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From Kardar-Parisi-Zhang scaling to explosive desynchronization in arrays of limit-cycle oscillators

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Phase oscillator lattices subject to noise are one of the most fundamental systems in nonequilibrium physics. We have discovered a dynamical transition which has a significant impact on the synchronization dynamics in such lattices, as it leads to an explosive increase of the phase diffusion rate by orders of magnitude. Our analysis is based on the widely applicable Kuramoto-Sakaguchi model, with local couplings between oscillators. For 1D lattices, we observe the universal evolution of the phase spread that is suggested by a connection to the theory of surface growth, as described by the Kardar-Parisi-Zhang (KPZ) model. Moreover, we are able to explain the dynamical transition both in 1D and 2D by connecting it to an apparent finite-time singularity in a related KPZ lattice model. Our findings have direct consequences for the frequency stability of coupled oscillator lattices.

I. INTRODUCTION

Networks and lattices of coupled limit-cycle oscillators do not only represent a paradigmatic system in nonlinear dynamics, but are also highly relevant for potential applications. The reason is that the coupling can serve to counteract the effects of the noise that is unavoidable in real physical systems. Synchronization between oscillators can drastically suppress the diffusion of the oscillation phases and can therefore improve the overall frequency stability. Experimental implementations of coupled oscillators include laser arrays [1], coupled electromagnetic circuits, e.g. [2, 3], and the modern recent example of coupled electromechanical and optomechanical oscillators [4–8]. In this work, we will deal with the experimentally most relevant case of 1D and 2D lattices.

Naive arguments indicate that the diffusion rate of the collective phase of N synchronized oscillators is suppressed as $1/N$, which leads to the improvement of frequency stability mentioned above. However, it is far from guaranteed that this ideal limit is reached in practice [9, 10]. The nonequilibrium nonlinear stochastic dynamics of the underlying lattice field theory is sufficiently complex that a more detailed analysis is called for. In this context, it has been conjectured earlier that there is a fruitful connection [11] between the synchronization dynamics of a noisy oscillator lattice and the Kardar-Parisi-Zhang (KPZ) theory of surface growth [12, 13].

We have been able to confirm that this is indeed true, particularly for 1D lattices. However, the most important prediction of our analysis is that a certain dynamical instability can take the lattice system away from KPZ-like behavior during the time evolution. As we will show, this instability is related to an apparent finite-time singularity in the related KPZ lattice model. It has a significant impact on the phase dynamics because the phase spread is increased by several orders of magnitude. As such, this phenomenon represents an important general feature of

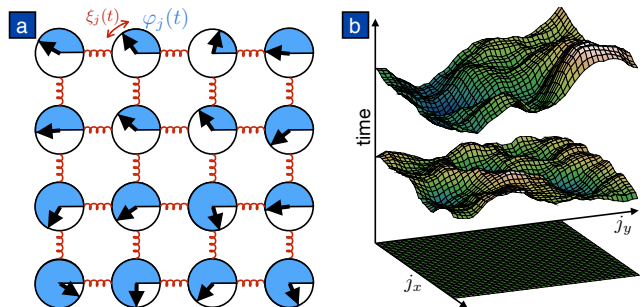


Figure 1. (a) Scheme of an oscillator array. We consider one- and two-dimensional lattices of limit-cycle oscillators, which are described individually by their phases $\varphi_j(t)$. These phases are influenced by noise $\xi_j(t)$ and by the coupling to their nearest neighbors, see Eq. (1). (b) Stochastic time evolution of the phase field in a 2D array of coupled oscillators (smoothed for clarity). The field is flat initially and roughens with time.

the dynamics of oscillator lattices.

II. EFFECTIVE PHASE MODEL

We will describe the time evolution of the coupled limit-cycle oscillators by the Kuramoto-Sakaguchi model [14, 15]. This is an effective model for the slow phase dynamics in the system (see Fig. 1), which describes the physics well if the amplitude fluctuations are small. The model can actually be derived by integrating out those fluctuations in the microscopic equations, which has recently been done for electromechanical [16] and optomechanical [17–19] oscillators. Effective phase models are studied widely in the context of synchronization and pattern formation, see [11, 14, 15, 20–24].

Here, we focus on the ideal case of a non-disordered lattice, with uniform natural frequencies and local coupling. Thus, our system is described by the follow-

ing noisy Kuramoto-Sakaguchi model for the oscillator phases $\varphi_j(t)$:

$$\dot{\varphi}_j = S \sum_{\langle k,j \rangle} \sin(\varphi_k - \varphi_j) + C \sum_{\langle k,j \rangle} \cos(\varphi_k - \varphi_j) + \xi_j, \quad (1)$$

where $\xi_j(t)$ is a Gaussian white noise term with correlator $\langle \xi_j(t) \xi_k(0) \rangle = 2D_\varphi \delta(t) \delta_{jk}$, and S and C are the coupling parameters. The sums run over nearest neighbors. We will often call this model the “phase model”. In this article, we focus on the time evolution from a homogeneous initial state ($\varphi_j(0) = 0$), to track the roughening of the phase field.

