tag{28}$$

with $\alpha \neq 1$, which is well-defined. Obviously, this vanishes for all α's if $\mathcal{D}_\sigma = 1$, as required. We are next interested in $S_1(\hat{\rho})$, which is equivalent to the von-Neumann entropy $S_{\mathrm{vN}}(\hat{\rho})$. By applying L'Hopital's rule to (28), we can easily acquire

$$S_1(\hat{\rho}) = \left\{\frac{(\mathcal{D}_\sigma)^{1/2}+1}{2}\right\}\ln\left\{\frac{(\mathcal{D}_\sigma)^{1/2}+1}{2}\right\} - \left\{\frac{(\mathcal{D}_\sigma)^{1/2}-1}{2}\right\}\ln\left\{\frac{(\mathcal{D}_\sigma)^{1/2}-1}{2}\right\}\,. \tag{29}$$

As a special case, $S_1(\hat{\mathsf{R}}) \to S_v = (v+1/2)\,\ln(v+1/2) - (v-1/2)\,\ln(v-1/2)$ for a coupled oscillator in the thermal equilibrium state [cf. (26)], being a well-known expression (e.g., [25]). In fact, it is also possible to arrive at the expression (29) independently by using Eq. (14) with (25) and $\mathcal{M}_\mu = \sum_{j=0}^{\mu} \binom{\mu}{j} (-\overline{p})^{\mu-j}\, \overline{p^j}$. This accordance implies the validity of our analytic continuation.

IV. RÉNYI ENTROPIES OF N-MODE NON-GAUSSIAN STATES

Motivated by the fact that the Gaussian state is a generalization of the ground state $|0\rangle$, we now intend to discuss a generalization of the excited states $|n\rangle$ where $n = 1, 2, 3, \cdots$. We first note that the Wigner function of the eigenstate $|n\rangle$ for a single mode is explicitly given by [14, 28, 29]

$$W_{|n\rangle}(q,p) = \frac{(-1)^n}{\pi\hbar}\, \exp\left\{-\Xi(q,p)\right\}\; L_n\left\{2\,\Xi(q,p)\right\}\,, \tag{30}$$

in which the symbol $\Xi(q,p) = \{(\hbar\kappa)^{-1}\, p\}^2 + (\kappa q)^2$, and the Laguerre polynomial $L(x) = \sum_{k=0}^{n}(-1)^k\,(k!)^{-1}\binom{n}{k}\,x^k$ [27]. Also, the Wigner functions $W_{|n\rangle}$ of pure states ($n \neq 0$) are not Gaussian and thus can be negative-valued [30]. If not restricted to pure states, however, it is possible to discuss non-Gaussian mixed states with their non-negative Wigner functions. To do so, we rewrite (30) as the linear combination

$$W_{|n\rangle}(q,p) = \sum_{k=0}^{n} a_{nk}\; W_{\underline{k}}(q,p)\,, \tag{31}$$

where the coefficient $a_{nk} := (-1)^n\,(-2)^k \binom{n}{k}$ satisfying $\sum_{k=0}^{n} a_{nk} = 1$, and the Wigner function $W_{\underline{k}}(q,p) := \Xi^k\,(k!)^{-1}\, W_{|0\rangle}$ (non-Gaussian for $k \neq 0$) satisfying $\int dq\, dp\, W_{\underline{k}}(q,p) = 1$. Then for the first several values of n, the coefficients are explicitly given by $a_{00} = 1$; $(a_{11} = 2, a_{10} = -1)$; $(a_{22} = 4, a_{21} = -4, a_{20} = 1)$; $(a_{33} = 8, a_{32} = -12, a_{31} = 6, a_{30} = -1)$, as well as the Wigner functions $W_{\underline{0}} = W_{|0\rangle}(q,p)$; $W_{\underline{1}} = (\pi\hbar)^{-1}\,\Xi(q,p)\,\exp\{-\Xi(q,p)\} =: W_{\rho_1}(q,p) \geq 0$ corresponding to the mixed state $\hat{\rho}_1 = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$; $W_{\underline{2}} = (2\,\pi\hbar)^{-1}\,\{\Xi(q,p)\}^2\,\exp\{-\Xi(q,p)\} =: W_{\rho_2}(q,p) \geq 0$ corresponding to the mixed state $\hat{\rho}_2 = (|0\rangle\langle 0| + 2\,|1\rangle\langle 1| + |2\rangle\langle 2|)/4$.

Therefore we next consider, as a generalization of the single-mode Wigner functions $W_{\underline{k}}(q,p)$, the N-mode non-Gaussian states in form of

$$W_{\rho_\nu}(\vec{x}) = \frac{\exp\{-(\vec{x})^T\,(\hat{\sigma})^{-1}\,\vec{x}\}}{(N)_\nu\;\pi^N\,\{\det(\hat{\sigma})\}^{1/2}}\,\{(\vec{x})^T\,(\hat{\sigma})^{-1}\,\vec{x}\}^\nu\,, \tag{32}$$

where $\nu = 1, 2, 3, \cdots$, and the Pochhammer symbol $(N)_\nu = \Gamma(N+\nu)/\Gamma(N)$ [27] (note that an N-mode generalization of $W_{|n\rangle}(q,p)$ will be discussed below in terms of the linear combination of $W_{\rho_\nu}(\vec{x})$'s). This Wigner function (32) also is fully determined by the matrix elements σ_{jk}. For the normalizing (i.e., $\int d^{2N}\vec{x}\; W_{\rho_\nu}(\vec{x}) \stackrel{!}{=} 1$), we here applied the Gaussian integral (19) with the help of $e^{-y^2}\,y^{2\nu} = (-\partial_a)^\nu\, e^{-ay^2}|_{a=1}$ and $(-\partial_a)^\nu\, a^{-N}|_{a=1} = (N)_\nu$. To acquire the first moment of probability $\overline{p}_\nu = \mathrm{Tr}(\hat{\rho}_\nu^2)$ in closed form, we now discuss the integral

