Phase diagram and criticality of the two-dimensional prisoner's dilemma model

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Phase diagram and criticality of the two-dimensional prisoner's dilemma model

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The stationary states of the prisoner's dilemma model are studied on a square lattice taking into account the role of a noise parameter in the decision making process. Only first neighboring players, defectors and cooperators, are considered in each step of the game. Through Monte Carlo simulations we determined the phase diagrams of the model in the plane noise versus temptation to defect for a large range of values of the noise parameter. We observed three phases: cooperators and defectors absorbing phases and a coexistence phase between them. The phase transitions as well as the critical exponents associated with them were determined using both static and dynamical scaling laws.

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I. INTRODUCTION

The study of simple models in Physics has been and remains a powerful tool to understand basic and fundamental aspects of natural phenomena. Despite their simplicity they succeed to describe quantitative and qualitatively how some real systems work. One model belonging to this set is the named prisoner's dilemma model (PD). From the physical point of view, even its simpler versions can present chaotic behaviour, different absorbing phases, coexistence of states and phase transitions between the coexistence and absorbing phases. The PD model was first proposed by Flood and Dresher in the fifties [1] in the context of the Game Theory [2, 3] and more recently it was used in studies of emergence of cooperation in Social Sciences by Axelrod [4]. In a evolutionary spatial version of the model, Nowak and May [5–7] observed several spatial and temporal chaotic patterns. However, it was a pioneering article by Szabó and Tóke [8] who brought it to the statistical physics community [9–21].

The model contains just four ingredients, that conveniently defined, can mimic real systems in a broad area of research [20]: the players, their strategies and the reward received when they play a round (payoff matrix) and an updating evolutionary dynamics. The players are placed on a lattice or on a network and the opponents of each player are their nearest neighbour players. Consider that N identical players can assume only two types of strategies: cooperation (C) and defection (D). Every round all players must decide if they keep their current strategy or change to another one. For a direct encounter between two players the payoff received is obtained from the payoff matrix Π defined as

$$\Pi = \begin{pmatrix} R & S \\ T & P \end{pmatrix},$$

where $R(P)$ is the reward if two players decide cooperate (defect), $T(S)$ is the reward received by a defector (cooperator) exploiting (exploited) a cooperator (by a defector). To characterize a prisoner's dilemma the conditions $T > R > P > S$ and $2R > T + S$ must be satisfied. The model will depend on only one parameter if we take $R = 1, T = b$ and $S = P = 0$. Then $1 < b < 2$.

The evolutionary rule to obtain a new configuration in the next round can be specified, for instance, by choosing a player and an opponent. If the total payoff received by the player is equal or greater than the value of its opponent payoff it decides to keep the same strategy. Otherwise, it changes to the strategy of its opponent. In this situation the decision is completely rational. However, it is possible to include some irrationality if we define a probability of transition from one strategy to another depending on both payoff values and of some parameter able to mimic noise in this decision. As an example we present below the expression proposed by Szabó and Tóke [8] for the transition probability

$$w = \frac{1}{1 + \exp\left[(E_p - E_o)/K\right]}, \tag{1}$$

where E_p and E_o are the total payoff values of the chosen player and of its opponent, respectively, and K is a noise parameter. It is important to point out that different ways can be adopted to choose the player's opponent and to calculate the total payoffs. So, an opponent can be randomly chosen or alternatively it can be that one with the highest total payoff within a certain neighbourhood, and the total payoff may include also a self-interaction or not. Besides, the total payoff can be accumulated during the game (memory) or be instantaneous. Other aspects that can be considered in the PD model are the type of lattice (triangular, square, complex, etc.) and network topology (mesh, fully or partially connected, etc.) and the way in which all players are updated (synchronously like in a cellular automata or asynchronously). The model shows an interesting behaviour as a function of the temptation to defect b and noise parameter K. The PD model can present two absorbing states (only cooperators or defectors) separated by a region where cooperators and defectors coexist. In the rational model, the transition (jump) to absorbing states appears to be discontinuous while in the noisy case it is continuous and belongs to the directed percolation universality class (DP).