How does the interplay of noise and coupling affect the frequency stability of the oscillators? This is a central question for synchronization and metrology. It can be discussed in terms of the *average frequencies* $\Omega_j(t) = t^{-1} \int_0^t dt' \dot{\varphi}_j(t') = \varphi_j(t)/t$. Here the $\varphi_j(t)$ are the *phases accumulated* during the full time evolution (see also [15, 25]). They have an important physical meaning in the present setting, essentially indicating the number of cycles that have elapsed. Important insights can be obtained from studying the evolving spread of the average frequencies, $w_\Omega(t) = \langle N^{-1} \sum_{j=1}^N [\Omega_j(t) - \bar{\Omega}(t)]^2 \rangle^{1/2}$, where $\bar{\Omega}(t) = N^{-1} \sum_{j=1}^N \Omega_j(t)$ is the mean average frequency of a lattice with N sites. This spread turns out to be directly related to the spread of the phase field, $w_\varphi(t)$, with

$$w_\varphi^2(t) = \left\langle \frac{1}{N} \sum_{j=1}^N [\varphi_j(t) - \bar{\varphi}(t)]^2 \right\rangle = t^2 w_\Omega^2(t), \quad (2)$$

where $\bar{\varphi}(t) = \bar{\Omega}(t)t$ is the mean (spatially averaged) phase. The angular brackets denote an ensemble average over different realizations of the noise.

For the simple case of uncoupled identical oscillators subject to noise, one finds $w_\varphi(t) = \sqrt{2D_\varphi t}$ and hence $w_\Omega(t) \sim t^{-1/2}$. This indicates a strong tendency towards synchronization because there is no disorder. We will see that the coupling between the oscillators can lead to different exponents, depending on the parameter regime, and that it can either enhance or hinder the synchronization process. We expect that this finding translates also to systems with small disorder in the natural frequencies.

III. RELATION TO THE KARDAR-PARISI-ZHANG MODEL

Much of our discussion of the initial stages of evolution will rely on small phase differences between neighboring sites. Then the phase model, Eq. (1), is well approximated by a second-order expansion (see also [11]), which can be recast in dimensionless form using a *single* parameter $g_{1D,2D} = 4D_\varphi C^2/S^3$. In a one-dimensional array,

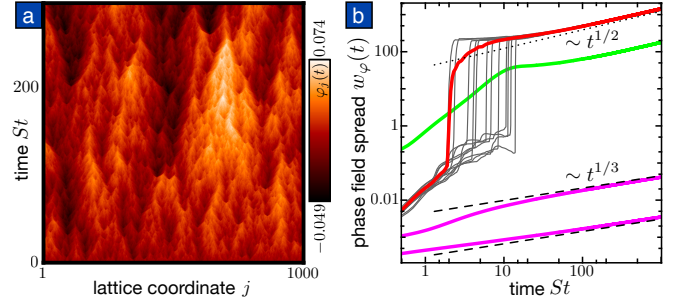


Figure 2. Dynamics in the one-dimensional Kuramoto-Sakaguchi model, Eq. (1). (a) Typical time evolution of the phase field from homogeneous initial conditions. We subtracted a trivial global drift of the phases. (b) Time evolution of the phase spread $w_\varphi(t)$. The magenta curves (two lowest solid lines) show the results for $g_{1D} = 8$ (upper curve) and $g_{1D} = 1$ (lower curve). After initial transients, they approach an asymptotic KPZ scaling of $w_\varphi(t) \propto t^{1/3}$ (dashed black lines). For $g_{1D} = 50$, we plot examples of the phase spread from single simulations as thin gray lines. The red curve (thick dark gray line) shows a small-ensemble average. After a rapid increase, it eventually approaches diffusive behavior, $w_\varphi(t) \propto t^{1/2}$ (dotted black line; $S/C = 0.001$ in all simulations). For comparison, we show the green curve (thick light gray line), where $S/C = 0.1$ and $g_{1D} = 25$. Note the logarithmic scale of the axes. (See Appendix D for more details.)

the resulting model reads

$$\begin{aligned} \frac{\partial h_j}{\partial \tau} &= (h_{j+1} + h_{j-1} - 2h_j) \\ &+ \frac{1}{4}[(h_{j+1} - h_j)^2 + (h_{j-1} - h_j)^2] + \sqrt{g_{1D}} \eta_j, \end{aligned} \quad (3)$$

where we have rescaled both the time, $\tau = St$, and the phase field, $h_j = -(2C/S)(\varphi_j - 2Ct)$. The noise correlator is $\langle \eta_j(\tau) \eta_k(0) \rangle = 2\delta_{jk} \delta(\tau)$.