$$I_{\mu\nu} \;=\; (2\pi\hbar)^N \int d^{2N}\vec{x}\; W_{\rho_\mu}(\vec{x})\; W_{\rho_\nu}(\vec{x}) \;=\; \frac{1}{C_{\mu\nu}\,(\mathcal{D}_\sigma)^{1/2}}\,, \tag{33}$$

in which $C_{\mu\nu} = 2^{\mu+\nu}\,(N)_\nu/(N+\mu)_\nu = C_{\nu\mu}$. It then follows that $\overline{p}_\nu = I_{\nu\nu}$ and $S_2(\hat{\rho}_\nu) = 2^{-1}\,\ln(\mathcal{D}_{\sigma,\nu})$ where $\mathcal{D}_{\sigma,\nu} = (C_{\nu\nu})^2\,\mathcal{D}_\sigma = \{\det(\hat{\sigma}_\nu)\}/h^{2N}$ expressed in terms of the matrix $\hat{\sigma}_\nu := (C_{\nu\nu})^{1/N}\,\hat{\sigma}$. For a later purpose, we briefly take into consideration the simplest case of $N = 1$. Then we have the entropy in reduced form, $S_2(\hat{\rho}_\nu) \to 2^{-1}\,\ln(\mathcal{D}_\sigma) + \ln\{\pi^{1/2}\,\Gamma(\nu+1)/\Gamma(\nu+1/2)\}$, where we applied the identity, $\Gamma(2x) = (2\pi)^{-1/2}\,2^{2x-1/2}\,\Gamma(x)\,\Gamma(x+1/2)$ [27]. As seen, this reduced form is positive-valued so long as $\nu \geq 1$, showing that $\hat{\rho}_\nu$ denotes a mixed state, even for the case of $\mathcal{D}_\sigma = 1$, by construction. Figs. 3 and 4 plot the behaviors of $S_2(\hat{\rho}_\nu)$ versus ν and N, respectively.

Interestingly, we recognize here that the above result, valid for the non-Gaussian state (32), can exactly be recovered by introducing an effective Gaussian state (of that non-Gaussian state) with its covariance matrix $\underset{\sim}{\hat{\sigma}}_\nu = 2^{-1}\,\hat{\sigma}_\nu$, for which the Wigner entropy is obviously well-defined and explicitly given by $S_W(\hat{\rho}_{\text{eff}}) = N + N\ln(\pi) + 2^{-1}\ln(\mathcal{D}_{\sigma,\nu})$, thus coinciding with $S_2(\hat{\rho}_\nu)$ up to a constant again. This coincidence, valid also for the non-Gaussian states in an extended sense, may be regarded as a generalization of the same coincidence, as is well-known, valid for the Gaussian states.

Next we consider an arbitrary linear combination of the states given in (32) such that $W_\rho(\vec{x}) = \sum_\nu a_\nu\, W_{\rho_\nu}(\vec{x})$ where $a_\nu \in \mathbb{R}$ and $\sum a_\nu = 1$ but a_ν's are not necessarily non-negative and so the resulting Wigner function $W_\rho(\vec{x})$ can be negative-valued. Applying (33), it is straightforward to find that $S_2(\hat{\rho}) = 2^{-1}\ln(\overline{\mathcal{D}}_\sigma)$

where $\overline{\mathcal{D}}_\sigma := (\sum_{\mu,\nu} a_\mu\, a_\nu / C_{\mu\nu})^{-2}\, \mathcal{D}_\sigma$. Likewise, the Wigner entropy of its effective Gaussian state with $\hat{\sigma} \to (\sum_{\mu,\nu} a_\mu\, a_\nu / C_{\mu\nu})^{-1/N}\, \hat{\sigma} =: \underline{\hat{\sigma}}$ is exactly given by the expression of $S_W(\hat{\rho}_{\text{eff}})$ in the preceding paragraph but simply with $\mathcal{D}_{\sigma,\nu} \to \overline{\mathcal{D}}_\sigma$.