All versions of the model can be analytically studied by writing the master equation and obtaining time dependent equations, in some level of the mean-field approximation, for the density of cooperators and defectors, as a function of the control parameters b and K [14]. However, it has been observed that only for a high level of the cluster approximation (six-point or more) the mean field theory results are able to agree with the Monte Carlo simulations [15]. Recent advances including dynamic aspiration into the evolution were found in square and sclae-free networks by also using the master equation and Monte Carlo simulation [16]

In this work we have considered the evolutionary PD model on a square lattice in which the opponent of a player is one of its four nearest neighbours. From one round to another all players try to change their strategies (asynchronous update) and the instantaneous total payoff is calculated taking into account the numbers of cooperators in the neighbourhood in two distinct cases, with and without self-interaction. Our main goal is to determine the phase

diagram of the model in the plane b versus K in these two situations, and also to characterize the phase transitions between the cooperation and defection absorbing phases to the coexistence phase. We have used extensive Monte Carlo simulation [22] in order to obtain both phase diagrams, and the dynamical and steady-state scaling theories [23, 24] to determine two critical exponents associated with the transitions. To our knowledge only the phase transition between the D-absorbing and the coexistence phases was studied in a case where self-interaction was neglected, and only for a small range of the noise parameter values. The nature of the transition was predicted to be continuous and belonging to the DP universality class. However, this conclusion was based on the value of the critical exponent associated to the steady-state order parameter for only a single value of K. Here, we have extended that analysis to a broad range of the noise parameter values. Besides, we determined two different critical exponents in two different approaches (dynamical and stationary) for several values of K in the two transitions (C-absorbing $\rightarrow$ coexistence $\rightarrow$ D-absorbing) in the cases with and without self-interaction.

Our work is organized as follow: in Section II we present our version of the PD model and we detail the Monte Carlo simulation procedures. In Section III we present and discuss our results for the critical behaviour of the model. Finally, in Section IV, we address our conclusions.

II. THE MODEL AND THE MONTE CARLO SIMULATIONS

In our version of the evolutionary PD model the players are placed on the sites of a square lattice of linear length L and a state variable $S_i = +1$ ($S_i = -1$) is signalled for a cooperator (defector) i with $i = 1, .., N = L^2$. Initially, the cooperator and defector densities are equal and the values of the parameters b and K are fixed. One player p and a first neighbour opponent o are randomly selected. If they are in the same state, another pair of neighbouring sites are chosen. If not, we determine the instantaneous total payoff for a cooperator using

$$E^C_{p,o} = a + \sum_{i=1}^{4} \delta_{S_i,1}, \tag{2}$$

and

$$E^D_{p,o} = b \sum_{i=1}^{4} \delta_{S_i,1}, \tag{3}$$

for a defector, where δ is the Kronecker's delta symbol. The parameter a is included in order to consider self-interaction for cooperation ($a = 1$) or not ($a = 0$). We have taken into account these two cases because a cooperator can represent a cluster of cooperators [5–7, 14]. We calculate the transition probability w given by Equation (1) and we compare this value with a random number $0 \leq r \leq 1$. If $w < r$ the player remains in its state and for $w \geq r$ it immediately assumes the opponent's strategy. In a round we try to change the strategy of players N times, defining one Monte Carlo step per particle (MCs) as our time unit. At each MCs we determine the density of cooperators and defectors. After a transient time these quantities attain stationary states in which it is possible to calculate their time average values. In general, we have used up to 10^5 rounds to calculate these stationary mean values after neglecting 10^5 MCs to attain the stationary regime. In order to decrease statistical errors we also repeat the experiment over $M = 200$ samples for $L = 100$ and $L = 200$. The largest value of L was used to study the critical behaviour of the model. In two special cases, that will be considered in this work, we use $M = 400$. The time dependent densities of cooperators and defectors were calculated using only surviving samples whose density of cooperators or defectors are not equal to one.

III. RESULTS AND DISCUSSIONS

We have distinguished three different cases, depending on the values of b and K for both values of a: absorbent phases of cooperation or defection and a coexistence phase between cooperators and defectors . In fact, Fig. 1 shows the behaviour of the mean value of the density of cooperators ρ_C as a function of b for $a = 0$ (Fig. 1(a)) and $a = 1$ (Fig. 1(b)) and several values of K. For values of $b \leq b_1$ a cooperator absorbing phase is present and for $b \geq b_2$ only the defector absorbing phase is seen. This Figure includes the $K = 0$ cases in which the values of b_1 are 3/4 (the inset in Fig. 1(a)) and 1 for a equal zero or one, respectively.