Eq. (3) can be readily identified as a lattice version of the Kardar-Parisi-Zhang (KPZ) model [12, 13, 26], a universal model for surface growth and other phenomena. This nonlinear stochastic continuum field theory describes the evolution of a height field $h(\vec{r}, t)$,

$$\dot{h} = \nu \Delta h + \frac{\lambda}{2} (\nabla h)^2 + \eta, \quad (4)$$

with white noise $\eta(\vec{r}, t)$, where $\langle \eta(\vec{r}_1, t) \eta(\vec{r}_2, 0) \rangle = 2D\delta^d(\vec{r}_2 - \vec{r}_1)\delta(t)$. The diffusive term smoothes the surface, while both the noise and the nonlinear gradient term tend to induce a roughening. Surface growth dynamics has been found to obey universal scaling laws [12, 27].

The continuum KPZ model in one dimension can be rescaled to become parameter-free. Its lattice version, however, depends on one dimensionless coupling constant $g_{1D} = aD\lambda^2/\nu^3$ [13, 28, 29], where a is the lattice spacing (see also Appendix A).

The relation of the KPZ model to coupled oscillator lattices has been pointed out before [11]. However, up

to now it has remained unclear how far this formal connection is really able to predict universal features of the synchronization dynamics. We will tackle this question in the following, focussing on the regime $S/C \ll 1$, which is the regime where KPZ dynamics turns out to be most fruitful for understanding the behavior of the full phase model (see Appendix A).

IV. DYNAMICS IN ONE-DIMENSIONAL SYSTEMS

First insights can be gained by numerical simulations of the phase model. The outcome of a single simulation of a one-dimensional array is displayed in Fig. 2a. The typical time evolution of the phase spread $w_\varphi(t)$ is shown in Fig. 2b. We can distinguish two parameter regimes from the long-time evolution. In one regime, we find $w_\varphi(t) \sim t^{1/3}$ after initial transients (see magenta curves (two lowest solid lines)). This power-law growth can be identified as universal KPZ behavior: In the continuum KPZ model, the scaling exponent can be calculated analytically in 1D and turns out to be $1/3$ [12]. Hence, we conclude that 1D arrays of limit-cycle oscillators, as described by the noisy Kuramoto-Sakaguchi phase model, indeed show KPZ scaling in certain parameter regimes.

Far more surprising is the other regime (red and green curves (thick gray lines)), where one observes diffusive growth, $w_\varphi(t) \sim t^{1/2}$, for long times. This clearly deviates from any KPZ predictions. Additionally, the short-time behavior is remarkable: In the trajectories of single simulations, we see an explosive growth of $w_\varphi(t)$ at some random intermediate time (thin gray lines). At this time, the variance of the phase field suddenly grows by several orders of magnitude. This corresponds to an explosive desynchronization of the oscillators.

To understand this important dynamical feature better, we now briefly turn away from the full phase model and study the evolution of the related lattice KPZ model, Eq. (3). The result of a single simulation is shown in Fig. 3a. We see that an instability develops, which now leads to an apparent (numerical) finite-time singularity.

The occurrence of such an instability is a random event. In Fig. 3b, we plot the probability of an instability during the evolution, as a function of the time τ and the coupling g_{1D} . In principle, instabilities can occur at all coupling strengths, but we find that for the lattice size employed here (1000 sites) they become much less likely (happen much later) for $g_{1D} < 40$. To extrapolate to larger lattices, we may assume that the stochastic seeds for the instabilities are planted independently in different parts of the system. Hence, one could calculate the probabilities for any N from our results.

It is worthwhile to note that divergences had been identified before in numerical attempts to solve the KPZ dynamics on a lattice [30–32] (see also [33, 34]). In those simulations, this behavior was considered to be a numerical artefact depending on the details of the discretization

