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Ellipsoidal Brownian self-driven particles in a magnetic field

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We study the two-dimensional Brownian dynamics of an ellipsoidal paramagnetic microswimmer moving at low Reynolds number and subject to a magnetic field. Its corresponding mean-square displacement showing the effect of particles's shape, activity, and magnetic field on the microswimmer's diffusion is analytically obtained. The comparison **between** analytical and computational results **display good agreements**. In addition, the effect of self-propulsion on the transition time from anisotropic to isotropic diffusion of the ellipse is also **investigated**.

I. INTRODUCTION

The study of active (self-propelled) Brownian particles (ABPs) has recently received considerable attention [1–4]. These micro-particles are able to self-propel due to an internal mechanism which may be based on light stimulus (thermophoresis), gradients of concentrations (diffusiophoresis) or by the motion of actuated synthetic filaments [5]. **ABPs are interesting from a fundamental point of view due to their out of equilibrium nature**. For example, when ABPs interact among each other they may form clusters [6], thus these simple systems may shed light on the way active matter organizes, which eventually will lead to technology based on active materials [3, 7]. In addition ABPs can be considered as minimalistic models for studying swimming microorganisms [8, 9]. **Substantial efforts have also focused on the potential biomedical and environmental applications for these small self-driven systems [10–12], such as targeted drug delivery, microsurgery, decontamination, environmental monitoring, among others [2].**

Related works have often considered spherical ABPs [9, 13–18]. However, it is very common to find in nature [19] and in synthetic devices anisotropic particles. Along this line, Han *et al* [20] studied the Brownian motion of passive (not self-propelled) ellipsoidal particles. They **experimentally** and theoretically characterized its anisotropic diffusion. Grima and Yaliraki [21] derived the equations of motion for an ellipsoidal Brownian passive particle under an external force. In particular, they applied their general results to the case of forces derived from a linear and a harmonic potential. Later on, Han *et al* [22] experimentally investigated the long time effective diffusion coefficients along the short and long axis of an ellipsoid as well as wall confinement effects on its diffusion.

Inspired by finding methods able to change the direction of microengines, Tierno *et al* [23] characterized the dynamics of overdamped paramagnetic ellipsoidal particles immersed in a rotating field and found nonlinear

motion of ellipsoids under certain circumstances. Their analytical results were also experimentally tested. **A subsequent work [24] then demonstrated** the ability of external magnetic fields to guide self-driven microellipsoids by means of a torque created by the external field and the magnetic anisotropy of the ellipsoids. The inclusion of thermal forces on similar systems has been undertaken by Güell and coworkers [25]. They found approximated analytical expressions for the diffusion of an ellipsoidal paramagnetic passive particle under the effect of a rotating field. More recently, ten Hagen *et al* [26] found analytical expressions, in two and three dimensions, for the mean-square displacement (MSD) of a self-propelled ellipsoidal particle. **Technological applications of these magnetically torqued ellipsoidal particles have been highlighted in a recent review [27].**

Other related studies to the present work are: The theoretical and experimental study on the dynamics of a microswimmer with a L -shape made by Kummel *et al* [8], the dynamics of living active particles with a permanent magnetic moment (magnetotactic bacteria) in a rotating field treated by Cebers and coworkers [28, 29], and the diffusive behavior of magnetotactic bacteria due to the random switching of the microorganisms flagellar motors reported in [30].

Motivated by the latter research, we extend previous investigations to the case of active (self-propelled) ellipsoidal Brownian particles under the presence of a magnetic field, that is, we will analyze the effect of particles's shape, self-propulsion and magnetic field, on the diffusion of an ellipsoidal self-driven particle. The effect of activity on the transition time from anisotropic to isotropic diffusion will also be discussed.