Despite the qualitatively very similar plots for the two cases, with two phase transitions, we observe that they are quantitatively different with respect to the values of b and K. If self-interaction is taken into account the range of

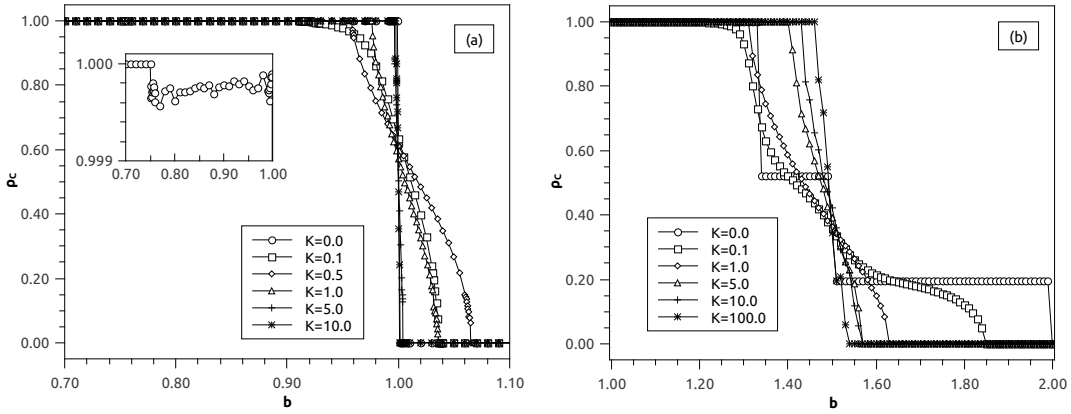

FIG. 1. Normalized density of cooperators as a function of b for several values of K for (a) $a = 0$ and (b) $a = 1$. The inset in (a) corresponds to details of the transition for $K = 0$.

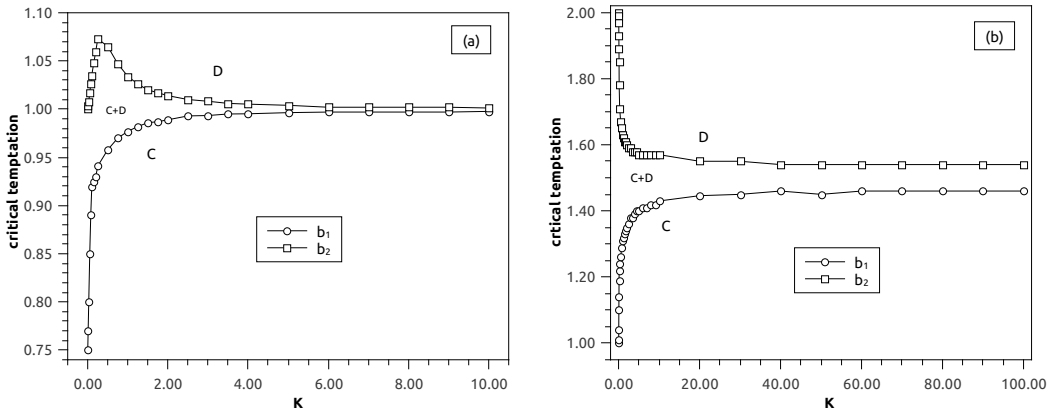

FIG. 2. Phase diagram in the plane b versus K for (a) $a = 0$ and (b) $a = 1$. Open circles represent the critical point b_1 from the absorbing C to the mixed phase $(C + D)$, while open squares represent the critical points b_2 from the mixed phase to D absorbing phase.

values of b in which the coexistence phase occurs is larger than one without self-interaction. Besides, we also note that the coexistence phase holds on a large range of the noise parameter values K. To compare these two cases we present in Fig. 2 the phase diagrams in the plane temptation to defect versus noise parameter. That is, we have plotted the values of b_1 and b_2 for a broad range of values of K for $a = 0$ and $a = 1$ in Figs. 2(a) and 2(b), respectively. To determine the points in which the transitions occur we have used the stationary values of the densities of cooperators ρ_C and defectors ρ_D because one of them goes to zero at the transitions, but they are finite in the active (coexistence) phase. That is, $\rho_C = 0$ for $b \geq b_2$ and $\rho_D = 0$ for $b \leq b_1$.

Below the line connecting the critical points b_1 the lattice is completely filled with cooperators and, above the line connecting the points b_2, it is poisoned with defectors. Between these two lines we have a coexistence phase. As we can see, a qualitative difference appears between the two cases in the transition to the defector's absorbing phase both for small and high values of K. While in the self-interaction case the critical value b_2 continuously decreases from two to 3/2 when K increases, in its absence we first observe a slight increase in the values of b_2 until it attains a maximum value and starts to decreasing to one. The critical points for the non self-interaction case was described in reference [20] and it is reproduced here for a broad range of K values. In the opposite limit (larger values of K) we observe that the coexistence phase tends to disappear in the non self-interaction case, while it remains existing for $a = 1$. This last observation is in contrast to the schematic phase diagram proposed in Ref. [20], in which the coexistence phase also disappears for large values of K when the self-interaction is taken into account.