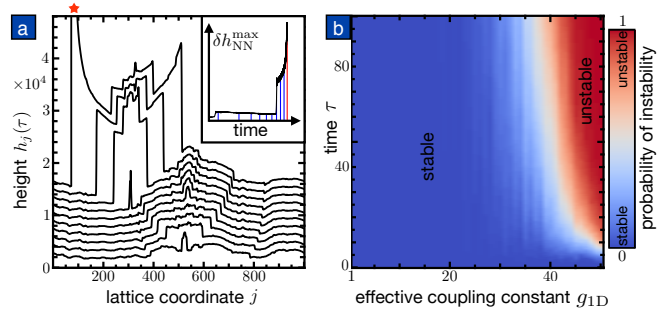


Figure 3. Instabilities in the one-dimensional lattice KPZ model as given by Eq. (3). (a) Typical time evolution of the height field $h_j(\tau)$ for large coupling parameter $g_{1D} = 50$, on a lattice with 1000 sites. We plot the height field for increasing times from bottom to top. The curves are vertically offset for clarity. The numerical divergence occurs at the point marked with a red star. The selected time points approach the divergence time logarithmically, as indicated in the inset. There, we also show the evolution of the maximum nearest neighbor difference, δh_{NN}^{\max} , just before the divergence. (b) The probability of encountering an instability up to time τ , as a function of the coupling g_{1D} . We see that an instability is more likely to occur earlier for increasing values of g_{1D} . Note that the probability of instabilities depends on the lattice size.

because it does not show up in the continuum model, at least in one dimension [31]. On the contrary, our phase model, describing synchronization in discrete oscillator lattices, is a genuine lattice model from the start. Hence, the onset of instabilities has to be taken seriously (see also [35]). In the full phase model, Eq. (1), the incipient divergences are eventually cured because the trigonometric functions in Eq. (1) are bounded. Instead of resulting in a finite-time singularity, they will lead the system away from KPZ-like behavior and make it enter a new dynamical regime.

To find out for which parameters this happens, we have determined numerically the probability of encountering large growth of nearest-neighbor phase differences. We find that we can distinguish between a “stable” regime, where no large phase differences ($> \pi$) show up in most simulations, and an “unstable” regime, where large differences occur with a high probability. We indeed get quantitative agreement with the results discussed above for the lattice KPZ model, Fig. 3b, for small S/C (< 0.001).

In a single simulation in the unstable parameter regime of the stochastic phase model (from homogeneous initial conditions), we typically observe that the phase field develops as in the corresponding KPZ lattice model initially. Then, a KPZ-like instability induces large nearest-neighbor phase differences within a certain region. There, turbulent dynamics takes over, which is a feature already present in the purely deterministic phase model. Moreover, in this region the average phase velocity is very large. The turbulent region then expands over the whole lattice quickly. Eventually, we observe turbulence everywhere, which leads to a rapid phase diffusion (see Ap-

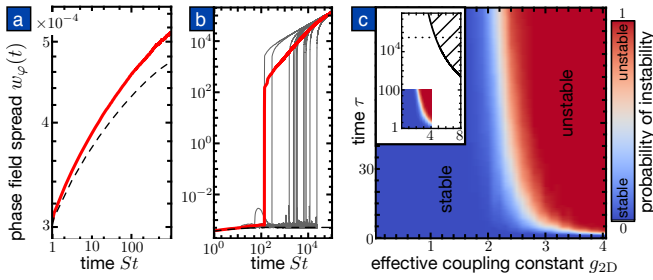


Figure 4. Dynamics in two-dimensional models. (a) The phase model, Eq. (1), displays a slow logarithmic growth of the phase spread for $g_{2D} = 1$ (red curve (solid line); $S/C = 0.001$). The linear theory would lead to a slightly different behavior, see the dashed black lines in (a) and (b). (b) Same quantity for a slightly larger coupling, $g_{2D} = 1.5$ (red curve (thick dark gray line); $S/C = 0.001$). Due to explosive instabilities (single trajectories shown as thin gray lines), there is a rapid increase to much larger values. (c) Lattice KPZ: Probability of instability in the lattice KPZ model, the 2D version of Eq. (3). The inset shows that the power-law 2D KPZ scaling (hatched region) would be expected at much later times than the instabilities (note the logarithmic scaling of the time axis). This makes the scaling unobservable also in the phase model, where the instabilities induce a different dynamical regime.

pendix C for more details). This time evolution is reflected in the phase spread, see the thin gray lines in Fig. 2b: The expansion of the turbulent region leads to an explosive growth, whereas the subsequent diffusion leads to the asymptotic scaling $w_\varphi \sim t^{1/2}$.

We emphasize that the phenomenon of explosive desynchronization is not rooted in simple stochastic phase slips (where the phase difference between two neighboring sites increases by roughly 2π). Rather, the instabilities we find consist in rapidly growing phase differences that occur already when the phase differences themselves are still much smaller than π . Thus, phase slips are not yet relevant during this stage of the dynamics. Eventually, at a later stage, the instabilities lead to phase differences comparable to π , and then also to phase slips. In other words, the instabilities responsible for explosive desynchronization precede phase slips, not the other way around.