Now we restrict our discussion into the particular case that the coefficients $a_\nu \to a_{nk}$ given in (31) for a given value of n. Then the resulting Wigner function $W_\rho(\vec{x}) = \sum_k a_{nk}\, W_{\rho_k}(\vec{x})$ can be regarded, by construction, as a generalization of $W_{|n\rangle}(q,p)$ in (31), i.e., $W_{\underline{k}}(q,p)$ for a single mode replaced by $W_{\rho_k}(\vec{x})$ in (32) for N modes; as a simple example, $W_\rho(\vec{x}) = a_{11}\, W_{\rho_0}(\vec{x}) + a_{10}\, W_{\rho_1}(\vec{x})$ for $n = 1$, which is a generalization of $W_{|1\rangle}(q,p)$. The explicit expressions of a_{nk} and $C_{\mu\nu}$ will give, after some algebraic manipulations, the reduced expression, $(\sum_{\mu,\nu} a_\mu\, a_\nu / C_{\mu\nu})^{-1} \to \binom{n+N-1}{N-1}$. This immediately yields the entropy $S_2(\hat{\rho}) = 2^{-1} \ln \mathcal{D}_\sigma + \ln \binom{n+N-1}{N-1}$ in reduced form, in which the second term increases with n and N. It is also instructive to note that for the case of $N = 1$, the quantity $\binom{n+N-1}{N-1}$ reduces to unity, regardless of the value of n, therefore the entropy $S_2(\hat{\rho}) \to 2^{-1} \ln(\mathcal{D}_\sigma)$. This entropy will vanish not only for the pure Gaussian state $|0\rangle$ with $\mathcal{D}_\sigma = 1$, but also, remarkably, for the pure non-Gaussian state $|n\rangle$ with $n \neq 0$, as required, due to the fact that in this case, by construction, the effective covariance matrix $2^{-1}\, \underline{\hat{\sigma}}$ reduces to its Gaussian-state counterpart $2^{-1}\, \hat{\sigma}$ indeed, independently of n! This shows a robust structure of the current generalization into the non-Gaussian states. It may also be legitimate to say that this generalization of the basis states $|n\rangle$'s will shed new light on the study of the Rényi entropies for a broader class of non-Gaussian states to be studied later.

Next let us briefly discuss higher-order moments $\overline{p^j}$ with $j = 2, 3, 4, \cdots$, and the resulting entropies $S_{j+1}(\hat{\rho})$ for the non-Gaussian states. To apply the same techniques including the Gaussian integral (19) as for $S_2(\hat{\rho})$, we express the Wigner function as

$$W_\rho(\vec{x}) = \frac{\exp\{-a\, (\vec{x})^T (\hat{\sigma})^{-1}\, \vec{x}\}}{\pi^N \{\det(\hat{\sigma})\}^{1/2}} \tag{34}$$

up to the normalizing, where the parameter a will be set unity. Then, the Fourier transform is in form of

$$\widetilde{W}_\rho(\vec{x}) = \frac{\exp\{-(4a\hbar^2)^{-1}\, (\vec{x})^T \hat{\Lambda}\, \hat{\sigma}\, \hat{\Lambda}\, \vec{x}\}}{(2\pi\hbar\, a)^N}\,. \tag{35}$$

Using (34) and (35) in place of (17) and (18), it is straightforward to acquire

$$\overline{p^j} = (2\hbar)^{-jN}\, (\mathcal{D}_\sigma)^{-1/2}\, \{\det(\hat{D}_{j+1})\}^{-1/2}\, (a_1)^{-N}\, (a_2)^{-N} \cdots (a_j)^{-N} \tag{36}$$

[cf. (20)] where $\det(\hat{D}_{j+1})$ given in Tab. I but simply with the replacement of $\hat{\sigma} \to (a_\mu)^{-1}\, \hat{\sigma}$ where $\mu = 1, 2, \cdots j$, as well as with $(\hat{\sigma})^{-1} \to a\, (\hat{\sigma})^{-1}$ in its diagonal elements. Eq. (36) can be evaluated exactly the same way (20) was evaluated in Sec. III. Then we perform the differentiation $(-\partial_a)^\nu\, (-\partial_{a_1})^\nu\, (-\partial_{a_2})^\nu \cdots (-\partial_{a_j})^\nu$ and set $a, a_\mu = 1$, followed by inserting the normalizing constants $\{(N)_\nu\}^{-1}$'s. Its explicit expression and that of the resulting Rényi entropy are straightforward to evaluate but simply too large in size, and so here, e.g., only for $N = 1$ and $\nu = 1, 2, 3$,

$$\begin{aligned}
\mathrm{Tr}\{(\hat{\rho}_1)^3\} &= \frac{4}{3^2\, z^4}\, (2z^3 + 3z^2 - 12z + 16) \\
\mathrm{Tr}\{(\hat{\rho}_2)^3\} &= \frac{4}{3^4\, z^7}\, (10z^6 + 12z^5 - 3z^4 - 74z^3 + 456z^2 - 960z + 640) \\
\mathrm{Tr}\{(\hat{\rho}_3)^3\} &= \frac{4}{3^8\, z^{10}}\, (560z^9 + 630z^8 + 180z^7 - 1221z^6 + 252z^5 + 30960z^4 - 190560z^3 \\
&\quad + 524160z^2 - 645120z + 286720)\,,
\end{aligned} \tag{37}$$

and so on, where $z := 3\mathcal{D}_\sigma + 1$ (cf. Fig. 5 for $N = 1$ and Fig. 6 for $N = 2$). For comparison, $\mathrm{Tr}\{(\hat{\rho}_0)^3\} = 4/z$ (cf. Tab. II).

Finally, we study the probability moments expressed in terms of the Husimi function briefly. In fact, it is highly tempting to do so, especially for non-Gaussian states, due to its inherent non-negative feature. For simplicity, we primarily restrict our discussion to the case of a single mode. The Husimi function can be understood as the Weierstrauss transform of the Wigner function, i.e., the convolution of the Wigner function with a Gaussian filter such that [13]

$$Q_\rho(q,p) \;=\; \frac{1}{\pi\hbar} \int dq' dp'\; W_\rho(q', p')\, \exp\left\{-\left(\{\kappa\, (q' - q)\}^2 + \left(\frac{p' - p}{\hbar\kappa}\right)^2\right)\right\}\,, \tag{38}$$