II. MODEL

Let us analyze a self-propelled paramagnetic ellipsoidal particle (swimmer) of semi-major axis a and semi-minor axis b . This particle moves in a two-dimensional fluctuating environment at temperature T , and is subject to an external magnetic field of the form $\mathbf{H}(t) = H_0[\cos(\omega_H t), \sin(\omega_H t)]$ (See Fig. 1). In this model, the

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FIG. 1. (Color online) Model of a paramagnetic Brownian swimmer of magnetic moment $\mathbf{m}$ and swimming velocity U , under a rotating magnetic field $\mathbf{H}(t) = H_0[\cos(\omega_H t), \sin(\omega_H t)]$.

swimmer is able to rotate along the azimuthal direction θ , and we represent its center of mass position (in cartesian coordinates) as $\mathbf{x}(t) = [x_1(t), x_2(t)]$. The self-propulsion of the particle is included by imposing a swimming velocity (along its long axis) of the form $\mathbf{U}_s = U\mathbf{e}(t)$, where $\mathbf{e}(t) = [\cos \theta(t), \sin \theta(t)]$ is the instantaneous unit vector in the direction of swimming with origin at the center of the particle, and U is the magnitude of the swimming velocity, which is taken to be constant here for simplicity. Thermal fluctuations in translation, $\mathbf{f}(t) = [f_1(t), f_2(t)]$, and rotation, $\mathbf{g}(t) = g(t)\mathbf{k}$, are modeled as zero-mean random variables whose correlations are given by [20]

$$\langle f_i(t)f_j(t') \rangle = 2D_{ij}(t)\delta(t-t'), \quad (1)$$

$$\langle g(t)g(t') \rangle = 2D_\Omega\delta(t-t'), \quad (2)$$

where

$$D_{ij}(t) = \bar{D}\mathbf{I} + \frac{1}{2}\Delta D \begin{bmatrix} \cos 2\theta(t) & \sin 2\theta(t) \\ \sin 2\theta(t) & -\cos 2\theta(t) \end{bmatrix}, \quad (3)$$

Note that $\bar{D} = 1/2(D_a + D_b)$ and $\Delta D = D_a - D_b$, where $D_a = k_B T/R_a$, $D_b = k_B T/R_b$ and $D_\Omega = k_B T/R_\Omega$; here R_a and R_b are the resistance coefficients to translation along the semi-axis a and b respectively, while R_Ω is the resistance coefficient to rotation and k_B is the Boltzmann constant.

In a low Reynolds number environment, and from the laboratory reference frame, the balance of forces on the ellipsoidal Brownian particle leads to the following Langevin equation for translation [26]

$$\frac{d\mathbf{x}}{dt} = U\mathbf{e}(t) + \mathbf{f}(t). \quad (4)$$

Similarly, the director vector, $\mathbf{e}$, follows the dynamics [31]

$$\frac{d\mathbf{e}}{dt} = \boldsymbol{\Omega}(t) \times \mathbf{e}(t), \quad (5)$$

where $\boldsymbol{\Omega}(t)$ represents the swimmer's angular velocity. The latter equation can be further simplified by using the torque-balance condition on the ellipse: $\boldsymbol{\tau}_m + \boldsymbol{\tau}_H = \tilde{\mathbf{g}}$, where $\boldsymbol{\tau}_H = -R_\Omega \boldsymbol{\Omega}(t)\mathbf{k}$ represents the hydrodynamic torque, $\boldsymbol{\tau}_m = \mu_w \mathbf{m} \times \mathbf{H}$ is the magnetic torque (μ_w is the magnetic susceptibility of water), and $\tilde{\mathbf{g}} = \tilde{g}(t)\mathbf{k}$ is the stochastic torque ($\tilde{g}/R_\Omega = g(t)$). Note that the magnetic moment is related to the magnetic field by $\mathbf{m} = V\chi\mathbf{H}$, where $V = (4/3)\pi ab$ and χ is a second order tensor representing the susceptibility of the ellipse. For the situation when there is a preferred magnetization direction, the susceptibility tensor may be expressed as $\chi = \chi_\perp \mathbf{I} + \Delta\chi \mathbf{e}\mathbf{e}$, where $\Delta\chi = \chi_\parallel - \chi_\perp$ and $\chi_\parallel$ ($\chi_\perp$) is the susceptibility component parallel (normal) to $\mathbf{e}$. The mentioned torque balance condition together with Eq. (5) finally leads to

$$\frac{d\theta}{dt} = \omega_c \sin[2(\omega_H t - \theta(t))] + g(t), \quad (6)$$

where we have defined $\omega_c = \mu_w V \Delta\chi H_0^2 / 2R_\Omega$. Note that the stochastic differential equations (4) and (6) describe the dynamics of the active ellipsoidal swimmer. The solution of those equations will enable us to find the swimmer's effective diffusion, as detailed in the following section.