To make this point even clearer, note that the relevant parameter for distinguishing the stable regime from the unstable regime in the phase model is $g_{1D} = 4D_\varphi C^2/S^3$. In particular, this means that one can get large phase differences also for small noise strength D_φ , if g_{1D} is chosen to be large. If the KPZ-like dynamics were not present, one would only get occasional phase slips in this case, and the time scale for such slips would be very long.

Hence, we conclude that in the unstable regime of the one-dimensional phase model, the onset of KPZ-like instabilities triggers deterministic turbulence and thus induces an explosive desynchronization of the oscillators. This is followed by diffusive growth of $w_\varphi(t)$ with a large diffusion coefficient.

V. DYNAMICS IN TWO-DIMENSIONAL SYSTEMS

We now consider 2D lattices. This promises to be interesting because the related physics of surface growth depends crucially on the dimensionality. This can be seen in the continuum KPZ model: An appropriately rescaled form contains a single dimensionless parameter (in contrast to 1D), which is $g_{2D} = D\lambda^2/\nu^3$. As a consequence, there are different time regimes in the growth of the surface width [36]. In particular, KPZ power-law scaling $w \sim t^\beta$ sets in beyond a time scale t^* that becomes exponentially large at small couplings, $t^* \sim \exp(16\pi/g_{2D})$. This is important in numerical attempts to observe this scaling, as in [37].

The lattice version of the 2D KPZ model, as obtained by extending Eq. (3) to two dimensions, also contains the single coupling parameter g_{2D} . We study the probability of encountering instabilities, see Fig. 4c, and find that it increases rapidly with larger coupling. This is qualitatively the same as in the 1D situation. However, there are additional, crucial consequences: the instabilities occur much earlier than the exponentially late onset of KPZ power-law scaling. We note that there are discretizations of the KPZ model which show different behavior, allowing the observation of KPZ scaling [37, 38], but these do not correspond to physical models of coupled phase oscillators. The early onset of instabilities in our model is illustrated in the inset of Fig. 4c, where the hatched region is the KPZ scaling regime from the continuum theory for infinite systems. Moreover, in finite systems, the surface width saturates eventually, for times $(\lambda^2/\nu)t \gg (\lambda L/\nu)^z$. This implies that for small g_{2D} , saturation sets in long before the projected onset of KPZ scaling for any reasonable lattice size. As an example, the dotted line shows the saturation time for $N = 10^6$. Overall, we predict that in 2D the power-law KPZ scaling regime will be irrelevant for the synchronization dynamics of oscillator lattices.

This is confirmed in simulations of the full phase model, Eq. (1), in 2D (Fig. 4a and b). Like in 1D, we focus on small values of S/C . As long as the phase differences remain small and no instabilities occur, which is the case for small $g_{2D} = 4D_\varphi C^2/S^3$, the behavior is analogous to the lattice KPZ model, see Fig. 4a. However, as suggested above, the KPZ scaling regime cannot be reached. Instead, we observe slow, logarithmic growth of the phase field spread, similar to the expectation from the linearized KPZ equation (dashed line; this is called Edwards-Wilkinson scaling, see also [36, 39] and Appendix D). According to Eq. (2), the slow growth of w_φ implies quick synchronization, $w_\Omega \sim \sqrt{\ln(t)/t}$.

As long as phase slips do not come into play, the Edwards-Wilkinson scaling of the phase field spread also shows up in simulations of the phase model with parameter $C = 0$, where there is no term corresponding to the KPZ nonlinearity in the equation of motion. This special limiting case of our model is just the XY model [40], which has been studied thoroughly.

At larger couplings g_{2D} , we observe explosive desynchronization (see Fig. 4b, red curve (thick dark gray line)), like in 1D, and the diffusive growth for long times (not shown here). This behavior is also displayed in the extreme case $S = 0$, which corresponds to $g_{2D} \rightarrow \infty$. The deterministic model with $S = 0$ has been studied in detail in [35].

VI. CONCLUSION

In conclusion, we have studied the phase dynamics of 1D and 2D lattices of identical limit-cycle oscillators, described by the noisy Kuramoto-Sakaguchi model. We have shown that, depending on parameters, the coupling can either enhance or hinder the synchronization. In 1D, for sufficiently small noise and at short times, one can observe roughening of the phase field as in the Kardar-Parisi-Zhang model, with the corresponding universal power-law scaling. At larger noise, or for larger times, explosive desynchronization sets in, triggering a transition into a different dynamical regime. We have traced back this behavior to an apparent finite-time singularity of the approximate (KPZ-like) lattice model. This is especially relevant for two dimensions, where it will occur before the long-term KPZ scaling sets in, although the initial slow logarithmic growth still makes 2D arrays favorable for synchronization.