as well as

$$W_\rho(q,p) \;=\; \exp\{-(a\,\partial_q^2 + b\,\partial_p^2)\}\, Q_\rho(q,p)$$
$$= \int_{-\infty}^{\infty} \frac{dq'dp'}{2\pi\hbar}\, \exp\left\{\frac{a(p')^2}{\hbar^2} + i\frac{p'q}{\hbar}\right\} \exp\left\{\frac{b(q')^2}{\hbar^2} + i\frac{q'p}{\hbar}\right\} \widetilde{Q}_\rho(q',p') \tag{39}$$

where $a = (4\kappa^2)^{-1}$ and $b = (\hbar\kappa)^2/4$, and the Fourier transform

$$\widetilde{Q}_\rho(q',p') = \int \frac{dqdp}{2\pi\hbar}\, Q_\rho(q,p)\, \exp\left\{\frac{-i(qp'+pq')}{\hbar}\right\}. \tag{40}$$

Then a useful identity will follow,

$$\widetilde{W}_\rho(q,p) = \exp\left(\frac{ap^2 + bx^2}{\hbar^2}\right)\, (\widetilde{Q}_\rho)^*(q,p)\,, \tag{41}$$

where the complex conjugate $(\widetilde{Q}_\rho)^*(q,p) = \widetilde{Q}_\rho(-q,-p)$. Substituting (39) and (41) into (13), we can obtain

$$\overline{p} \;=\; (2\pi\hbar) \int dq'dp'\, e^{\{2a\,(p')^2 + 2b\,(q')^2\}/\hbar^2}\, |(\widetilde{Q}_\rho)^*(q',p')|^2 \tag{42}$$

$$\overline{p^2} \;=\; (2\pi\hbar) \int dq_1\, dp_1\, dq_2\, dp_2\; e^{(2a\,\{(p_1)^2+(p_2)^2+p_1\,p_2\}\,+\,2b\,\{(q_1)^2+(q_2)^2+q_1\,q_2\})/\hbar^2} \times$$
$$\widetilde{Q}_\rho(q_1+q_2, p_1+p_2)\cdot(\widetilde{Q}_\rho)^*(q_1,p_1)\cdot(\widetilde{Q}_\rho)^*(q_2,p_2)\cdot\exp\left\{\frac{i(q_1p_2 - p_1q_2)}{2\hbar}\right\}, \tag{43}$$

and so on.

Several additional comments are appropriate here. First, from the single-mode results such as (42) and (43), the N-mode counterparts will follow straightforwardly. Second, it is easy to show, with the help of (38), that if the Wigner function is Gaussian, then the Husimi function is Gaussian, too. Third, while the von-Neumann entropy of any pure state is zero, this is not the case for the Wehrl entropy defined as $(-1)\int dqdp\; Q_\rho(q,p)\, \ln\{Q_\rho(q,p)\}$ (for a non-Gaussian state) in terms of the Husimi function $Q_\rho = \langle\beta|\hat{\rho}|\beta\rangle/\pi$ where $\beta = 2^{-1/2}\,(\kappa q + ip/\hbar\kappa)$, fundamentally due to the non-orthogonality $|\langle\beta|\gamma\rangle|^2 = e^{-|\beta-\gamma|^2}$ [31]. Exploring the Rényi entropies in terms of the Wehrl entropy for the wild world of non-Gaussian states remains an open question.

V. CONCLUDING REMARKS

We have studied the Rényi-α entropies $S_\alpha(\hat{\rho})$ for both Gaussian and non-Gaussian states in N modes. We have first derived the entropies of Gaussian states in closed form for arbitrary positive integers $j \leftarrow \alpha$ and then acquired them also for real values of $\alpha > 0$ with the help of a recurrence relation between those entropies for two consecutive values of j's and then the analytic continuation. In the literature, on the other hand, the entropy $S_2(\hat{\rho})$ has been the primary quantity thus far. In fact, this entropy $S_2(\hat{\rho})$ results from the first-order moment of probability $\overline{p}$, as shown, and accordingly contains a "coarse-grained" information only. Although the statistical behaviors of Gaussian states are, as is well-known, determined essentially by its covariance matrix alone, thus giving rise to $S_2(\hat{\rho})$ directly, the concrete shapes of higher-order moments $\overline{p^\alpha} = \text{Tr}(\hat{\rho}^{\alpha+1})$ have been unknown therefore have not yet been intensively explored.

Subsequently, we have studied the Rényi-α entropies for an interesting class of non-Gaussian states which can be regarded as a generalization of the eigenstates of a single harmonic oscillator. By introducing the effective Gaussian states for the non-Gaussian states, we have also generalized the same relation between the first-moment entropy $S_2(\hat{\rho})$ and the Wigner entropy of its (effective) Gaussian state as other investigators have shown previously only for the case of Gaussian states. Because the dynamics of an N-oscillator system is Gaussian, our result will contribute to a systematic study of the entropy dynamics when the current form of a non-Gaussian state is initially prepared. The next subject will include an investigation of entanglement in terms of the Rényi entropies, also for a wider class of non-Gaussian states, as well as a systematic analysis of the quantum-classical transition for those entropies in the semiclassical limit, which could also provide an interesting and novel contribution to the study of open systems.

Acknowledgments

The author appreciates all comments and constructive questions of the referees which made him clarify the paper and improve its quality, especially in relation to the two comments after Eq. (27). He gratefully acknowledges the financial support provided by the U.S. Army Research Office (Grant No. W911NF-15-1-0145).

