Generation of anisotropy in turbulent flows subjected to rapid distortion

Timothy T. Clark, Susan Kurien, and Robert Rubinstein

Phys. Rev. E **97**, 013112 — Published 23 January 2018

DOI: 10.1103/PhysRevE.97.013112

Generation of anisotropy in turbulent flows subjected to rapid distortion

Timothy T. Clark,[1] Susan Kurien,[2] and Robert Rubinstein[3]

[1] *Department of Mechanical Engineering, University of New Mexico, Albuquerque, NM 87131, USA*
[2] *Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[3] *Distinguished Research Associate, NASA Langley Research Center, Hampton VA 23681, USA*

A novel computational tool for the anisotropic time-evolution of the spectral velocity correlation tensor is presented. We operate in the linear, rapid distortion limit of the mean-field-coupled equations. Each term of the equations is written in the form of an expansion to arbitrary order in the basis of irreducible representations of the SO(3) symmetry group. The computational algorithm for this calculation solves a system of coupled equations for the scalar weights of each generated anisotropic mode. The analysis demonstrates that rapid distortion rapidly but systematically generates higher-order anisotropic modes. To maintain a tractable computation, the maximum number of rotational modes to be used in a given calculation is specified *a priori*. The computed Reynolds stress converges to the theoretical result derived by Batchelor and Proudman, [*Quart. J. Mech. and Applied Math*, **7**-1, 83–103, (1954)] if a sufficiently large maximum number of rotational modes is utilized; more modes are required to recover the solution at later times. The emergence and evolution of the underlying multidimensional space of functions is presented here using a 64-mode calculation. New implications for modeling strategies are discussed.

PACS numbers: 47.27.Gs,47.27.eb,47.27.E-

I. INTRODUCTION

A mathematically appropriate decomposition of the anisotropic two-point turbulence correlation function utilizes the irreducible representations of the SO(3) rotational symmetry group. This has been demonstrated successfully in a series of papers beginning with [1] and reviewed in [2, 3]. Among the primary outcomes of those studies was a refinement of the Kolmogorov 1941 local isotropy hypothesis – that is, at sufficiently high Reynolds number, the small scales recover isotropy not in some absolute sense, but because the higher-order anisotropic modes in an SO(3) basis expansion decay more rapidly in scale than does the isotropic part.

In the present work we use the SO(3) group decomposition to develop a computational scheme for the *evolution* of the general anisotropic two-point velocity correlation function in the linear problem of mean-flow coupled turbulence. This approach is distinct from previous uses of the SO(3) decomposition for turbulence in that we apply the method to the equations of motion to derive a computable model, instead of using the basis functions as a diagnostic tool in a post-processing step.

In order to expose the utility of the two-point correlation dynamics in the development of the model, we review some known features of single-point turbulence modeling. The pressure-strain correlations that arise in the (single-point) Reynolds stress evolution equations for incompressible flow are typically discussed in terms of a "rapid part", that is, the terms that couple directly to the mean-flow gradients, and a "slow part" that is expressed purely in terms of moments of velocity fluctuations ([4]). Despite its linearity in the Reynolds stresses and in the mean velocity, the rapid pressure-strain (RPS) correlation represents a significant challenge to engineering turbulence modeling efforts due to its integro-differential nature; its representation in the single-point equations is unclosed and requires a model assumption to achieve closure. In the two-point description of the Reynolds stress evolution equation the RPS appears in closed form, which can alleviate this issue. However, efforts to simplify these two-point descriptions by describing the turbulence spectra as an average over all angles in Fourier space (so that the spectra are functions of wavenumber, rather than wave-vectors) are still confronted with the need to model the rapid part, despite its apparent simplicity ([5, 6]). If, as we propose here, a full wave-vector representation of the dynamics is permitted, it allows these terms to be represented exactly, and thus requires no model for the rapid part of the pressure strain correlation.

An advantage of such an approach is that it permits the model to handle the "Rapid Distortion Theory" (RDT) problem without approximation. RDT is based on the assumption that, when a large mean-field distortion is imposed on a turbulent field, the early time response to the distortion can be described by ignoring the effects of higher-order velocity correlations. "Early time" in defined in essence as the time whilst the turbulence time-scales associated with the triple (and higher) correlations remain much larger than the time-scales associated with the mean-field. The rapid distortion problem has a deep history in both theoretical and experimental turbulence research [7, 8]. Of relevance to the present work is the derivation by Batchelor and Proudman (BP54 [9]) of the exact form of the spectral correlation tensor as a function of the so-called extension factor, a surrogate for the time over which the mean-strain acts on a homogeneous turbulence. For the purposes of demonstrating the efficacy and accuracy our algorithm, we will show that our computed results approach the BP54 theory to arbitrarily long times and with any desired accuracy.

The potential disadvantage of such an approach is that it is generally computationally costly. For the homogeneous case, it requires the solution of a fully three-dimensional wave-vector problem. This difficulty is overcome to a degree by using the SO(3) decomposition in our approach which, as we will show, reduces a 3D problem of tensor functions of wavevector into a 1D problem of computing scalar functions of the wavenumber. That is, the complexity of a 3D grid is substituted for a series expansion of a 1D function to arbitrary accuracy by using in principle as many anisotropic rotational modes in the SO(3) decomposition as the problem requires. Our longer-term goal is to generate a computationally tractable model of the Fourier-transformed fluctuating-velocity tensor as a function of wave-vector that permits the exact solution of the rapid part of the pressure strain correlations. Such an approach would then allow the "modeling" aspects to be restricted entirely to those associated with the third-order (and higher) correlations of the fluctuating velocity.

The ability to formulate and treat this important problem exactly permits an unbiased critique of the limitations of the simpler practical models and perhaps may provide insights into how to improve them. To achieve this goal, Zemach [10] developed a system of vector algebra as a means of reducing the dynamical equation for a second-rank tensor field, dependent on wave-vector only, to a system of coupled dynamical equations for scalar fields. The outcome of [10] for the second-rank tensor field for velocity correlations is identical, up to recombinations of terms to ensure solenoidality, to the the irreducible representations of the SO(3) symmetry group given in [1, 11, 12]. This representation is then used to formulate a model (Local Wave Vector (LWV)) model [13] by applying the framework of [10] to the full (nonlinear) mean-field coupled equations of motion for the second-order correlations in turbulent flow. In [13] are derived the elements of the coupling matrix between initial and final states for the linear problem. This latter report involves, in addition to the mean-field coupled terms which are treated exactly by the rotational mode decomposition, assumptions such as closure approximations, and a return-to-isotropy model. Both of these reports delve deeply and systematically into mathematical foundations and detailed derivations which lie beyond the scope of this paper. The code used to compute the results presented in this paper is built on the strategy developed in [13]; we will only consider the linear component and will give explanations of the development as needed. However we will not review here the great deal of extra detail in those reports that are freely available to the interested reader. We will instead focus on the outcome of the computations and their implications.

In Section II we consider the evolution of the second-order spectral velocity correlation tensor in the mean-flow coupled system. Following [13] the dynamical equation for a second-rank tensor field, dependent on wave-vector only is reduced to a system of coupled dynamical equations for scalar fields. For homogeneous turbulence of an incompressible fluid, three such scalar fields is sufficient. Rather than solving the three coupled dynamical field equations in 3D wave-vector space, we elect to expand the scalar fields in spherical harmonics. The computational problem is then of a differential equation in the radial (wavenumber k) variable, with rotational mode functions coupled in a discrete and large, but computationally feasible matrix. We show that each solenoidal term in the equations of motion may be decomposed into known basis functions in the SO(3) group representation. Crucially, the differential operators that are the elements of the coupling matrix act on the basis functions such that the resulting terms remain within the SO(3) basis space but with different and predictable weights. This fact allows us to efficiently code the dynamics and prescribe arbitrarily many modes for the calculation. In Section III we verify the accuracy or our computation against analytical results for single-point quantities of [9]. We show that the systematic generation of anisotropic modes in the linear problem can be performed to arbitrary accuracy with sufficiently many modes, and discuss some of the properties of the emergent modes. Finally we summarize and discuss the implications of our approach in Section IV.

II. THEORY

The SO(3) group representation theory identifies quantities that transform into themselves under rotation. The theory specifies the simplest, or irreducible, sets of such quantities, into which any scalar, vector or tensor function, may be decomposed. Isotropy, which we normally think of as invariance under rotation, becomes just one piece of a systematic representation for arbitrarily anisotropic functions. The two-point spectral correlation of velocity $E_{ij}(\boldsymbol{k})$ in a homogeneous, incompressible flow may be represented in the SO(3) symmetry group basis by nine scalar fields and nine tensor dyadic operators [14]. These may be constructed in different ways [1, 10–12] but must satisfy the same symmetry and parity properties to span the basis space for the second-rank tensor. For non-helical flows which are the focus of this paper, index symmetry, and even-parity in wavevector $\boldsymbol{k}$, $E_{ij}(\boldsymbol{k}) = E_{ji}(\boldsymbol{k}) = E_{ij}(-\boldsymbol{k})$, implies a reduction to a space of six scalar fields and corresponding six tensor operators. Finally, the solenoidal condition $k_i E_{ij}(\boldsymbol{k}) = k_j E_{ij}(\boldsymbol{k}) = 0$ reduces the characterization to three scalar fields which multiply appropriate rotationally-invariant, solenoidal constant tensors.

$$E_{ij}(\boldsymbol{k}) = S^{\kappa}_{ij}(\boldsymbol{k})\kappa(\boldsymbol{k}) + S^{\lambda}_{ij}\lambda(\boldsymbol{k}) + S^{\chi}_{ij}\chi(\boldsymbol{k}). \tag{1}$$

We here and henceforth use the notation in [10] and its companion report [13] in which the computational algorithm to be described below was developed. The scalar fields may be represented in terms of expansions in the scalar spherical harmonics

$$\kappa(\boldsymbol{k}) = \sum_{\substack{\text{even } \ell\geq 0\\ m}} \kappa^{\ell m}(k) Y^{\ell m}(\theta,\phi),$$
$$\lambda(\boldsymbol{k}) = \sum_{\substack{\text{even } \ell\geq 2\\ m}} \lambda^{\ell m}(k) Y^{\ell m}(\theta,\phi),$$
$$\chi(\boldsymbol{k}) = \sum_{\substack{\text{odd } \ell > 1\\ m}} \chi^{\ell m}(k) Y^{\ell m}(\theta,\phi). \qquad (2)$$