Since the order parameter of the model has only one-component and a single absorbing state at each transition, it must obey the conjecture of Jansen [25] and Grassberger [26]. So, we expect that the two transitions are continuous for any value of K and to belong to the Directed Percolation (DP) universality class. Besides, in the particular case studied by Szabó and Tóke [8] (coexistence$\rightarrow$ D-absorbing phase transition for $a = 0$), they found a typical power-law

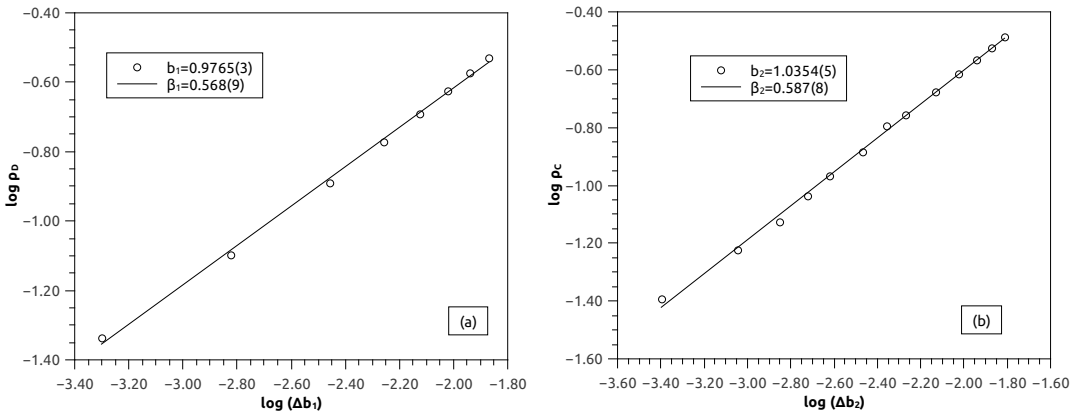

FIG. 3. Log-log plot of the stationary order parameter as a function of the distance from the critical point for $a = 0$ and $K = 1.0$. Open circles represent the simulation data and the straight lines the best fit to them. (a) Transition to the C absorbing phase,(b) Transition to the D absorbing phase.

behaviour for the stationary cooperator density near the transition point b_2 for only one value of K. The value determined for the associated critical exponent is very close to that of the DP.

Now, we turn to investigate the critical properties of the present model for all values of the noise parameter K. We employ two different procedures to find the critical behavior of the model. Then, two critical exponents were determined allowing a better understanding of the universality class of the model. Using the steady-state scaling theory for the absorbing phase transition [23, 24] we expect that the order parameter scales algebraically as

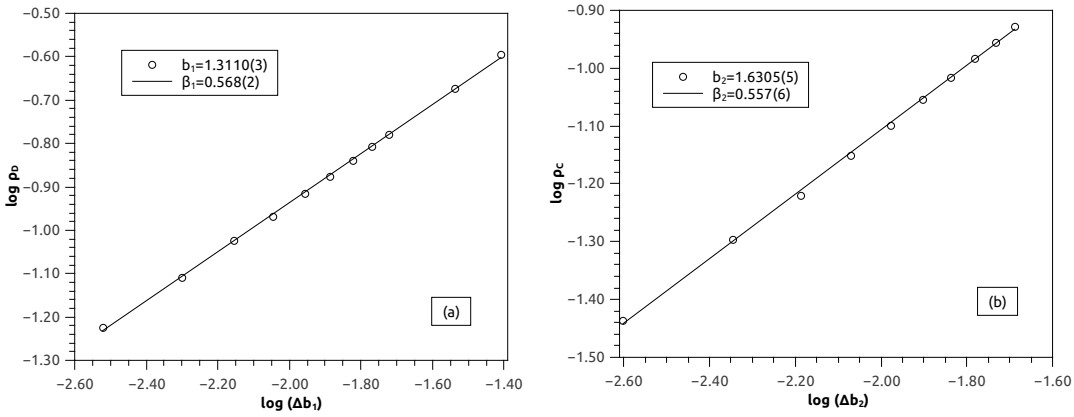

FIG. 4. Same as in Fig. 3 for $a = 1$.

$$\rho_{C,D} \sim (\Delta b_i)^{\beta_i}, \tag{4}$$

where $i = 1$ indicates the transition $\rho_D \to 0$ for $\Delta b_1 = b - b_1 > 0$ with b close to b_1 and $i = 2$ the transition $\rho_C \to 0$ for $\Delta b_2 = b_2 - b > 0$ with $b \approx b_2$, where β_1 and β_2 are the order parameter critical exponents for the two transitions.