Our predictions will be significant for all studies of synchronization in locally coupled oscillator arrays, when the phase-only description is applicable. This can be the case in optomechanical arrays (e.g. in extensions of the work presented in [5]). Our results may also become important for the study of nonequilibrium driven-dissipative condensates, described by the stochastic complex Ginzburg-Landau equation or Gross-Pitaevskii-type equations, where a connection to the KPZ model has been explored recently [41–47] for the continuum case. Once these studies are extended to lattice implementations of such models (e.g. in optical lattices), one may encounter the physics predicted here. Indeed, for one-dimensional systems, dynamical instabilities similar to the ones discussed in our work have been identified to play an important role recently (see [48]).

ACKNOWLEDGMENTS

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Appendix A: Rescaling the models

In the main text, we used rescaled versions of the various models. Here, we will explain how to arrive at these

rescaled equations.

In the continuum KPZ model, Eq. (4), in one dimension, we can construct the length scale $x_0 = \nu^3/D\lambda^2$, the time scale $t_0 = l_0^2/\nu$, and a scale for the height, $h_0 = \nu/\lambda$. Using this, we get the dimensionless quantities $\tilde{x} = x/x_0$, $\tilde{t} = t/t_0$ and $\tilde{h} = h/h_0$. In terms of those variables, the one-dimensional KPZ equation becomes parameter-free and reads

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} + \frac{1}{2} \left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 + \tilde{\eta}, \quad (\text{A1})$$

with the rescaled noise term $\tilde{\eta}(\tilde{x}, \tilde{t})$ with correlator $\langle \tilde{\eta}(\tilde{x}, \tilde{t}) \tilde{\eta}(0, 0) \rangle = 2\delta(\tilde{x})\delta(\tilde{t})$. In the KPZ model in two dimensions, we cannot construct independent scales for the height and the space variables. Hence, after rescaling, we are left with one parameter $g_{2D} = D\lambda^2/\nu^3$.

In contrast to the continuum model, discretized versions of the KPZ equation in one dimension will contain one dimensionless parameter. To arrive at our particular lattice model, Eq. (3), we define a lattice constant a and rescale time, $\tau = (\nu/a^2)t$, and height, $h_j(\tau) = (\lambda/\nu)h(x, t)$, in Eq. (4). Additionally, we discretize the derivatives according to $\Delta h \rightarrow h_{j+1} + h_{j-1} - 2h_j$ and $(\nabla h)^2 \rightarrow (h_{j+1} - h_j)^2 + (h_{j-1} - h_j)^2$. The resulting model contains the coupling constant $g_{1D} = aD\lambda^2/\nu^3$.

A similar procedure was also performed in [28]. Note, however, that in this article (and for other numerical studies) the derivatives were discretized in a different way, which also leads to drastically different stability properties. In our study, we want to compare the behavior of the phase model to the one of the lattice KPZ model. Because Eq. (3) was derived as an approximation of the phase model, this dictates the way of discretizing the KPZ model.

In the derivation of Eq. (3) from the phase model, Eq. (1), we rescaled the time, $\tau = St$, and the phase field, $h_j = -(2C/S)(\varphi_j - 2Ct)$. If this rescaling is done in the full phase model in one dimension, we arrive at

$$\begin{aligned} \frac{\partial h_j}{\partial \tau} = & 2\frac{C}{S} \sum_{\langle k, j \rangle} \sin \left[\frac{S}{2C}(h_k - h_j) \right] \\ & - 2\frac{C^2}{S^2} \sum_{\langle k, j \rangle} \left\{ \cos \left[\frac{S}{2C}(h_k - h_j) \right] - 1 \right\} + \sqrt{g_{1D}} \tilde{\xi}_j, \end{aligned} \quad (\text{A2})$$

with the rescaled noise term $\tilde{\xi}_j(\tau)$ with correlator $\langle \tilde{\xi}_j(\tau) \tilde{\xi}_k(0) \rangle = 2\delta(\tau)\delta_{jk}$. We see that this equation contains two parameters, $g_{1D} = 4D_\varphi C^2/S^3$ and S/C .

In our studies, the nearest-neighbor differences $h_k - h_j$ are initially small and evolve according to Eq. (3). With time, those differences increase (in general). The equation displayed here, Eq. (A2), shows that Eq. (3) is a particularly good approximation to the phase model (i.e. it is viable up to longer times) for small values of the parameter S/C . This is because (for given differences $h_k - h_j$), the arguments of the trigonometric functions will be small. Hence, we choose to focus our analysis on

the regime $S/C \ll 1$. For example, this enables us to observe KPZ scaling $w_\varphi(t) \sim t^{1/3}$ in simulations of the one-dimensional phase model (see Fig. 2b). We note that this is not possible for arbitrary values of S/C because the KPZ scaling might not set in before phase differences become large and the resemblance to KPZ dynamics in general is therefore lost.

Appendix B: Scaling behavior of the KPZ surface width

In the theory of surface growth, the surface width $w(L, t)$ is defined as

$$w^2(L, t) = \langle \frac{1}{L^d} \int d^d r [h(\vec{r}, t) - \bar{h}(t)]^2 \rangle, \quad (\text{B1})$$

with the average surface height $\bar{h}(t) = L^{-d} \int d^d r h(\vec{r}, t)$ in a system of linear size L . The surface width has been found to obey a scaling law $w^2(L, t) \sim L^{2\zeta} F(t/L^\zeta)$ [27]. In particular, for times $(\lambda^2/\nu)t \ll (\lambda L/\nu)^z$, we have $w^2(L, t) \sim t^{2\beta}$ with $\beta = \zeta/z$. In one dimension, the scaling exponent β can be calculated analytically and is $\beta = 1/3$ [12].

Appendix C: Details on the development and consequences of instabilities

We discussed the development and consequences of instabilities both in the lattice KPZ model and in the phase model in the main text. Here, we will give some more details.

For the KPZ model, we displayed the time evolution in a single simulation with large parameter g_{1D} in Fig. 3a. In addition to the normal roughening process, which we expect from the continuum theory, we see the rapid growth of single peaks. Those can send out shocks of large height differences, which then propagate through the system, as can be seen in the center of Fig. 3a. The collision of such shocks can produce larger peaks. We commonly observe that eventually very large shocks grow during propagation, which leads to the singularity in the numerical evolution (marked with a red star in the figure). The details of the instability development depend on the lattice size and the coupling parameter. For example, for very small lattices with periodic boundary conditions, the shocks which were sent out from a single peak might collide after crossing the boundaries. This can produce a divergence easily. We note that for larger lattices, this somewhat trivial self-amplification is typically not the process which leads to the divergences.

We now turn to the one-dimensional phase model with large parameter g_{1D} . Fig. 5 shows snapshots of the phase field from a simulation with $g_{1D} = 50$ for different points in time. As explained in the main text, the onset of KPZ-like instabilities induces large phase differences locally. Because of the deterministic dynamics, this part of the

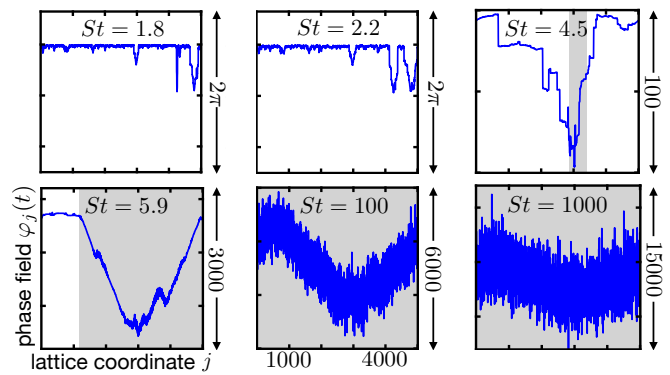


Figure 5. Phase field time evolution in the one-dimensional phase model, Eq. (1), in a simulation where instabilities occur. We show snapshots of the phase field for increasing times. After a KPZ-like time evolution with small phase differences in the beginning, we see the onset of instabilities. This leads to the rapid development of a triangular structure, which gets diffused on very long time scales. This phenomenon can be explained by turbulent deterministic dynamics (in the gray regions). Note the very different scale of the vertical axes in the subsequent panels. Parameters: $S/C = 0.001$, $D_\varphi/S = 1.25 \times 10^{-5}$ (resulting in $g_{1D} = 50$), $S\Delta t = 10^{-4}$.

phase field becomes turbulent. As a consequence of the large phase differences, the turbulent region will have a very different phase velocity from the KPZ-like region (on average). At the same time, the turbulent part of the lattice (shaded areas in the plots of Fig. 5) grows in space, roughly linearly with time. These two processes lead to a triangular phase field shape covering the whole lattice. Additionally, the turbulent dynamics keeps increasing the phase differences, including wrap-arounds by 2π .

The consequences of this time evolution are evident in the evolution of the average phase field spread, see the red curve (and also the individual gray curves) in Fig. 2b: The formation of the triangle is responsible for a rapid increase. Afterwards, the ubiquitous turbulence leads to phase diffusion. To be more precise, the behavior of w_φ (red curve) after the rapid increase can be fitted very well with $w_\varphi(t) = \sqrt{A + Bt}$, with fit parameters A and B . The diffusion coefficient B turns out to be much larger than the noise strength D_φ . We found the same value of the diffusion coefficient in simulations of the deterministic Kuramoto-Sakaguchi model with random initial conditions (and the same parameter value S/C), which shows turbulent behavior. This indicates that the diffusive behavior of $w_\varphi(t)$ for long times in the stochastic simulations is indeed due to deterministic turbulence.