Appendix: Derivation of the generator in closed form: Eq. (24)

We explicitly evaluate the generator $\det(\hat{G}_j) = \det\{\hat{A} - \hat{B}_j\,(\hat{D}_j)^{-1}\,\hat{C}_j\}$, starting from the case of $j = 1$, followed by $j = 2, 3, 4, \cdots$, first for a single mode, followed by multi modes (cf. Tab. I and Fig. 1). We can then find the recurrence relation

$$\det(\hat{G}_j) = \frac{\mathcal{D}_\sigma}{(2\hbar)^{2N}} \left\{\frac{1 + U(j)}{2}\right\}^2 \tag{A.1}$$

where $\mathcal{D}_\sigma = (\det\hat{\sigma})/\hbar^{2N}$ and

$$U(j) = 1 + \frac{(\mathcal{D}_\sigma)^{-1} - 1}{U(j-1) + 1}\,. \tag{A.2}$$

Consider $U(j) + c = \lambda\,\{U(j-1) + b\}$, where $\lambda = (1+c)/\{U(j-1)+1\}$, and $b = \{c + (\mathcal{D}_\sigma)^{-1}\}/(1+c)$. Requiring now that $b \overset{!}{=} c$. Then it follows that $c = (\mathcal{D}_\sigma)^{-1/2} =: c_0$. Let $Y(j) = \{U(j) + c_0\}^{-1}$. This satisfies the recurrence relation

$$Y(j) = \alpha\,Y(j-1) + \beta\,, \tag{A.3}$$

where $\alpha = \{(\mathcal{D}_\sigma)^{1/2} - 1\}/\{(\mathcal{D}_\sigma)^{1/2} + 1\}$, and $\beta = (\mathcal{D}_\sigma)^{1/2}/\{(\mathcal{D}_\sigma)^{1/2} + 1\}$. This means that $Y(j) - \beta/(1-\alpha)$ is a geometric sequence with the common ratio $r = \alpha$, which will result in

$$Y(j) = \frac{(\mathcal{D}_\sigma)^{1/2}}{2} \left\{1 - \left(\frac{(\mathcal{D}_\sigma)^{1/2} - 1}{(\mathcal{D}_\sigma)^{1/2} + 1}\right)^j\right\}\,. \tag{A.4}$$

Consequently, we can determine the closed form of $U(j)$ and then that of $\det(\hat{G}_j)$ in (A.1) and (24).

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Fig. 2: (Color online) The Rényi-α entropies for N modes in Tab. II, $y = S_\alpha(\hat{\rho})$ versus $x = \mathcal{D}_\sigma$ for $\hat{\rho}$. The values $\alpha = 2, 3, 4, 5, 6, 7$, in sequence from top to bottom. As required, $y = 0$ at $x = 1$ for all α's.

Fig. 3: (Color online) $y = S_2(\hat{\rho}_\nu) - S_2(\hat{\rho}_0) = \ln\{4^\nu\,(N)_\nu/(N+\nu)_\nu\}$ versus $x = \nu$, in discussion after Eq. (33). Here the state $\hat{\rho}_0$ denotes a Gaussian state with $S_2(\hat{\rho}_0) = 2^{-1}\ln(\mathcal{D}_\sigma)$, and the substitution of x for integer ν may be considered the analytic continuation. The values $N = 1, 2, 3, 4, 5, 6$, in sequence from bottom to top. For a given value of $\mathcal{D}_\sigma$, as seen, each curve increases with x; especially if $\mathcal{D}_\sigma = 1$ (denoting a pure Gaussian state), then the non-Gaussian state $\hat{\rho}_\nu$ with $\nu > 0$ and $W_{\rho_\nu}(\vec{x}) \geq 0$ [cf. (32)] denotes a non-pure state by construction, so its entropies should be larger than those of the Gaussian counterparts. Note an explicit N-dependence of y here, while it is not the case for Gaussian states ($\nu = 0$).

Fig. 4: (Color online) $y = 0.2\,\{S_2(\hat{\rho}_\nu) - S_2(\hat{\rho}_0)\} = 0.2\ \ln\{4^\nu\,(N)_\nu/(N+\nu)_\nu\}$ (re-scaled) versus $x = N$, as for Fig. 3. The values $\nu = 1, 2, 3, 4, 5, 6$, in sequence from bottom to top. As seen, $y \to 0.2\ \ln(4^\nu)$ with $N \to \infty$.

Fig. 5: (Color online) The Rényi entropies $y = S_\alpha(\hat{\rho}_\nu)$ versus the $x = \mathcal{D}_\sigma$ for the non-Gaussian states with $N = 1$ [cf. Eqs. (36) and (37)]. From top to bottom, 1) the solid curves with $\nu = 3$: $\alpha = 2, 3, 4, 5$, in sequence from top to bottom; 2) the dash curves with $\nu = 2$, in the same way; 3) the dashdot curves with $\nu = 1$, in the same way.

Fig. 6: (Color online) (Color online) The Rényi entropies $y = S_\alpha(\hat{\rho}_\nu)$ versus the $x = \mathcal{D}_\sigma$ for $N = 2$. Otherwise, the same parameters as for Fig. 5.