As an example of the determination of the critical exponents β_i for the two phase transitions we have built a log-log plot of the order parameter as a function of Δb_i in Fig. 3 for $a = 0$ and in Fig. 4 for $a = 1$ for the noise parameter $K = 1.0$. The values of b_i were accurately determined through the best linear fit to the log-log plot for the simulation data for a set of values of b near the transitions, and the exponents β_i are calculated as the slopes of the best fits.

As we can see, for the two transitions the values we have obtained for the critical exponents $\beta_1 = 0.569$ and $\beta_2 = 0.587$ for $a = 0$, and $\beta_1 = 0.568$ and $\beta_2 = 0.557$ for $a = 1$ are close to the values found for DP ($\beta_{DP} = 0.58$) [23, 24].

We have also considered the critical behaviour for other values of the noise parameter, and we have found a good agreement with the DP universality class only for $K \leq 2$ ($a = 0$) and $K \leq 10.0$ ($a = 1$). For other values of the

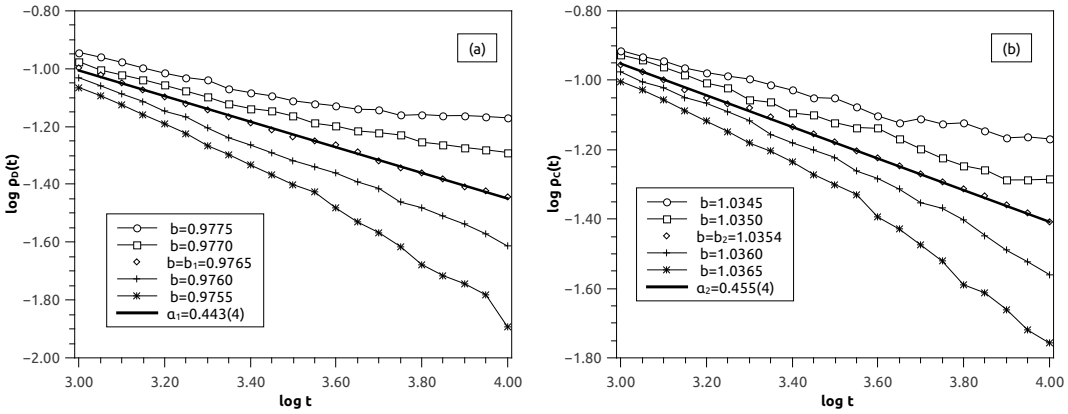

FIG. 5. Log-log plots of the order parameter versus time for $K = 1.0$ and $a = 0$ and several values of b close to the critical points (a) $b_1 = 0.9765$ and (b) $b_2 = 1.0354$. The values of b are indicated in the legend of the figures as well as the slope of the linear fit (bold straight line) at criticality.

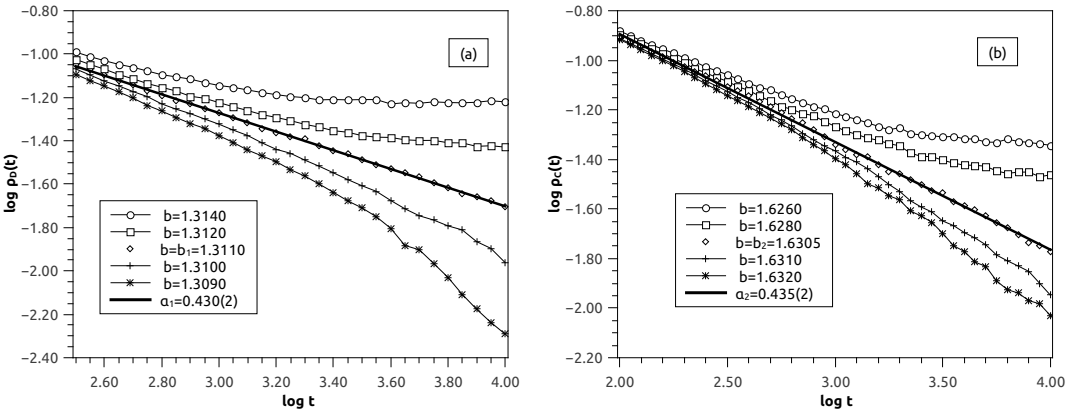

FIG. 6. Log-log plots of the order parameter versus time for $K = 1.0$ and $a = 1$ and several values of b close to the critical points (a) Best value $b_1 = 1.3110$ and (b) Best value $b_2 = 1.6305$. The values of b are indicated in the legend of the figures as well as the slope of the linear fit (bold straight line) at criticality.