III. MEAN-SQUARE DISPLACEMENT

To find analytical expressions for the swimmer's effective diffusion, we need to calculate its MSD. Let us integrate Eq. (4) with respect to time, square it, and perform an ensemble average to that result. The resulting equations in components read

$$\begin{aligned} \langle x_1^2(t) \rangle &= U^2 \int_0^t \int_0^t \langle \cos \theta(t') \cos \theta(t'') \rangle dt'' dt' \\ &\quad + 2\bar{D}t + \Delta D \int_0^t \langle \cos 2\theta(t') \rangle dt' \end{aligned} \quad (7)$$

and

$$\begin{aligned} \langle x_2^2(t) \rangle &= U^2 \int_0^t \int_0^t \langle \sin \theta(t') \sin \theta(t'') \rangle dt'' dt' \\ &\quad + 2\bar{D}t - \Delta D \int_0^t e^{-4D_\Omega t'} \langle \cos 2\theta(t') \rangle dt'. \end{aligned} \quad (8)$$

Clearly, one can see that in order to integrate Eqs. (7)-(8), the explicit expressions for the correlations $\langle \cos 2\theta(t') \rangle$, $\langle \cos \theta(t') \cos \theta(t'') \rangle$, and $\langle \sin \theta(t') \sin \theta(t'') \rangle$ are needed. One way of calculating these correlations is to use the corresponding joint probability distribution,

$G(\theta_1, t_1; \theta_2, t_2) = P(\theta_1, t_1 | \theta_2, t_2) P(\theta_2, t_2)$, which is obtained after solving the respective Smoluchowski equation for Eq. (6). In this way one can use that

$$\begin{aligned} \langle \cos \theta(t') \cos \theta(t'') \rangle &= \langle \cos \theta_1 \cos \theta_2 \rangle \\ &= \int \int \cos \theta_1 \cos \theta_2 P(\theta_1, t_1 | \theta_2, t_2) \\ &\quad \times P(\theta_2, t_2) d\theta_1 d\theta_2, \end{aligned} \quad (9)$$

where the conditional and unconditional PDF are $P(\theta_1, t_1 | \theta_2, t_2)$, and $P(\theta_2, t_2)$ respectively. **Unfortunately, the joint probability distribution G** for the non-linear Eq. (6) cannot be explicitly obtained. Hence, a further approximation is required to make some progress. Based on numerical observations to the solution of Eq. (6), it is proposed that its solution is approximately given by [25]

$$\theta(t) \approx \theta_d(t) + \sigma B_t, \quad (10)$$

where $\theta_d(t)$ is the corresponding deterministic solution to Eq. (6), B_t is the classical Brownian motion, and $\sigma = \sqrt{2D_\Omega}$. One should notice that depending on the ratio $\gamma \equiv \omega_c/\omega_H$, the rotational deterministic dynamics of the ellipsoidal particle is different. For example, for the case $\gamma > 1$, the deterministic solution to Eq. (6) is

$$\theta_d(t) = \omega_H t - \arctan[\gamma + c_1 \tan(c_1 \omega_H t - c_3)], \quad (11)$$

where $c_1 = \sqrt{1 - \gamma^2}$ and $c_3 = \arctan(\tan \theta_0 + \gamma/c_1)$. The behavior of Eq. (11) can be appreciated in Fig. (2)(a). This dynamics represents the case when the ellipsoidal particle rotates synchronously with the magnetic field. For the case $\gamma < 1$, the deterministic solution to Eq. (6) is

$$\theta_d(t) = \omega_H t + \arctan[-\gamma + c_2 \tanh(c_2 \omega_H t + c_4)], \quad (12)$$

where $c_2 = \sqrt{\gamma^2 - 1}$ and $c_4 = \tanh^{-1}(\tan \theta_0 + \gamma/c_2)$. The behavior of Eq. (12) is shown in Fig. (2)(b). For this case, the ellipsoidal swimmer performs a back-and-forth rotational motion while it is trying to follow the rotation of the magnetic field. This state can be considered as an asynchronous state.