Appendix D: Methods

The numerical time integration of the coupled Langevin equations on the lattice was performed with the algorithm presented in [49]. In the following, we provide further details on the parameters employed for the

simulations whose results are shown in the figures.

For the simulations of the full phase model in one dimension in Fig. 2, we employed the following parameters. Fig. 2a: $S/C = 0.001$, $D_\varphi/S = 2 \times 10^{-6}$, (resulting in $g_{1D} = 8$), $S\Delta t = 0.01$, $N = 5 \times 10^3$. We only show a part of the phase field. Fig. 2b: Parameters for the upper magenta curve: $S/C = 0.001$, $D_\varphi/S = 2 \times 10^{-6}$, $S\Delta t = 0.01$, $N = 10^4$. Lower magenta curve: $S/C = 0.001$, $D_\varphi/S = 2.5 \times 10^{-7}$, $S\Delta t = 0.1$, $N = 10^4$. For both magenta curves, the average was taken over 300 simulations. For the red curve: $S/C = 0.001$, $D_\varphi/S = 1.25 \times 10^{-5}$, $S\Delta t = 0.001$, $N = 10^3$. For the green curve: $S/C = 0.1$, $D_\varphi/S = 0.0625$, $S\Delta t = 0.001$, $N = 10^3$. For those two curves, the average was taken over 120 simulations.

We now turn to the simulations of the KPZ model. In general, direct numerical simulations of this model where the scaling properties are extracted are always performed for stable evolution. Hence, they are done in the small-coupling regime, also for slightly different lattice realizations with quantitatively different stability properties, see [28]. There, it is also found that the parameter g_{1D} has an influence on the transient dynamics in one dimension (see also [29]) which explains the transients that we observed in the phase model, in Fig. 2b (magenta curves).

In Fig. 3b, we plot the probability of encountering instabilities in the 1D KPZ lattice model as given by Eq. (3), for a wide range of the coupling parameter g_{1D} . The data is extracted from 300 simulations for each value of $g_{1D} = 1, 2, \dots, 50$, running up to time $\tau = 100$, with a time step $\Delta\tau = 10^{-4}$. The probability of instability is just the ratio of unstable simulations. We checked that the results for this quantity do not change at $g_{1D} = 50$ if we go to a smaller time step of $\Delta\tau = 10^{-5}$. A simulation was considered unstable when one of the nearest-neighbor height differences at one lattice site exceeded a large value, which was chosen to be 10^5 . We used a lat-

tice size of $N = 1000$. The probability of an instability generally increases for larger lattices. An exception are very small lattices, where boundary effects can become important (see the example above).

Fig. 4c shows the results for the probability to find an unstable simulation in the 2D KPZ lattice model. The data for the plot is from 300 simulations for each value of $g_{2D} = 0.1, 0.2, \dots, 4$, on a lattice of size $N = 64^2$ with time step $\Delta\tau = 0.01$. A simulation was considered unstable when one of the nearest neighbor height differences at one lattice site exceeded a large value, which was chosen to be 10^8 . As in 1D, the probability of instability depends on the lattice size.

Regarding the results for the two-dimensional phase model, the red curve in Fig. 4a shows the phase field spread from an average over 300 simulations with the following parameters: $S/C = 0.001$, $D_\varphi/S = 2.5 \times 10^{-7}$, $N = 256^2$, $S\Delta t = 0.1$. For the 300 simulations for the red curve in Fig. 4b, we used the parameters $S/C = 0.001$, $D_\varphi/S = 3.75 \times 10^{-7}$, $N = 64^2$, $S\Delta t = 0.1$.

We commented in the main text on the relation of the results for the phase model, which are shown in Fig. 4a, to predictions from the linearized KPZ equation. This linear model is the Edwards-Wilkinson model [39], which produces a slow logarithmic growth of the surface width in a continuum system [36]. Because the model is linear, we can also straightforwardly take into account the effects of the lattice discretization and the finite size of the lattice. The resulting analytical prediction is shown as the dashed line in Fig. 4a. The red curve shows a good initial fit and some deviations only at later times. Further investigation reveals that the two-dimensional lattice version of the KPZ model (in analogy to Eq. (3)) shows the same deviations. We checked that another lattice version of KPZ (as in [28]) does indeed agree with the result from the linear equation. The reason for the discrepancy in different lattice models might be more subtle influences of the nonlinearity, as also reported in [50].

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