noise parameter the gap (jump) in the values of the order parameters at criticality is high enough to invalidate this approach. From the time dependence of the order parameters of the model we can also use a dynamical scaling approach [23, 24]. We have measured the density of cooperators and defectors as a function of time for fixed values of K and b close to the transition points. At criticality the order parameters decay according to power laws

$$\rho_{C,D}(t) \sim t^{-\alpha_i} F_{\pm}(t|\Delta b_i|^{(\nu_{\shortparallel})_i}), \tag{5}$$

from which we can determine the exponents α_i when $F_{\pm}(0)$ is constant. The exponents α_i are related to β_i through $\alpha_i = \beta_i/(\nu_{\shortparallel})_i$, where $(\nu_{\shortparallel})_i$ are critical exponents associated with the temporal correlation lengths. In this way, if we build the logarithm scaled plot of the order parameter as a function of time we can obtain the critical exponents α_i and compare them with the same exponent of the DP ($\alpha_{DP} = 0.45$) [23, 24].

In Figs. 5 and 6, for the non self-interaction and self-interaction cases, respectively, and $K = 1.0$ we have shown the log-log plots of the time dependence of the order parameter for some values of b near criticality. Figs. $5(a)$ and $6(a)$ refer to the transition point b_1 while Figs. $5(b)$ and $6(b)$ close to transition point b_2. In the legend of these figures we also write the values of the critical exponents obtained by a linear fit at criticality (bold straight lines). As expected, all values of the critical exponents α_i are close to the α_{DP} value, indicating again that the model belongs to DP universality class. However, as we have mentioned before, we have found some difficulties to determine the static critical behavior for high values of K. Fortunately, this is not the case when we consider the dynamical scaling laws. We show in Fig. 7 for $a = 0$ and $K = 10.0$ and in Fig. 8 for $a = 1$ and $K = 100.0$, results for the exponent α_i, which are in agreement with the DP universality class.

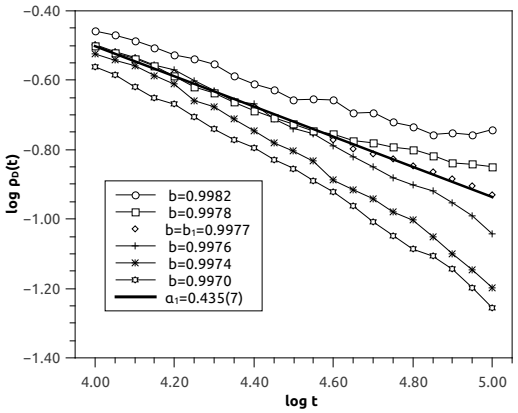

FIG. 7. Log-log plot of the order parameter versus time for $K = 10.0$ and $a = 0$ for several values of b close to the critical point $b_1 = 0.9977$. The values used for b are indicated in the legend of the Figure as well as the slope of the linear fit (bold straight line) at criticality.

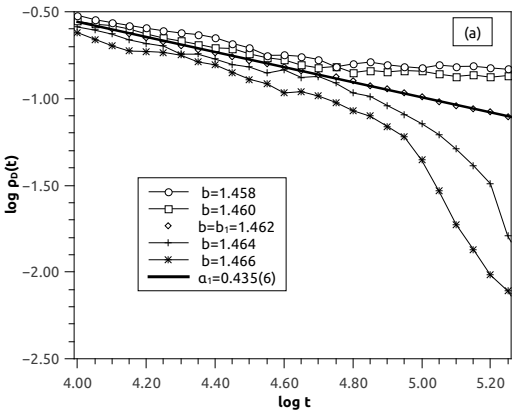

FIG. 8. Log-log plot of the order parameter versus time for $K = 100.0$ and $a = 1$ for several values of b close to the critical point $b_1 = 1.462$. The values used for b are indicated in the legend of the Figure as well as the slope of the linear fit (bold straight line) at criticality.

By comparing these two figures with Figs. $5(a)$ and $6(a)$ we note that the scale on the t-axis is one order of magnitude larger, indicating that greater the value of the noise parameter more difficult to achieve the steady state. Besides, the fluctuations of the order parameter on time are very pronounced for small number of samples and, due to this fact, we used $M = 400$ to make these plots.

The theory of the dynamical scaling for the non-equilibrium systems predicts the scaling law form (Eq. 5) for the time-dependence of the order parameter [17]. In accordance with this theory a plot of $\rho_{C,D(t)}.t^{\alpha}$ versus $t|\Delta b_i|^{\beta/\alpha}$, near criticality, shows data collapse into two curves depending on the signal of $\Delta b_i = b - b_i$. We searched for this behaviour for several values of K using the previous results obtained for α_i and β_i at both transition points. In Fig. 9 we show the data collapse for two values of K in the self-interaction and non self-interaction cases: $K = 1.0$ at the b_2 critical point for $a = 0$ (Fig. $9(a)$) and $K = 10.0$ at the b_1 critical point for the case where $a = 1$ (Fig.$9(b)$). The values we used for the critical exponents were obtained through the scaling forms given by Eqs. 4 and 5, and they are given in the legend of the figures. Clearly, the rescaled data split into two scaling functions: one for the cooperator/defector coexistence phase and another to the absorbing phase (C or D).