Once we have established Eqs. (11)-(12), we use Eq. (10) to show that

$$\langle \cos 2\theta(t) \rangle = e^{-4D_\Omega t} \cos 2\theta_d(t). \quad (13)$$

In the same way,

$$\begin{aligned} \langle \cos \theta(t') \cos \theta(t'') \rangle &= \langle \cos [\theta_d(t') + \sigma B_{t'}] \\ &\quad \times \cos [\theta_d(t'') + \sigma B_{t'']} \rangle. \end{aligned} \quad (14)$$

After certain manipulation and by defining $\Theta \equiv \theta_d(t'') + \theta_d(t') + 2\theta_0$, it can be shown that

$$\begin{aligned} \langle \cos \theta(t') \cos \theta(t'') \rangle &= \frac{e^{-\sigma^2(t'-t'')/2}}{2} \left[e^{-2\sigma^2 t''} \cos \Theta(t) \right. \\ &\quad \left. + \cos [\theta_d(t'') - \theta_d(t')] \right]. \end{aligned} \quad (15)$$

FIG. 2. (Color online) Deterministic rotational dynamics induced by the magnetic field and its dependence on $\gamma = \omega_c/\omega_H$. For $\gamma > 1$, the ellipse rotates synchronously with the magnetic field. For $\gamma < 1$, the ellipse performs a back-and-forth rotational motion due to the magnetic field.

Similarly, the correlation

$$\begin{aligned} \langle \sin \theta(t') \sin \theta(t'') \rangle &= \langle \sin [\theta_d(t') + \sigma B_{t'}] \\ &\quad \times \sin [\theta_d(t'') + \sigma B_{t'']} \rangle \end{aligned} \quad (16)$$

can be simplified as

$$\begin{aligned} \langle \sin \theta(t') \sin \theta(t'') \rangle &= \frac{e^{-\sigma^2(t'-t'')/2}}{2} \left[-e^{-2\sigma^2 t''} \cos \Theta \right. \\ &\quad \left. + \cos [\theta_d(t'') - \theta_d(t')] \right]. \end{aligned} \quad (17)$$

The latter results allow us to have implicit analytical expressions for the effective diffusion coefficients as we will see in the next section.

IV. EFFECTIVE DIFFUSION

By definition, the diffusion coefficient along the x_i -direction ($i \in \{1, 2\}$) is

$$D_{x_i}(t, \theta_0) = \frac{\langle x_i^2(t) \rangle}{2t}. \quad (18)$$

Thus if one includes Eq. (7) together with Eqs. (13) and (15) in Eq. (18), we get

$$\begin{aligned} D_{x_1}(t, \theta_0) &\approx \bar{D} + \frac{\Delta D}{2t} \int_0^t e^{-4D_\Omega t'} \cos 2\theta_d(t') dt' \\ &\quad + \frac{U^2}{2t} \int_0^t \int_{t'}^t e^{-D_\Omega(t'-t'')} \\ &\quad \times \left\{ e^{-4D_\Omega t''} \cos \Theta \right. \\ &\quad \left. + \cos [\theta_d(t'') - \theta_d(t')] \right\} dt' dt''. \end{aligned} \quad (19)$$

Similarly, using Eqs. (8) and (17) one can show that the diffusion along the x_2 direction is

$$D_{x_2}(t, \theta_0) \approx \bar{D} - \frac{\Delta D}{2t} \int_0^t e^{-4D_\Omega t'} \cos 2\theta_d(t') dt' + \frac{U^2}{2t} \int_0^t \int_{t''}^t e^{-D_\Omega(t'-t'')} \times \left\{ -e^{-4D_\Omega t''} \cos \Theta + \cos[\theta_d(t'') - \theta_d(t')] \right\} dt' dt'', \quad (20)$$

Notice that the latter two expressions provide a way of analytically quantifying the effects of activity (U), swimmer's shape ($\bar{D}$, ΔD), and strength of the magnetic field on the diffusion of a paramagnetic ellipsoidal swimmer. Therefore Eqs. (19)-(20) represent an extension to the works of Grima and Yaliraki [21], Güell *et al* [25], ten Hagen *et al* [26], among others, to the case of active matter under external fields.