FIG. 1: The structure of block matrix $\hat{D}_{j+1}$, used for an evaluation of its determinant.

$$\hat{D}_{j+1} = \begin{array}{|c|c|} \hline \hat{A} & \hat{B}_j \\ \hline \hat{C}_j & \hat{D}_j \\ \hline \end{array}$$

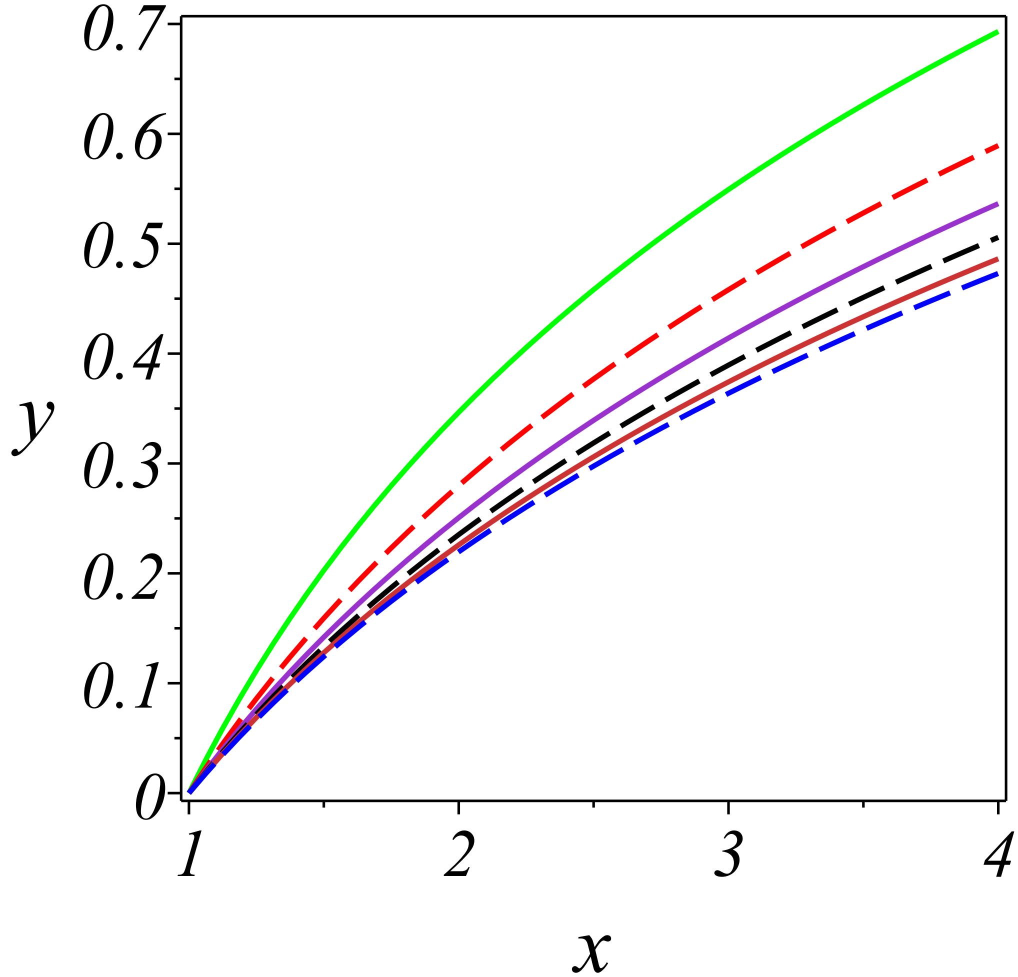

FIG. 2:

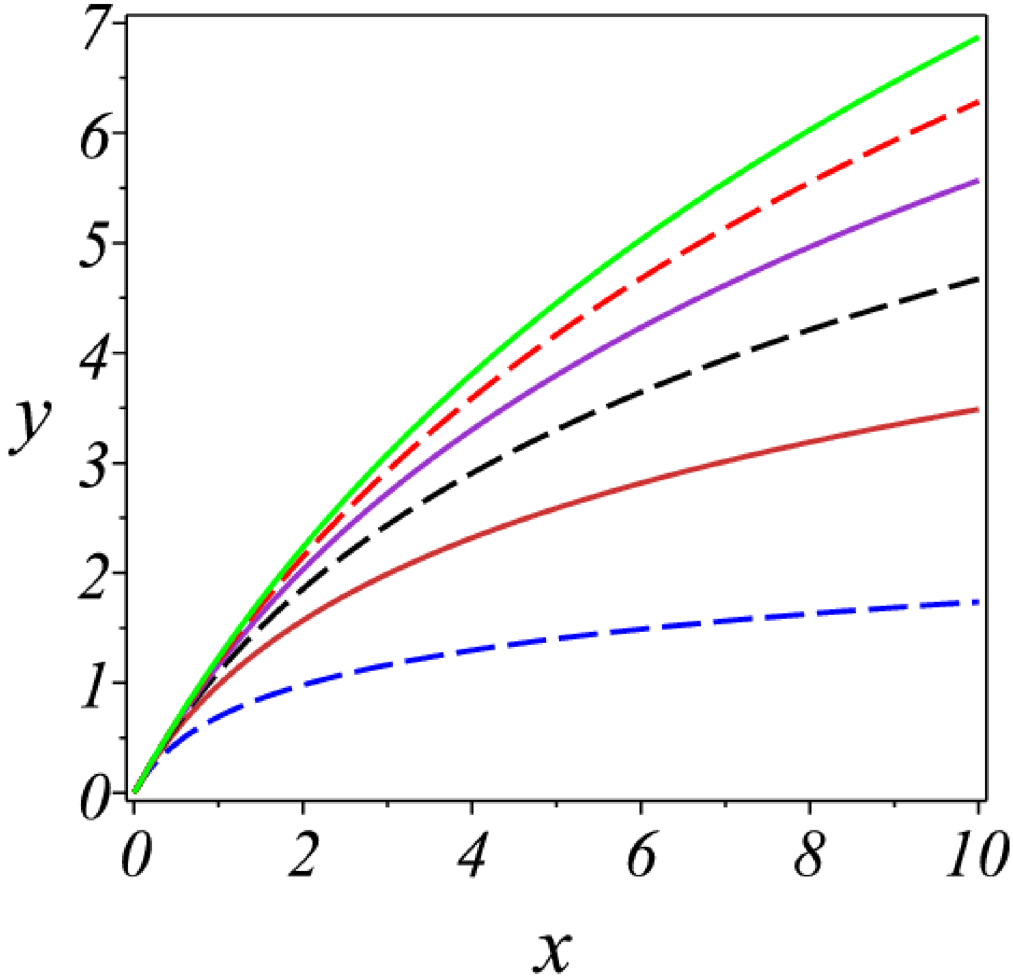

FIG. 3:

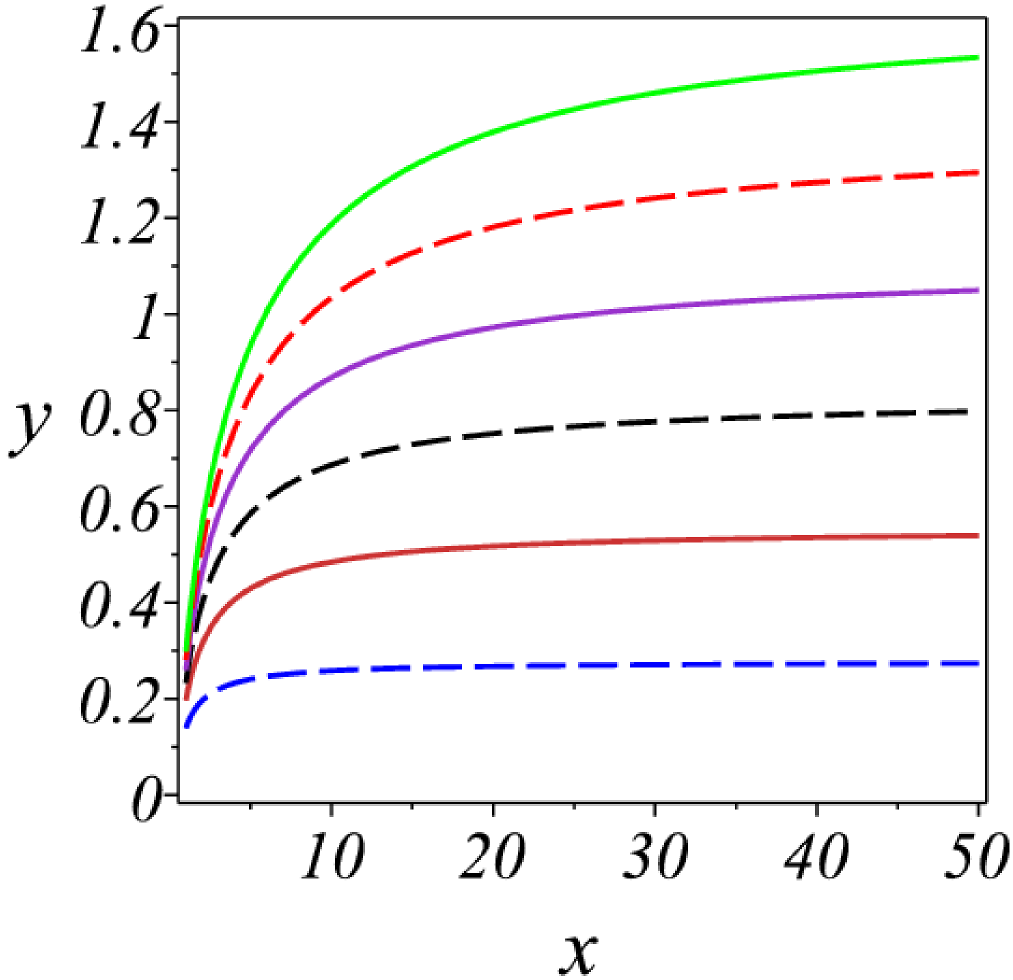

FIG. 4:

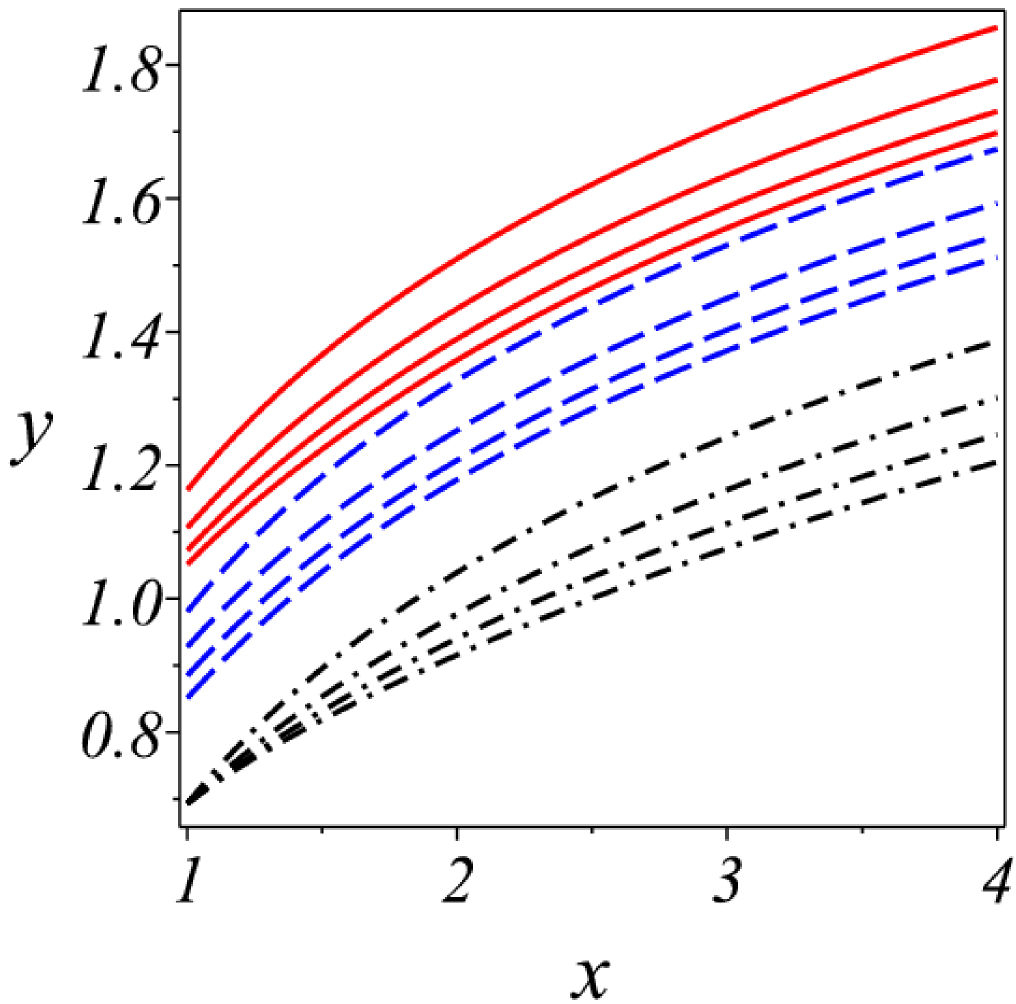

FIG. 5:

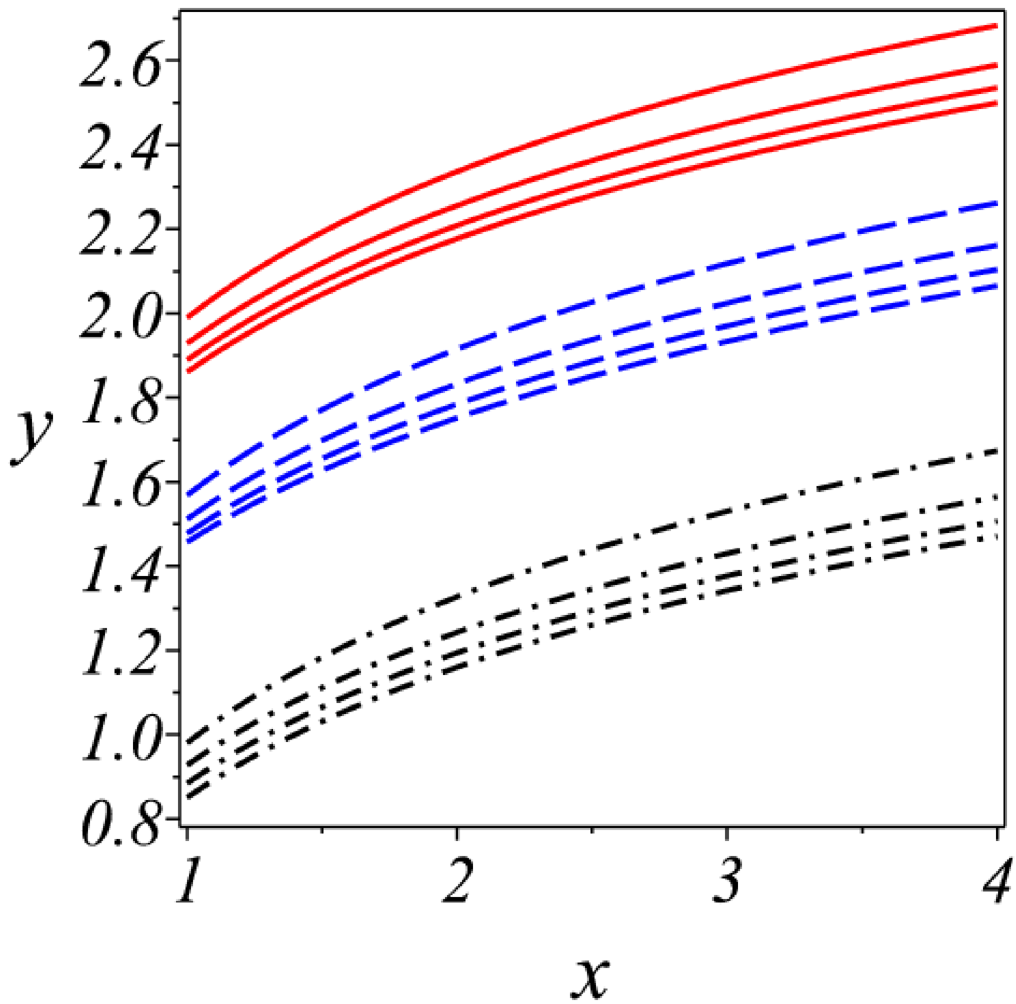

FIG. 6: