Reply to “Comment on ‘Defocusing complex short-pulse equation and its multi-dark-soliton solution’ ”

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Reply to "Comment on "A defocusing complex short pulse equation and its multi-dark soliton solution by Darboux transformation""

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Our paper [Phys. Rev. E 93, 052227 (2016)], proposing an integrable model for the propagation of ultra-short pulses, has recently received a Comment by Youssoufa et al. about a possible flaw in its derivation. We point that their claim is incorrect since we have explicitly stated that a term is neglected to derive our model equation in our paper. Further, the integrable model is validated by comparing with the normalized Maxwell equation and other known integrable models. Moreover, we show that a similar approximation has to be made in deriving the same integrable equation as explained in the Comment.

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With the significant progress of laser technology in the last two decades, the study of ultrashort optical pulses has attracted much attention [1]. However, the mathematical description of such ultrashort optical pulses requires a new approach beyond the conventional slowing varying envelope approximation (SVEA) [2]. As a matter of fact, some work have been done in the literature and several mathematical models have been proposed and studied [3–7]. Especially, In 2004, a short pulse (SP) equation was proposed by Schäfer and Wayne [8], which turns out to be integrable and admit Lax pairs [9] and multi-soliton solutions [10, 11]. Recently, an integrable complex short pulse equation was proposed in [12, 13] and [14] separately, was generalized to including the defocusing case [15].

The main criticism by Youssoufa et al. in the preceding Comment [16] concerns the approximation used in our papers [15]. To address their criticism, let us firstly reconfirm our derivation briefly here. Starting from Maxwell equation, and assuming an instantaneous Kerr effect, we obtain the following normalized equation (Eq. (21) in [15])

$$E_{zz} - E_{tt} = \pm E + \left(|E|^2 E\right)_{tt} . \qquad (1)$$

It is noted that the same equation with positive sign before E has been derived earlier in [4–6, 6, 13]. This is equation is considered as a full wave equation to model the ultra-short optical pulses with a few cycles, and has been investigated in [5, 6]. In seeking for a right-moving wave packet, we assume a multiple scales ansatz

$$E(z,t) = \epsilon E_0(\tau, z_1, z_2, \cdots) + \epsilon^2 E_1(\tau, z_1, z_2, \cdots) + \cdots , \qquad (2)$$

where ϵ is a small parameter, τ and z_n are the scaled variables defined by

$$\tau = \frac{t-x}{\epsilon}, \quad z_n = \epsilon^n z\,. \qquad (3)$$

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As a result we obtain the following partial differential equation for E_0 at the order $O(\epsilon)$:

$$-2\frac{\partial^2 E_0}{\partial\tau\partial z_1} = \pm E_0 + 2\frac{\partial}{\partial\tau}\left(|E_0|^2\frac{\partial E_0}{\partial\tau}\right) . \tag{4}$$

As pointed in [15], a term $E_0^2 E_{0,\tau}^*$, (misprinted by $E_0^2 E_{0,\tau}$ in [15]), is neglected for the purpose of obtaining an integrable nonlinear wave equation. Otherwise, an intermediate equation, which is non-integrable, turns out to be

$$-2\frac{\partial^2 E_0}{\partial\tau\partial z_1} = \pm E_0 + 2\frac{\partial^2}{\partial\tau^2}\left(|E_0|^2 E_0\right) . \tag{5}$$

Furthermore, by scale transformations

$$x = \frac{1}{\sqrt{2}}\tau, \quad t = \frac{1}{\sqrt{2}}z_1, \quad q = \sqrt{2}E_0 , \tag{6}$$

we arrive at

$$q_{xt} \pm q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0 , \tag{7}$$

$$q_{xt} \pm q + \frac{1}{2}\left(|q|^2 q\right)_{xx} = 0 , \tag{8}$$

respectively, from (4) and (5).

Thus, we disagree with the claim in the Statement [16] that a term (Eq. (17) in [16]) is missing since we have explicitly stated this neglected term in our paper [15]. Furthermore, to verify the validation of the approximation, in [15], we firstly compare the one-soliton solution for the focusing complex short pulse equation with the ones for the normalized Maxwell equation (1) with positive sign, the NLS equation and higher-order NLS equation. The results are shown in Fig. 1 in [15]. For the defocusing case, we also compared the one-soliton solution with the one for the normalized Maxwell equation (1) with negative sign. The result is Fig. 2 in [15].

From above comparisons, it can be seen that the solitary wave solutions for both the focusing and defocusing complex short pulse equation is consistent with the ones for the normalized Maxwell equation.

The physical background underlining the approach by Kuetche et al. mainly are based on the research done by Kozlov et. al [17–20]. In [14, 16], the nonlinear part of electric polarization $\mathbf{P}_{NL}$ as

$$\partial_t \mathbf{P}_{NL} = \alpha|\mathbf{E}|^2\mathbf{E}_t + \beta\mathbf{E}\times(\mathbf{E}\times\mathbf{E}_t). \tag{9}$$

As mentioned in [19], if a cubic nonlinearity $\mathbf{P}_{NL} = \chi|\mathbf{E}|^2\mathbf{E}$ is considered due to Kerr effect, then

$$\partial_t \mathbf{P}_{NL} = 3\chi|\mathbf{E}|^2\mathbf{E}_t + 2\chi\mathbf{E}\times(\mathbf{E}\times\mathbf{E}_t). \tag{10}$$

In [3, 20], the nonlinear contribution to the refractive index was studied in more details. By considering the contribution due to the electronic-vibrational nonlinearity beside the electronic nonlinearity, a more general expression for $\partial_t\mathbf{P}_{NL}$ takes the form of Eq. (9).

It was pointed in [3], $\alpha = \frac{3}{2}\beta$ for purely electronic nonlinearity, and some difference can be brought in a correlation between α and β by the electronic-vibrational mechanism of nonlinearity. However, the case $\beta = 0$ is too special which may not be true physically. In other words, an alternative approximation similar to our approach in [15] is used in deriving an integrable complex short pulse equation (Eq. (27) in [16]). Without this approximation, the derivation will lead to a non-integrable complex short pulse equation (8).

In summary, the integrable complex short pulse equation (7) can be derived to model the propagation of ultrashort pulse in nonlinear media either from our approach or the approach by Youssoufa et al. [16] by certain approximations. The integrability of this model equation allows us to obtain more mathematical properties and various exact solutions [21, 22]. Meanwhile, a non-integrable complex short pulse equation (8) can also been used to describe the ultrashort pulses. To conclude, we argue that the criticism in the Comment [16] is incorrect and Youssoufa et al. used an alternative but similar approximation in deriving the same complex short pulse equation.

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