As we have stressed before the values of β_i can not always be obtained from the steady state values of the order parameters through Eq. 4. However, if we use Eq. 5 for the time dependence of the order parameters all exponents α_i can be found. Then, if we assume that the scaling law form for the time-dependent order parameter is valid for any value of the noise parameter, we can find the values for β_i adjusting them to a given data set close to criticality, in order to obtain the best data collapse for a fixed value of K. Figs. 10 and 11 support this idea for large values of K at the two critical points for $a = 0$ and $a = 1$, respectively.

We summarize the results we found for the the critical exponents in Fig. 12, in which we plot the critical exponents

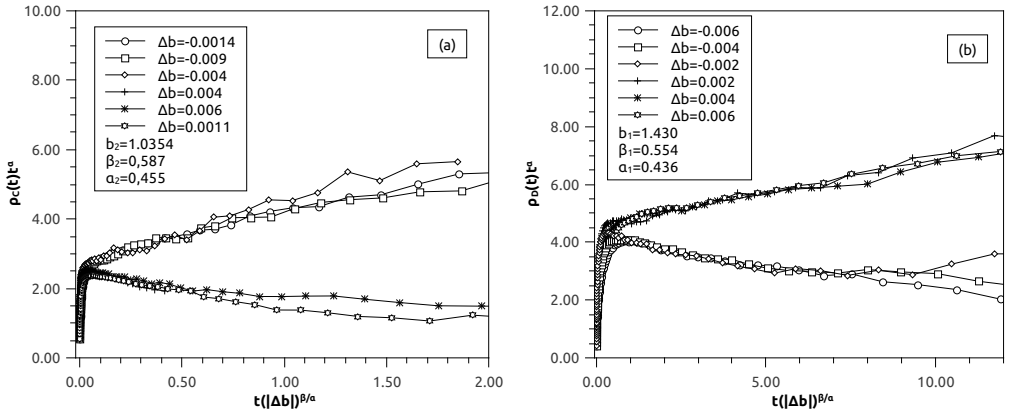

FIG. 9. Dynamical scaling law for some values of Δb close to the critical points. (a) $a = 0$, $K = 1.0$ and $b_2 = 1.0354$. (b) $a = 1$, $K = 10.0$ and $b_1 = 1.430$.

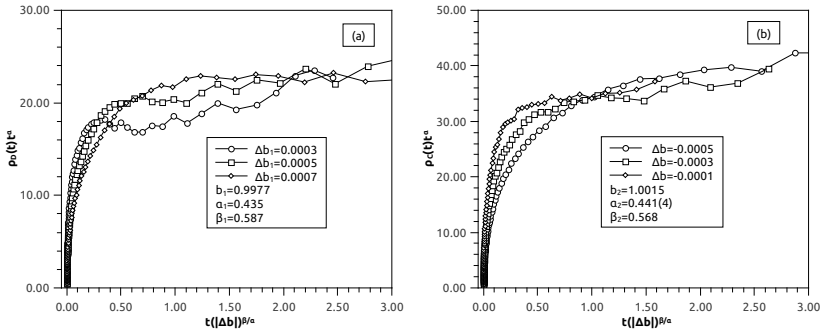

FIG. 10. Dynamical scaling law for some values of Δb close to the critical points for $a = 0$ and $K = 10.0$. (a) $b_1 = 0.9977$ and (b) $b_2 = 1.0015$.

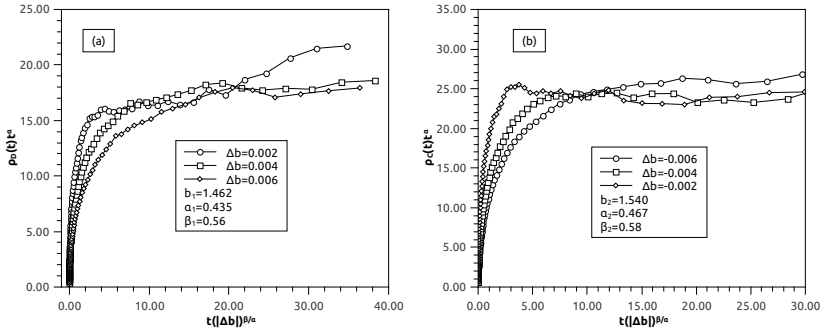

FIG. 11. Dynamical scaling law for some values of Δb close to critical points for $a = 1$ and $K = 100.0$. (a) $b_1 = 1.462$ and (b) $b_2 = 1.540$.

as a function of K. As we can see, all values found are close to and fluctuate around the well known DP values.