For the case $\gamma > 1$ it is possible to find an asymptotic expression for its long-term effective diffusion coefficient. To do this, one can show that for long times, Eq. (6) reduces to the following Langevin equation $d\theta/dt = \omega_H + g(t)$ with a given initial orientational angle θ_0 . The latter Langevin equation corresponds to a conditional PDF of the form

$$P(\theta_1, t_1 | \theta_2, t_2) = \frac{1}{\sqrt{4\pi D_\Omega \tau}} e^{-\frac{(\theta_1 - \theta_2 - \omega_H \tau)^2}{4\pi D_\Omega \tau}}, \quad (21)$$

where $\tau = t_1 - t_2$, here $t_1 > t_2$. Thus with the use of Eq. (9), one can find the swimmer's orientation correlations exactly. After performing the indicated integrals in Eqs. (7)-(8), using Eq.(18), and applying the limit $t \rightarrow \infty$, one gets that the ellipsoidal swimmer long-time diffusive behavior is

$$D_{x_1} = D_{x_2} = \bar{D} + \frac{D_\Omega U^2}{D_\Omega^2 + \omega_H^2}, \quad (22)$$

which is exactly the same expression as for a spherical particle under an external torque [32]. In other words, after a long time the ellipsoidal particle statistically behaves like a sphere. For comparison purposes, Eq. (22) is plotted in Fig. 3(a) and discussed in the next section.

V. COMPARISON AMONG NUMERICAL AND ANALYTICAL RESULTS

In order to check how accurately the analytical Eqs. (19)-(20) describe the effective diffusion coefficients, we perform Brownian dynamics simulations (A stochastic Runge-Kutta algorithm was used. The particle always started at the origin with an initial angular orientation $\theta_0 = 0$, and 20,000 realizations were used to perform the averages). In these simulations, we consider an ellipsoidal swimmer with translational diffusivities $D_a = 0.18 \mu\text{m}^2/\text{s}$, $D_b = 0.044 \mu\text{m}^2/\text{s}$ (typical experimental values [20]), immersed in water at $T = 300 \text{ K}$,

FIG. 3. (Color online) Effective diffusion coefficients along the x_1 direction, (D_{x_1}), and along the x_2 direction, (D_{x_2}). (a): The case of a paramagnetic ellipsoidal particle where ω_c is larger than ω_H , ($\gamma > 1$). The long-time asymptotic limit given by Eq. (22) is also plotted for comparison. (b): The case of a paramagnetic ellipsoidal particle where ω_c is smaller than ω_H , ($\gamma < 1$).

and simulate the cases $\gamma > 1$ and $\gamma < 1$. Fig. 3(a) shows the effective diffusion coefficients along the x_1 and x_2 directions of an ellipsoidal particle swimming at speed $U = 3 \mu\text{m/s}$ with $\gamma > 1$. The theoretical results appear as (green and black) dashed lines, while the Brownian dynamics results appear in blue and red lines. A good agreement between theory and numerics can be appreciated. The asymptotic long time diffusive coefficient dictated by Eq. (22) is also plotted as a (light red) dashed line for comparison. Since the rotating nature of the magnetic field does not break the overall isotropy of the problem and the initial anisotropic translational motion of the particle will eventually be washed out by rotational diffusion, after a period of time, isotropic diffusion is recovered and both diffusive coefficients (D_{x_1}, D_{x_2}) tend to the same constant value dictated by Eq. (22). Fig. 3(b) shows the same particle swimming at the same speed ($U = 3 \mu\text{m/s}$) but with $\gamma < 1$. Once again the theoretical results appear as (green and black) dashed lines, while the Brownian simulations appear as (red and blue) lines. A good agreement between theory and numerical results is also obtained in this regime. From the latter comparisons one can see that Eqs. (19)-(20) provide a good description of the effective diffusion of a magnetically torqued swimmer at both short and long times. One can also see that for long times, both cases ($\gamma > 1, \gamma < 1$) tend to a constant value, that is, the system tends to display an isotropic behavior as expected.

FIG. 4. (Color online) Effect of magnetic frequency on the transition time from anisotropic to isotropic effective diffusion.

FIG. 5. (Color online) Effect of self-propulsion on the transition time from anisotropic to isotropic effective diffusion. The solid lines represent quadratic fitting curves.

A. Transition time to isotropic diffusion

We finally show the effect of swimming velocity (U) and magnetic frequency (ω_H) on the transition time the ellipsoidal particle takes to develop isotropic diffusion. To this end, we use Eqs. (19)-(20) and fix the Brownian diffusive constants as $D_a = 0.18 \mu\text{m}^2/\text{s}$, $D_b = 0.044 \mu\text{m}^2/\text{s}$, as well as the frequency $\omega_c = 14.7 \text{ s}^{-1}$ for all the simulations. In the first numerical experiment, we vary the magnetic field frequency as $\omega_H = \{1 \text{ s}^{-1}, 3 \text{ s}^{-1} \dots 21 \text{ s}^{-1}, 23 \text{ s}^{-1}\}$ and repeat the procedure for three swimming velocities: $U = \{0 \mu\text{m/s}, 3 \mu\text{m/s}, \text{ and } 4 \mu\text{m/s}\}$. In