Each scalar field is expanded in spherical harmonics; the coefficients $\kappa^{\ell m}(k)$, $\lambda^{\ell m}(k)$ and $\chi^{\ell m}(k)$ are the rotationally invariant *basis functions* defining $E_{ij}(\boldsymbol{k})$. Rotational modes indexed by ℓ form a $(2\ell+1)$ dimensional space indexed by m, $-\ell \leq m \leq \ell$. The spherical harmonic functions $Y^{\ell m}(\theta,\phi) = k^{-\ell} Y^{\ell m}(\boldsymbol{k})$ are expressed in terms of the conventional polar and azimuthal angles. We will denote index ℓ variously as the '*spin*', '*rotational mode index*', or '*sector*' of the representation in keeping with previous work [1, 12]. Each basis function belongs to a rotationally invariant subspace corresponding to an irreducible representation of the SO(3) symmetry group. That is, each (ℓm) contribution transforms into itself under SO(3) group rotations. The $\ell = 0$ sector of $\boldsymbol{\kappa}$ in this expansion is the isotropic part. Each basis function with $\ell > 0$ defines the strength of one mode of anisotropy, as expressed by the spherical harmonic it multiplies. The $\ell = 1$ contribution is not permitted due to the solenoidal condition in homogeneous incompressible flows [1].

The κ term in Eq. (1) contains the tensorially isotropic projection operator

$$S^{\kappa}_{ij} = P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j, \qquad (3)$$

where $\hat{k}_i = k_i/k$ for $i = 1, 2$ or 3 denoting the Cartesion coordinate indices. By *tensorially isotropic* we mean that S^{κ}_{ij} is spin-0, transforming identically to itself under true rotations. This operator does not change the spin of the scalar function that it operates on. Thus, all anisotropic contributions in this term arise from the scalar spherical harmonics $\kappa(\boldsymbol{k})$.

To formulate the tensor operators S^{λ}_{ij} and S^{χ}_{ij}, first define three *primary* vector operators, the unit vector $\hat{\boldsymbol{k}}$, the infinitesimal SO(3) generator [14] $\boldsymbol{L} = -i(\boldsymbol{k}\times\nabla) = +i(\nabla\times\boldsymbol{k})$ and its moment $\boldsymbol{M} = i(\hat{\boldsymbol{k}}\times\boldsymbol{L})$. Here the gradient operates with respect to $\boldsymbol{k}$ as $\nabla_i = (\partial/\partial k_i)$. Then [10] shows

$$S^{\lambda}_{ij} = \frac{1}{2}\Big(\hat{k}_i M_j + \hat{k}_j M_i) - (L_i L_j + L_j L_i)\Big) \qquad (4)$$
$$S^{\chi}_{ij} = \frac{1}{2}\Big(\hat{k}_i L_j + \hat{k}_j L_i) - (M_i L_j + M_j L_i)\Big). \qquad (5)$$

S^{λ}_{ij} is even-parity and has trace $(-L^2)$. It operates on even-order polynomials forming even-parity tensor functions. The S^{χ}_{ij} operator is odd-parity and trace-free and operates on odd-order polynomials forming even-parity tensor functions. The χ contributions permit symmetry-breaking under reflections in the azimuthal plane while still remaining index-symmetric and even-parity [12]. Both S^{λ} and S^{χ} tensors operate on the scalar functions that they multiply to change their rotational mode index, or spin. This property is essential to how higher-order anisotropic modes are generated and will be described further below. Equivalent representations derived in [12] using basis functions in [1] separate the trace and trace-free contributions in the so-called *directional* and *polarization* components which are separately rotationally invariant under SO(3) operations [6]. All representations [1, 10–12] are equivalent and invariant under SO(3) group rotations.

The mean-flow-coupled equations for the evolution of the energy spectral tensor in homogeneous incompressible flow may be derived from the Navier-Stokes equations as,

$$\begin{aligned}\dot{E}_{ij}(\boldsymbol{k}) = U_{ab}\Big(&-\delta_{ai}E_{bj}(\boldsymbol{k}) - \delta_{aj}E_{ib}(\boldsymbol{k})\\ &+ 2\hat{k}_a\hat{k}_i E_{bj}(\boldsymbol{k}) + 2\hat{k}_a\hat{k}_j E_{bi}(\boldsymbol{k})\\ &+ k_a\frac{\partial}{\partial k_b}E_{ij}(\boldsymbol{k})\Big),\end{aligned} \qquad (6)$$

where $\dot{E}_{ij}(\boldsymbol{k}) = \partial E_{ij}(\boldsymbol{k})/\partial t$. The mean-flow-gradient tensor $U_{ab} = \frac{\partial}{\partial x_b}U_a(\boldsymbol{x},t)$ is independent of $\boldsymbol{x}$ and traceless. The first, second, and third lines of the right-hand side (RHS) of Eq. (6) are the production, rapid pressure-strain and mean-flow distortion terms respectively. This equation is linear in both the mean-flow gradient and E_{ij}. However, it is not yet in a form that is useful for computation in the SO(3) basis. To achieve that, we rewrite the RHS in the form of an operator O acting on E_{ij}. Set $\partial/\partial k_b = k_b(\partial/\partial k) - M_b/k$ and define two new operators M and Δ:

$$\mathsf{M}_b Z_{ij} = M_b Z_{ij} - \hat{k}_i Z_{bj} - \hat{k}_j Z_{ib} \qquad (7)$$
$$\Delta_{ab} Z_{ij} = (-\delta_{ia} + \hat{k}_i\hat{k}_a)Z_{bj} + (-\delta_{ja} + \hat{k}_j\hat{k}_a)Z_{ib} \quad (8)$$

where Z_{ij} is any of the symmetric solenoidal tensors S^{κ}_{ij}, S^{λ}_{ij} or S^{χ}_{ij}. Then Eq.(6) can be rewritten in the desired form,

$$\dot{E}_{ij} = \mathsf{O}E_{ij} = U_{ab}\Big(\hat{k}_a\hat{k}_b(k\frac{\partial}{\partial k}) - \hat{k}_a\mathsf{M}_b + \Delta_{ab}\Big)E_{ij}, \quad (9)$$

where the wave-vector argument of $E_{ij}(\boldsymbol{k})$ is implicit. Note that with this recombination of terms, the different physical components of the equation are mixed Any linear operator acting on a solenoidal symmetric tensor field with the form in Eq.(1) of the linear combination of κ, λ and χ, will generate output scalar fields that are also linear combinations of κ, λ and χ. Therefore, the coupled scalar equations to be solved have the form:

$$\begin{bmatrix}\dot{\kappa}\\ \dot{\lambda}\\ \dot{\chi}\end{bmatrix} = \begin{bmatrix}O_{\kappa\kappa} & O_{\kappa\lambda} & O_{\kappa\chi}\\ O_{\lambda\kappa} & O_{\lambda\lambda} & O_{\lambda\chi}\\ O_{\chi\kappa} & O_{\chi\lambda} & O_{\chi\chi}\end{bmatrix}\begin{bmatrix}\kappa\\ \lambda\\ \chi\end{bmatrix} \qquad (10)$$

where the operator $\mathsf{O} = U_{ab}\Big\{\mathsf{A}(k\frac{\partial}{\partial k}) + \mathsf{B} + \mathsf{C}\Big\}$ is now an operator *connection matrix* with the assignment $\mathsf{A} =$

$\hat{k}_a\hat{k}_b$, $\mathsf{B} = -\hat{k}_a\mathsf{M}_b$ and $\mathsf{C} = \Delta_{ab}$. The components $O_{\mu\nu}$ have μ, ν taking on labels κ, λ and χ with any pair $(\mu\nu)$ indicating the change to μ due to ν. The evolution equation for κ from Eq. (10) then has the form:

$$\begin{aligned}\dot{\kappa} = U_{ab}(&\mathsf{A}_{\kappa\kappa}k\frac{\partial}{\partial k}\kappa + \mathsf{A}_{\kappa\lambda}k\frac{\partial}{\partial k}\lambda + \mathsf{A}_{\kappa\chi}k\frac{\partial}{\partial k}\chi \\ &+\mathsf{B}_{\kappa\kappa}\kappa + \mathsf{B}_{\kappa\lambda}\lambda + \mathsf{B}_{\kappa\chi}\chi \\ &+\mathsf{C}_{\kappa\kappa}\kappa + \mathsf{C}_{\kappa\lambda}\lambda + \mathsf{C}_{\kappa\chi}\chi),\end{aligned} \tag{11}$$

with companion equations for $\dot{\lambda}$ and $\dot{\chi}$. These are the formal equations solved in our model calculations.

The task remains to compute the linear operator connection matrices A, B and C. which depend on combinations of the three primary operators defined above. Note that up to the point of the general form derived in (10) the formal representation of the problem is independent of the choice of coordinate system. For the purposes of writing a computer code we choose to work in the basis of spherical harmonics, a natural choice for SO(3) representation, and to use proper components instead of normal (eg. Cartesian) components. We may write an arbitrary vector operator $\boldsymbol{V}$ in its proper components:

$$\begin{aligned}V_+ &= V_1 + iV_2 \\ V_0 &= V_3 \\ V_- &= V_1 - iV_2,\end{aligned} \tag{12}$$

where once again the subscripts 1, 2 and 3 are the Cartesian components x, y and z respectively. For spherical harmonic functions in unit normalization, $\int_\Omega Y^{\ell'm'}Y^{\ell m}d\Omega/(4\pi) = \delta_{\ell'\ell}\delta_{m'm}$, the action of proper components of $\boldsymbol{L}$ can be shown to leave ℓ unchanged and change m according to:

$$\begin{aligned}L_+Y^{\ell m} &= \sqrt{(\ell-m)(\ell+m+1)}Y^{\ell,m+1} \\ L_0Y^{\ell m} &= mY^{\ell m} \\ L_-Y^{\ell m} &= \sqrt{(\ell+m)(\ell-m+1)}Y^{\ell,m-1}.\end{aligned} \tag{13}$$

On the other hand, the action of $\hat{\boldsymbol{k}}$ results in a linear combination of lowered and raised spins ℓ with prescribed weights for each term:

$$\begin{aligned}\hat{k}_+Y^{\ell m} = &-\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}}Y^{\ell+1,m+1} \\ &+\sqrt{\frac{(\ell-m-1)(\ell-m)}{(2\ell+1)(2\ell-1)}}Y^{\ell-1,m+1}.\end{aligned} \tag{14}$$

$\hat{k}_0$ and $\hat{k}_-$ operate respectively to retain and lower m for mixed $(\ell+1)$ and $(\ell-1)$ indexed combinations of basis functions. It may similarly be shown that the action of *dyadic* tensor operators formed from pair combinations of $\boldsymbol{k}$, $\boldsymbol{L}$ and $\boldsymbol{M}$, as in Eqs. (3,4,5), on $Y^{\ell m}$ transforms it into a sum of terms depending on $Y^{\ell'm'}$ where the values of $\Delta\ell = \ell' - \ell = 0, \pm1$, or ±2 depending on the particular operator form, and $\Delta m = m' - m = 0, \pm1$, or ±2, depending on the coordinate indices. Similarly, the action on $Y^{\ell m}$ of pair combinations of $\boldsymbol{k}$, $\boldsymbol{L}$ and $\boldsymbol{M}$, results in a sum of terms depending on $Y^{\ell'm'}$ where $\Delta\ell = \ell' - \ell = 0, \pm1$, or ±2 depending on the particular operator form, and $\Delta m = m' - m = 0, \pm1$, or ±2, depending on the coordinate indices.

The main point in this description is that $\hat{\boldsymbol{k}}$, $\boldsymbol{L}$, and $\boldsymbol{M}$ and their dyadic operator products have prescribed effects, namely, raising, leaving unchanged or lowering the spin indices (ℓ, m) of the spherical harmonic basis functions upon which they operate, with scalar prefactors that are fixed functions of ℓ and m. The detailed results of such calculations for each of the elements in the connection matrix may be obtained from [10].

III. VALIDATION OF NUMERICAL STRATEGY AGAINST BP54 THEORY

The computational algorithm proposed allows for efficient numerical simulation of the equations of motion using a wavenumber-dependent form with the angular dependence contained implicitly in the weights of the spherical harmonics contributions. For a prescribed maximum number of rotational modes $\mathcal{L}_{max}$, one can compute the generation of anisotropy up to that maximum mode with deterministic weights computed from the operator connection matrix for each $Y^{\ell m}$ contribution to the κ, λ, and χ basis functions. As modes of order greater than the prescribed $\mathcal{L}_{max}$ are generated, they are discarded. The only approximation thus arises from the truncation of the series expansion at $\ell = \mathcal{L}_{max}$.

We now review the numerical strategy employed for calculation. We substitute $z = \log(k)$ and then compute over a range of $z \in [z_{min}, z_{max}]$ where, for the current calculations, $z_{min} = -16$ and $z_{max} = +16$, corresponding to minimum and maximum values of k of $k_{min} \approx -1.125 \times 10^{-7}$ and $k_{max} \approx 8.886 \times 10^{6}$. A total of 3201 mesh points (including end points) were used to discretize z, with a spacing of $\delta_z = 10^{-2}$. The boundary conditions in z were consistent with a k-space power law with an arbitrary exponent at low and high z. No special treatment of the ℓ, m modes beyond $\mathcal{L}_{max}$ was employed. The temporal and spatial (z-coordinate) differencing scheme was analogous to a MacCormack method (see [15]). This yields a scheme that is second order in time and in z.

To test the algorithm, we consider uniform plane strain such that $U_{11} = 0$, $U_{22} = -1$, and $U_{33} = 1$. We compute Eq. (11) and its companion equations for λ and χ for an initially isotropic homogeneous flow such that κ^{00} is the only non-zero contribution at time $t = 0$. The results presented in this paper are independent of the choice of the initial spectrum of κ^{00}, thus, for sake of brevity, we will not describe it. The total initial energy is normalized so that the initial large-eddy turnover time is $t = 1$ in the code units.

Batchelor and Proudman [9] obtained an analytical expression for the time-evolved Reynolds stress tensor for homogeneous turbulence subjected to uniform rapid distortion due to symmetric mean-strain. The latter implies that the distortion takes place on timescales much faster than those of the nonlinear terms in the equations of motion. For a given strain tensor, BP54 define the distortion factor

$$c(t) = \exp\Big(\int_0^t U_{33}\mathrm{d}t\Big). \tag{15}$$

which is a surrogate for the duration of the distortion. Then BP54 provide a general solution of the Reynolds tensor as a function of c. They reduce their result to the single-point quantities defined by the ratio of post- to pre-distortion value of a Reynolds stress component

$$\mu_n = \frac{R_{nn}(t)}{R_{nn}(0)} \tag{16}$$

where n is 1, 2 or 3, (the indices are not summed over).

The general solution of BP54 was specialized for the case of initially axisymmetric turbulence by [16], assuming reasonable low-order truncations of the spherical harmonics expansions. In [16], the goal was to obtain the solution for distortion of initially axisymmetric flow, which, in our notation corresponds to $m = 0$ for each j-sector. In the present work we demonstrate a calculation which generates anisotropic modes order-by-order to obtain the BP54 solution. As a validation study we use the BP54 exact solution for plane strain distortion, starting from isotropy; but we note that the computational method we have presented may be used for any distortion starting from any initial condition.

The BP54 solution for the plane strain distortion of initially isotropic flow is given by:

$$\begin{aligned}
\mu_1 &= \frac{3}{4}\frac{c^{-2}}{c^2-1}\Big(\frac{c(c^4-c^2+1)}{(c^4-1)^{1/2}}((c^2+1)y-x)-1\Big)\\
\mu_2 &= \frac{3}{4(c^2-1)}\Big(1+\frac{c(c^4-c^2-1)}{(c^4-1)^{1/2}}y\Big)\\
\mu_3 &= \frac{3}{4}c^{-2}\Big(1+\frac{c(c^4+c^2-1)}{(c^2-1)(c^4-1)^{1/2}}(x-y)\Big),
\end{aligned} \tag{17}$$

where

$$x = \int_0^{(1-c^{-4})^{1/2}} 1 - c^2t^2/(1-c^2)^{-1/2}(1-t^2)^{-1/2}\mathrm{d}t,$$

$$y = \int_0^{(1-c^{-4})^{1/2}} 1 - c^2t^2/(1-c^2)^{1/2}(1-t^2)^{-1/2}\mathrm{d}t$$

This solution is an rational function of c with no reference to the SO(3) basis functions whatsoever. Any computation of μ must converge to this exact solution for a sufficiently refined calculation. We show below that this is indeed true for the rotational mode calculation we have proposed given sufficiently many modes.

In order to recover the single-point quantities required for comparison with the BP54 solutions above, we first define the angular average of a function $f(\boldsymbol{k})$,

$$\langle f(\boldsymbol{k})\rangle_\Omega = \frac{1}{4\pi}\int f(\boldsymbol{k})d\Omega = \frac{1}{4\pi}\int f(\boldsymbol{k})\sin\theta d\theta d\phi \tag{18}$$

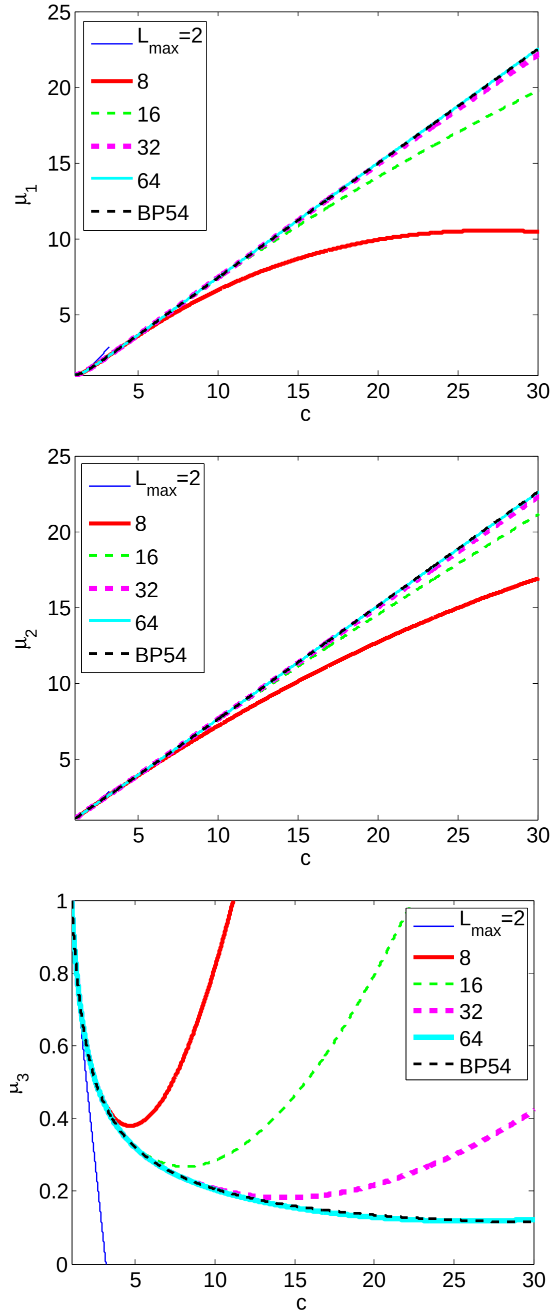

FIG. 1. [Color online] Ratio of post- to pre-distortion energy μ_3 for various values of $\mathcal{L}_{max}$ as a function of the distortion factor c. BP54 is the theoretical result from [9]. $c = 7.34$ corresponds to $t = 1$ which is the initial large eddy turnover time.

Then the Reynolds stress tensor is given by:

$$\begin{aligned}
R_{ij} &= \int k^2\langle E_{ij}(\boldsymbol{k})\rangle_\Omega dk = \frac{2}{3}k^2\delta_{ij}\kappa^{0,0}(k)\\
&\quad - \sum_m k^2\Big(\kappa^{2m}(k)+3\lambda^{2m}(k)\Big)\langle k_ik_jY^{2m}\rangle_\Omega
\end{aligned} \tag{19}$$

where the constant matrices $\langle k_ik_jY^{2m}\rangle_\Omega$ are straightforward to calculate [13]. In the above we have used

$\langle Y^{\ell m}(\theta,\phi)\rangle_\Omega = 1$ if $\ell = m = 0$ and vanishes otherwise. Therefore, when averaging over the sphere, only those contributions to κ, λ and χ which reduce to spin 0 upon operation with the associated tensor, survive. $\chi(\boldsymbol{k})$ (Eq. (2)) has no terms with $\ell \leq 2$ and therefore odd-spin quantities do not contribute to angular-averaged quantities. In terms of the SO(3) decomposition, the Reynolds stress is a truncation at $\ell = 2$ of the spherical harmonic expansion of the modal spectrum tensor. However, the dynamics dictate that contributions from higher spins feed directly and indirectly into the $\ell = 0$ and 2 modes due to the selection rules discussed above. Therefore, anisotropic contributions at higher orders in spin must be taken into account to recover the Reynolds stress accurately. This point is demonstrated in the results to follow.

The three panels of Fig. 1 show the computed value of μ_n for each n respectively. In each case we present model calculations with $\mathcal{L}_{max}$ ranging from 2 to 64 and plot these along with the exact BP54 result. The calculation is stopped if any of the Reynolds stress components becomes unphysical (unrealizable). Therefore the $\mathcal{L}_{max} = 2$ calculation ends at about $c = 2.5$ since μ_3 becomes negative. What is common to all n is that for any $\mathcal{L}_{max}$ the computed solution departs from the analytical solution after some finite time; the larger the number of modes permitted, the later the departure from the exact solution. For $\mathcal{L}_{max} = 64$ the computed solution agrees with the theoretical one for up to about 4 large-eddy turnover times. We observe that the improvement, as more modes are added, is systematic. The error over any desired interval $[1, c]$ can be made arbitrarily small as the number of modes is increased.

It must be noted further, that as $\mathcal{L}_{max}$ is increased, μ_2 appears to converge to the BP54 benchmark most rapidly, while μ_3 is the slowest to converge. This illustrates that nominally "near isotropic" behavior of a single component is possible, but could mask underlying anisotropic contributions. That is, one might be able to recover, for some components, a partial expression of the solution with fewer modes, but that is choice could adversely affect other components of the solution. It is therefore important to assess the impact of anisotropic contributions on the tensor as a whole.

More detailed structure of the rotational modes arising in this problem may be revealed by these calculations. Some redundancies and simplifications may be deduced *a priori*. First, the χ contributions, though included in the calculations, are identically zero because the flow configuration does not break symmetry in the $z(3)$-direction [12]. Within each even-ℓ sector it may be shown that odd-m do not contribute because the chosen mean-strain tensor is diagonal. Finally, given the definition of the spherical harmonics, the sign of m is an additional redundancy for the integrated quantities described below. It remains to compare the contributions of κ to those of λ. Define the volume-integrated quantity

$$Q_\gamma(\ell, m) = \int_0^{k_{max}} k^2 \gamma^{\ell m}(k) dk \tag{20}$$

where γ is either of κ or λ. These integrated rotational mode contributions among the different m for $\ell = 4$ and $\ell = 6$ are illustrated in Fig. 2. The data for these figures are taken from the $\mathcal{L}_{max} = 64$ calculation. Instead of the expected $l + 2$ independent contributions, we see one-half that since $Q_\kappa(\ell, m)$ and $Q_\lambda(\ell, m)$ are nearly indistinguishable up to and indeed for later than the times shown in the figure. For a given ℓ the different m contributions also have different growth rates which change over time.

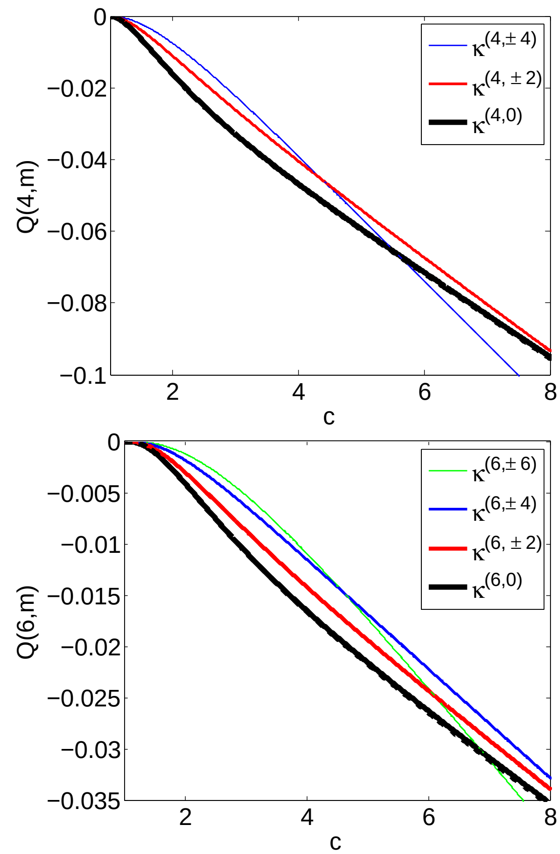

FIG. 2. [Color online] Evolution of volume integrated $\ell = 4$ modes (top) and volume integrated $\ell = 6$ modes (bottom).The κ (solid) and λ (dashed) contributions are essentially indistinguishable.