IV. CONCLUSIONS

The role of the noise parameter on the stationary states of the Prisoner's Dilemma model was studied through the Monte Carlo simulations. We determined the phase diagram of the model considering or not the self-interaction of cooperators. Depending on the values of the parameters of the model we observed three distinct states: an absorbing phase of cooperators, a coexistence phase of cooperators and defectors and an absorbing phase of defectors. Using

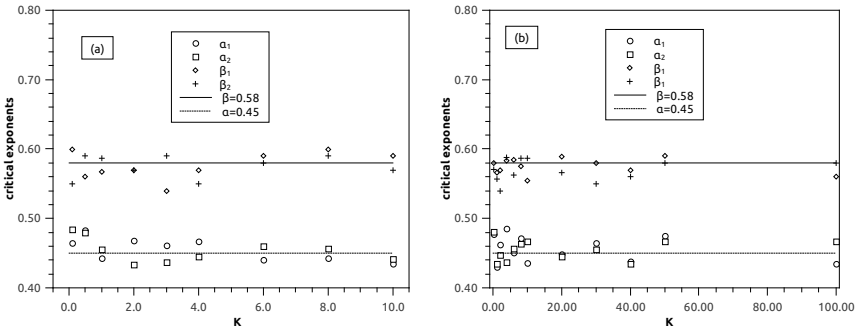

FIG. 12. Critical exponents as a function of K for (a) non self interaction and (b) self interaction cases.

the static and dynamical scaling laws we determined the critical exponents associated with these phase transitions. In accordance with our results, the prisoner's dilemma model belongs to the DP universality class, as conjectured in the literature.

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[1] W. Poundstone, *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb* (Doubleday New York, New York, 1992).
[2] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, 1982).
[3] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behaviour* (Princeton University Press, Princeton, 1944).
[4] R. Axelrod, *The Evolution of Cooperation* (Basic Books, New York, 1984)
[5] M. A. Novak and R. M. May, Nature **359**, 826 (1992).
[6] M. A. Novak and R. M. May, Int. J. Bifurcation Chaos **3**, 35 (1993).
[7] M. A. Nowak and R. M. May, Int. J. Bifurcation Chaos **4**, 33 (1993).
[8] G. Szabó and C. Töke, Phys. Rev. E **58**, 69 (1998).
[9] M. H. Vainstein and J. J. Arenzon, Phys. Rev. E, **64**, 051905 (2001).
[10] G. Abramson and M. Kuperman, Phys, Rev. E **63**, 030901R (2001).
[11] B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, Phys. Rev. E **66**, 021907 (2002).
[12] N. Masuda and K. Aihara, Physics Letters A **313**, 55 (2003).
[13] C. Hauert and G. Szabó, Am. J. Phys **73**, 405 (2005).
[14] A. L. Ferreira, A. Lipowski, T. B. Pedro, M. Santos and W. Figueiredo, J. Stat. Mech., 123405 (2016).
[15] G. Szabó, J. Vukov, and A. Szolnoki, Phys. Rev. E, **72**, 047107 (2005).
[16] M. A. Amaral, L. Wardil, M. Perc, and J. K. L. da Silva, Phys. Rev. E, **94**, 032317 (2016).
[17] J. Vukov and G. Szabó, Phys. Rev. E **71**, 036133 (2005).
[18] F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, Phys. Rev. E **72**, 056128 (2005).
[19] F. C. Santos, J. M. Pacheco, and T. Lenaerts, Proc. Nat. Acad. Sciences **103**, 3490 (2006).
[20] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).
[21] M. Lin, N. Li, L. Tian and D. Shi, Physica A, **389**, 1753 (2010).
[22] M. E. J. Newman and G.T. Barkema, *Monte Carlo Methods in Statistical Physics*, (Oxford University Press Inc., Oxford, 2001).
[23] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999).
[24] M. Henkel, H. Hinrichsen, and S. Lübeck, *Non-Equilibrium Phase Transitions, vol. 1, Absorbing Phase Transitions* (Springer, Dordrecht, 2008).
[25] H. K. Janssen, Z. Phys. B, **42**, 151 (1981).
[26] P. Grassberger, Z. Phys. B, **47**, 365 (1982).