FIG. 6. (Color online) Typical paths of paramagnetic ellipsoidal microswimmers during 150 s. A swimmer with $U = 6 \mu\text{m/s}$ and subject to a magnetic frequency $\omega_H = 7 \text{ s}^{-1}$ is shown in Fig. 6(a), while a swimmer with $U = 6 \mu\text{m/s}$ under $\omega_H = 21 \text{ s}^{-1}$ is shown in Fig. 6(b). Fig. 6(c) represents typical paths for a swimmer without magnetic field.

Fig. 4, for $\gamma > 1$, one can see that a larger swimming velocity and smaller magnetic frequency lengthen the transition time. Similar dependence on magnetic frequency has been reported for passive ellipsoidal particles [25]. The increase in transition time due to self-propulsion can be understood as a result of enhanced directionality memory of the particle's long axis along which the active particle translates for a longer time compared with a passive one. For $\gamma < 1$, while particle self-propulsion has the same effect of lengthening the transition time, the transition time also increases with the magnetic frequency due to the back-and-forth motion similar to the case of passive particles [25]. In our second numerical experiment, we further explore the dependence of transition time on swimming velocity. In Fig. 5, the transition time is plotted against typical microswimmer velocity from $U = 0 \mu\text{m/s}$ to $U = 10 \mu\text{m/s}$ at two different magnetic frequencies selected in the synchronous and asynchronous regimes, namely, $\omega_H = 7 \text{ s}^{-1}$ and 21 s^{-1} . The simulations for the case $\omega_H = 7 \text{ s}^{-1}$ is represented in red squares, while the case $\omega_H = 21 \text{ s}^{-1}$ is shown in blue circles. The solid lines (blue and red) represent a quadratic fitting curve, suggesting a quadratic scaling between the transition time and the swimming velocity as $t_{\text{trans}} \sim CU^2$, where C is a constant.

Finally, typical paths of paramagnetic ellipsoidal microswimmers during 150 s are shown in Fig. 6. A swimmer with $U = 6 \mu\text{m/s}$ and subject to a magnetic frequency $\omega_H = 7 \text{ s}^{-1}$ ($\gamma < 1$) is shown in Fig. 6(a), while a swimmer with $U = 6 \mu\text{m/s}$ under $\omega_H = 21 \text{ s}^{-1}$ ($\gamma > 1$) is shown in Fig. 6(b). Fig. 6(c) represents typical paths for a swimmer without magnetic field. One can appreciate the rotational motion the swimmers perform when they are subject to an external rotating magnetic field. This

rotation reduces the swimmer's effective diffusion. In the absence of external fields the ellipsoidal swimmer samples longer distances (See Fig. 6c). One can also observe from Figs. 6(a)-(b) that for higher magnetic frequencies ($\gamma > 1$), the swimmer reaches longer distances compared to lower frequencies $\gamma < 1$.

VI. CONCLUDING REMARKS

In summary, we have found analytical expressions able to characterize the effect of particle's shape, self-propulsion and external magnetic fields on the diffusion of ellipsoidal particles for both short and long times. A comparison among analytical and computational results was also made and good agreements were obtained. **The particle activity offers an additional mechanism to control the crossover from anisotropic to isotropic diffusion, and it was found that the transition time increases with the self-propulsion velocity.**

As a remark, although a paramagnetic microswimmer is considered in this work, similar dynamics is expected for ferromagnetic microswimmers (e.g. magnetotactic bacteria) under the same setup. Despite the differences in the ferromagnetic and paramagnetic moments,

the two-dimensional rotating magnetic field setup renders similar mathematical structures in the equation of motion in these different cases. Under a rotating magnetic field, a ferromagnetic microswimmer hence also displays synchronous and asynchronous behaviors similar to a paramagnetic microswimmer [29, 30]. Finally, it may be biologically relevant to investigate the effects of the run and tumble motility of bacterial cells on the transition from anisotropic to isotropic diffusion. Qualitatively, we would expect that the run and tumble behavior to shorten the transition time since it introduces an additional mechanism to wash out the directionality of the particle. The quantitative aspect may be an interesting topic for future analysis.

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