IV. DISCUSSION AND CONCLUSION

We have developed a computational tool using the SO(3) decomposition of the second-order correlation function equation for mean-field coupled turbulence. The nonlinear terms in the equations of motion are neglected under the assumption of rapid distortion physics and viscosity is also neglected. We are therefore in a physical regime that is described by exact, closed equations of motion. The study is restricted to symmetric distortions of non-helical flows, for which Batchelor and Proudman [9] derived the analytical solution for the evolution of the second-order correlation function. However the method itself can be used with any initial condition and for any distortion. Our calculations using sufficiently many rotational modes show convergence to the BP54 analytical result for non-trivial plane-strain distortion of initially isotropic flow.

We conclude that, for the exact linear problem, the evolution of statistical quantities, including low-order (single-point) quantities like the Reynolds stress, depend on arbitrarily many higher-order anisotropic sectors. It is apparent that even for times much less that one large-eddy turnover time, nominally a 'rapid distortion' regime, the number of rotational modes required for a converged solution proliferate rapidly. A single time-step generates contributions ranked 2 higher than the maximum ℓ at a given time.

Calculation of the mean-flow coupled problem to arbitrary accuracy in the manner proposed here points to strategies for improved modeling. As already noted, capturing anisotropy is critical in efforts to accurately model the rapid-pressure strain correlation. However, we show that the anisotropy arises from coupling of the various terms in the problem via the linear operator. An important conclusion therefore, is that the RPS correlation cannot be modeled in isolation from the other anisotropy-generating terms of the problem. On the other hand, our calculation shows a way to systematically generate anisotropy at the second-order level of description and therefore suggests a suitable truncation at order dictated by the problem itself. This has implications for the class of single-point models for the stress such as that of [4] (LRR) which *a priori* truncates the rotational-mode dependences down to a maximum of $\ell = 2$ by angle-averaging [5]. The error inherent in the LRR-class of models for anisotropic flow can in principle be explicitly quantified using the model presented here.

While we have focused on the rapid distortion problem in this study, the framework presented forms the basis for a more complete modeling approach for the fully nonlinear problem including turbulent diffusion and return-to-isotropy terms [13, 17]. These possibilities will be explored in future work. The discussion of the full nonlinear problem in the context of two-point second-order correlations will need to include closure (modeling) assumptions of the third-order correlations which are addressed separately [12] and will be integrated with this approach in future efforts.

ACKNOWLEDGMENTS

SK acknowledges support from the Mix and Burn project, ASC Physics and Engineering Models Program at LANL; work at LANL was performed under the auspices of the U.S. DOE Contract No. DE-AC52-06NA25396. TTC was supported by a Los Alamos National Laboratory subcontract to the University of New Mexico, No. 325696.

[1] I. Arad, V. Lv́ov, and I. Procaccia, Phys. Rev. E , 6753 (1999).
[2] S. Kurien and K. Sreenivasan, in *New Trends in Turbulence*, Proceedings of the Les Houches Summer School 2000, edited by M. Lesieur, A. Yaglom, and F. David (Springer-Verlag, Berlin, 2001) p. 53.
[3] L. Biferale and I. Procaccia, Physics Reports **414**, 43 (2005).
[4] B. E. Launder, G. J. Reece, and W. Rodi, Journal of Fluid Mechanics **68**, 537 (1975).
[5] D. Besnard, F. Harlow, R. Rauenzahn, and C. Zemach, Theoretical and Computational Fluid Dynamics **8**, 1 (1996).
[6] C. Cambon and L. Jacquin, J. of Fluid Mech. **202**, 295 (1989).
[7] L. Prandtl, National Advisory Committee for Aeronautics, Technical Memorandum 726 (1933).
[8] G. I. Taylor, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift fr Angewandte Mathematik und Mechanik **15**, 91 (1935).
[9] G. K. Batchelor and I. Proudman, Quart. J. Mech. App. Math. **7**, 83 (1954).
[10] C. Zemach, Los Alamos National Laboratory, Tech. Rep. LA-UR-16-28542 (2016).
[11] V. S. L'vov, I. Procaccia, and V. Tiberkevich, Phys. Rev. E **67**, 026312 (2003).
[12] R. Rubinstein, S. Kurien, and C. Cambon, Journal of Turbulence **16**, 1058 (2015), http://dx.doi.org/10.1080/14685248.2015.1051184.
[13] C. Zemach, Los Alamos National Laboratory, Tech. Rep. LA-UR-16-28800 (2016).
[14] M. Hamermesh, *Group Theory and Its Application to Physical Problems*, Dover Books on Physics Series (Dover Publications, 1989).
[15] D. Anderson, J. Tannehill, and R. Pletcher, *Computational Fluid Mechanics and Heat Transfer* (Hemisphere Publ. Corp. New York, 1984) pp. 102–103.
[16] K.R.Sreenivasan and R. Narasimha, J. Fluid Mech. **84**, 497 (1978).
[17] R. Rubinstein, T. Clark, and S. Kurien, Computers and Fluids (2016), 10.1016/j.compfluid.2016.07.009.