Stochastically gated local and occupation times of a Brownian particle

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Stochastically-gated local and occupation times of a Brownian particle

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We generalize the Feynman-Kac formula to analyze the local and occupation times of a Brownian particle moving in a stochastically-gated one-dimensional domain. (i) The gated local time is defined as the amount of time spent by the particle in the neighborhood of a point in space where there is some target that only receives resources from (or detects) the particle when the gate is open; the target does not interfere with the motion of the Brownian particle. (ii) The gated occupation time is defined as the amount of time spent by the particle in the positive half of the real line, given that it can only cross the origin when a gate placed at the origin is open; in the closed state the particle is reflected. In both scenarios, the gate randomly switches between the open and closed states according to a two-state Markov process. We derive a stochastic, backward Fokker-Planck equation (FPE) for the moment generating function of the two types of gated Brownian functional, given a particular realization of the stochastic gate, and analyze the resulting stochastic FPE using a moments method recently developed for diffusion processes in randomly switching environments. In particular, we obtain dynamical equations for the moment generating function, averaged with respect to realizations of the stochastic gate.

I. INTRODUCTION

An important quantity in the mathematical theory of stochastic processes is the *occupation time* [1], which is the time spent by a Brownian motion above the origin within a time window of size t. That is, given the Brownian motion $X(t)\in\mathbb{R}$, the occupation time T is

$$T=\int_0^t \Theta(X(\tau))d\tau, \tag{1.1}$$

where $\Theta(X)$ denotes the Heaviside function. In addition to being a fundamental quantity in the mathematical theory of random walks [2], occupation times have figured prominently in a variety of physical applications under the alternative name of *residence times.* Examples include the non-equilibrium dynamics of coarsening systems [3, 4], ergodicity properties of anomalous diffusion [5, 6], simple models of blinking quantum dots [7], fluorescent imaging [8], and branching processes [9]. The penultimate example involves a single fluorescent particle diffusing under the objective of a confocal microscope. Every time it enters the focus of the laser beam it is excited and emits photons, so that the total number of emitted photons is proportional to the mean residence time of the molecule in the laser beam's cross-section. If V denotes the volume occupied by the beam, then the residence time is defined according to

$$T=\int_0^t I_V(X(\tau)))d\tau, \tag{1.2}$$

where $X(t)\in\mathbb{R}^3$ is now three-dimensional Brownian motion, $I_V(x)$ denotes the indicator function of the set $V\subset\mathbb{R}^3$, that is, $I_V(x)=1$ if $x\in V$ and is zero otherwise. (Note that for one-dimensional (1D) motion, $\Theta(x)=I_{\mathbb{R}^+}(x)$.)

A related quantity is the *local time* [1], which characterizes the amount of time that a diffusion process such as Brownian motion spends in the neighborhood of a point in space. In probability theory, it plays an important role in the path-wise formulation of reflected Brownian motion [10]. Given the Brownian motion $X(t)\in\mathbb{R}$, let $T(A,t)$ denote the occupation time of the set $A\subset\mathbb{R}$ during the time interval $[0,t]$:

$$T(A,t)=\int_0^t I_A(X(\tau))d\tau. \tag{1.3}$$

From this definition, the local time density $T(a,t)$ at a point $a\in\mathbb{R}$ is defined by setting $A=[a-\epsilon,a+\epsilon]$ and taking

$$T(a,t)=\lim_{\epsilon\rightarrow 0^+}\frac{1}{2\epsilon}\int_0^t I_{[a-\epsilon,a+\epsilon]}(X(s))ds. \tag{1.4}$$

We thus have the following formal representation of the local time density:

$$T(a,t)=\int_0^t \delta(X(\tau)-a)d\tau, \tag{1.5}$$

where $T(a,t)da$ is the amount of time the Brownian particle spends in the infinitesimal interval $[a,a+da]$.

Occupation and local times are two examples of a Brownian functional. Suppose that $X(t)\in\mathbb{R}$ represents pure Brownian motion. A *Brownian functional* over a fixed time interval $[0,t]$ is formally defined as a random variable T given by

$$T=\int_0^t U(X(\tau))d\tau, \tag{1.6}$$

where $U(x)$ is some prescribed function or distribution such that T has positive support. Thus, $U(X)=\delta(x-a)$ for the local time density at $x=a$ and $U(x)=\Theta(x)$ for the occupation time in $\mathbb{R}^+$. Since $X(t)$, $t\geq 0$, is a Wiener process, it follows that each realization of a Brownian path will typically yield a different value of T, which means that T will be distributed according to some probability density $P(T,t|x_0,0)$ for $X(0)=x_0$. The statistical properties of a Brownian functional can be analyzed using path integrals, and leads to the well-known *Feynman-Kac* formula [11]. For a general review of Brownian functionals and their applications, see Ref. [12].

In this paper, we extend the notions of local and occupation times by considering 1D Brownian motion in a stochastically-gated domain. In the case of local time, suppose that a stochastically-gated target is located at the point a. Let $n(t)$ be the state of the gate at time t with $n(t)\in\{0,1\}$. The gate is said to be open at time t if $n(t)=1$ and closed if $n(t)=0$. The gate switches between the two states according to a two-state Markov process with rates α and β, that is,

$$0 \underset{\alpha}{\overset{\beta}{\rightleftharpoons}} 1.$$

We then define the stochastically-gated local time density according to

$$\mathcal{T}(a,t,t_0)=\int_{t_0}^{t} n(\tau)\delta(X(\tau)-a)d\tau. \tag{1.7}$$

Now $\mathcal{T}(a,t,t_0)da$ is the amount of time the Brownian particle spends in the interval $[a,a+da]$ when the gate is open during the time interval $[t_0,t]$. (The presence of the dynamic gate means that the system is time-inhomogeneous.) We thus have a doubly stochastic process driven by both the Brownian motion and the switching gate, see Fig. 1. One possible physical interpretation of this process is that each time the Brownian particle is in a small neighborhood of the gate, the target receives

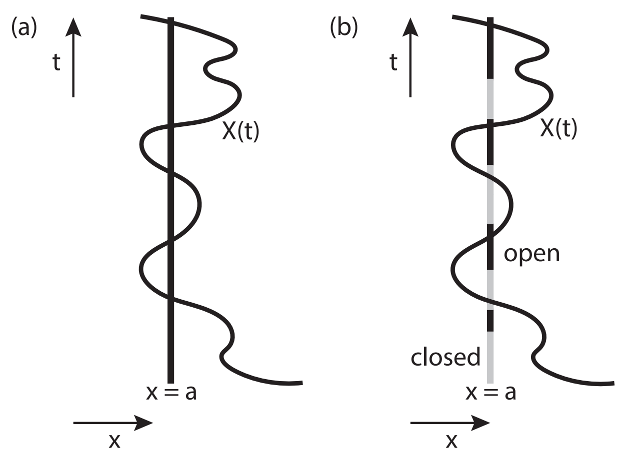

FIG. 1. Schematic diagram of Brownian motion for (a) no gate and (b) a switching gate at $x=a$. In the latter case the open (closed) state of the gate is indicated by black (grey) vertical bars. In (a) the local time density T characterizes the amount of time the particle spends in a neighborhood of the point $x=a$, whereas in (b) the stochastically-gated local time $\mathcal{T}$ only counts the time around $x=a$ when the gate is open. Note that the gate does not affect the motion of the Brownian particle.

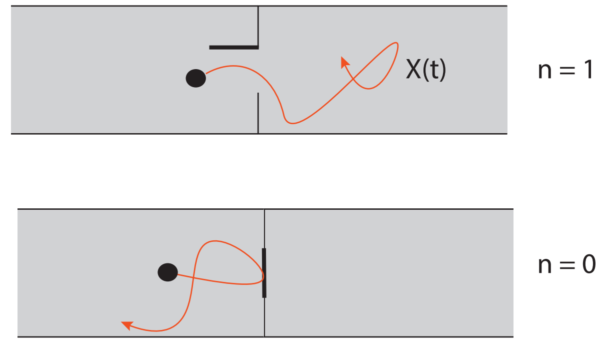

FIG. 2. Schematic diagram of two possible trajectories for gated Brownian motion. (a) The gate is open when the Brownian particle reaches the origin and the trajectory crosses over from the left to right domains. (b) The gate is closed when the particle reaches the origin resulting in the particle being reflected.

resources from the particle at a constant rate κ, but only when the gate is open. Hence the total amount of resources received by the target during the interval $[t_0,t]$ is $\kappa\mathcal{T}(a,t,t_0)$, where κ has units of velocity. Note, however, that in this example the gate does not affect the Brownian motion, that is, the two processes are independent.

Turning to the example of occupation time, since the latter concerns the amount of time that a Brownian particle spends in some bounded or partially bounded domain, it is natural to extend the notion of occupation time to the case of a stochastically-gated boundary. This type of scenario is common in cell biology, where a macromolecule diffuses in some bounded intracellular domain that contains one or more narrow channels within the boundary of the domain; each channel is controlled by a stochastic gate that switches between an open and closed state [13]. Applications in biological physics include diffusion-limited reactions [14], neurotransmission [15], insect physiology [16], stochastically gated gap junctions [17], and lateral membrane diffusion [18]. In our previous studies of diffusion in domains with randomly switching boundaries, we have focused on calculating mean first passage times and related quantities using a method of moments [19, 20]. In this paper, we turn to a different type of problem, namely, calculating the occupation time of a Brownian particle moving in a stochastically-gated one-dimensional domain. The occupation time determines the amount of time spent by the particle in the positive half of the real line, given that it can only cross the origin when a gate placed at the origin is open; in the closed state the particle is reflected. This is illustrated in Fig. 2. As in the case of local time, the gate randomly switches between the open and closed states according to a two-state Markov process. One major difference, however, is now the presence of the gate affects the Brownian motion. The stochastically-gated occupation time is thus

$$\mathcal{T}(t,t_0)=\int_{t_0}^t \Theta(X_\sigma(\tau))d\tau, \tag{1.8}$$

where X_σ denotes that the Brownian motion is gated.

In section II we derive a generalized Feynman-Kac formula for the moment generating function Q of the local time density or the occupation time, given a particular realization of the stochastic gate. This yields a stochastic, backward Fokker-Planck equation (FPE), which is then analyzed in section III using the moments method of Refs. [19]. In particular, we derive dynamical equations for $\mathbb{E}_\sigma[Q]$ with expectation taken respect to different realizations σ of the stochastic gate. These equations are then solved in sections IV and V for the local time density and occupation time, respectively. We obtain a number of specific results. First, we show that the expected local time density, averaged with respect to realizations of the Brownian motion and the stochastic gate, is given by the product of the local time density without switching and the fraction of time that the gate is open. This is a natural consequence of the fact that the two processes are independent. However, the stochastic gate has a non-trivial affect on higher-order moments of the local time density due to temporal correlations in the dynamics of the gate. Second, we use asymptotic analysis to show that the averaged probability density of the occupation time in the large time limit reduces to the same density as the non-switching case. Finally, in section VI we briefly discuss the interpretation of higher-order moments of the distribution of the moment generating function Q with respect to different realizations of the stochastic gate.

II. GENERALIZED FEYNMAN-KAC FORMULA

In this section we develop a mathematical framework for investigating the effects of a stochastic gate on local and occupation times based on generalized Feynman-Kac formulae. However, before proceeding, it is useful to consider the stochastic dynamics of the gate itself, which evolves independently of the Brownian motion. The discrete state $n(t)$ evolves according to a two-state Markov chain with matrix generator

$$W=\begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix}. \tag{2.1}$$

The right nullspace of the matrix W is spanned by the vector $\psi=(1,1)^\top$ and the left nullspace is spanned by the stationary density

$$\rho\equiv\begin{pmatrix}\rho_0\\ \rho_1\end{pmatrix}=\frac{1}{\alpha+\beta}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{2.2}$$

If $P_{nn_0}(t)=\mathbb{P}[N(t)=n|N(0)=n_0]$ then the corresponding master equation takes the form

$$\frac{dP_{nn_0}}{dt}=\sum_{m=0,1}P_{mn_0}W_{mn}$$

Using the fact that $P_{0n_0}(t)+P_{1n_0}(t)=1$ we can solve this pair of equations to give

$$P_{0n_0}(t)=\delta_{0,n_0}\mathrm{e}^{-t/\tau_c}+\alpha\tau_c(1-\mathrm{e}^{-t/\tau_c}),\quad \tau_c=\frac{1}{\alpha+\beta}.$$

It follows that τ_c is the relaxation time of the Markov process with $P_{mn_0}(t)\to\rho_m$ in the limit $t\to\infty$. The stationary autocorrelation function is then given by

$$\langle n(t)n(t')\rangle=\frac{D}{\tau_c}\mathrm{e}^{-|t-t'|/\tau_c}, \tag{2.3}$$

with noise amplitude $D=\alpha\beta\tau_c^3$. This shows that the two-state Markov process, also known as dichotomous noise [21], is a form of colored noise.

A. Gated local time density

Since the gate does not affect the Brownian motion in our definition of local time density, see Eq. (1.7) and Fig. 1, we could simply take expectations of Eq. (1.7) with respect to the Markov process $n(\tau)$ to give

$$\overline{\mathcal{T}}=\int_{t_0}^t\langle n(\tau)\rangle\delta(X(\tau)-a)d\tau=\rho_1 T,$$

where T is the ungated local time density (1.5). (This would not be possible in the case of the occupation time (1.8), since the gate affects the Brownian motion.) On the other hand, averaging $\mathcal{T}^2$ with respect to the gate yields

$$\begin{aligned}\overline{\mathcal{T}^2}&=\int_{t_0}^t\int_{t_0}^t\langle n(\tau)n(\tau')\rangle\delta(X(\tau)-a)\delta(X(\tau')-a)d\tau'd\tau\\ &=\frac{D}{\tau_c}\int_{t_0}^t\int_{t_0}^t\mathrm{e}^{-|\tau-\tau'|/\tau_c}\rangle\delta(X(\tau)-a)\delta(X(\tau')-a)d\tau'd\tau.\end{aligned}$$

We thus see that the colored noise process has a non-trivial affect on second-order (and higher-order) moments of the local time density averaged with respect to realizations of the gate. Hence, averaging $\overline{\mathcal{T}^2}$ with respect to the Brownian motion is non-trivial, and we cannot simply use results from the classical ungated case. Therefore, we will proceed by first averaging with respect to the Brownian motion given a particular realization of the stochastic gate on $[t_0,t]$, $\sigma=\{n(s),t_0\le s\le t\}$. This will yield a Feynman-Kac formula in the form of a stochastic Fokker-Planck equation (FPE), which we will then analyze in section IIIA.

Suppose that the initial state of the Brownian particle is $X(t_0)=x_0$. Let $P(\mathcal{T},t|x_0,t_0)$ be the corresponding probability density for $\mathcal{T}(a,t,t_0)=\mathcal{T}$. (For ease of notation, we suppress the explicit dependence on σ.) Since $\mathcal{T}\ge 0$, we can introduce the moment generating function (or Laplace transform with respect to $\mathcal{T}$)

$$Q(s,t|x_0,t_0)=\int_0^\infty \mathrm{e}^{-s\mathcal{T}}P(\mathcal{T},t|x_0,t_0)d\mathcal{T}. \tag{2.4}$$

Using the classical path-integral representation of pure Brownian motion, we have

$$Q(s,t|x_0,t_0) = \int_0^\infty \mathrm{e}^{-s\mathcal{T}} \int_{-\infty}^\infty \left[\int_{x(t_0)=x_0}^{x(t)=x} \delta\left(\mathcal{T} - \int_{t_0}^t n(\tau)\delta(x(\tau)-a)d\tau\right) P[x]\mathcal{D}[x]\right] dx d\mathcal{T}$$
$$= \int_{-\infty}^\infty \left[\int_{x(t_0)=x_0}^{x(t)=x} \exp\left(-s\int_{t_0}^t n(\tau)\delta(x(\tau)-a)d\tau\right) P[x]\mathcal{D}[x]\right] dx. \tag{2.5}$$

Since only the initial point x_0 is fixed, we are also integrating with respect to the final position x. Hence,

$$Q(s,t|x_0,t_0) = \int_{-\infty}^\infty G(s,x,t|x_0,t_0)dx, \tag{2.6}$$

with

$$G(s,x,t|x_0,t_0) = \left\langle \exp\left(-s\int_{t_0}^t n(\tau)\delta(x(\tau)-a))d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t)=x}$$
$$= \int_{x(t_0)=x_0}^{x(t)=x} \exp\left(-\int_{t_0}^t \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + sn(\tau)\delta(x(\tau)-a))\right]d\tau\right)\mathcal{D}[x], \tag{2.7}$$

where $\langle\cdots\rangle$ denotes averaging over realizations of the Brownian motion.

The next step is to note that

$$G(s,x,t+\Delta t|x_0,t_0) = \left\langle \exp\left(-s\int_{t_0}^{t+\Delta t} n(\tau)\delta(x(\tau)-a)d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t+\Delta t)=x}$$
$$\approx \int p(\Delta x)\left\langle \exp\left(-s\int_{t_0}^t n(\tau)\delta(x(\tau)-a)d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t)=x-\Delta x} \exp\left(-sn(t)\delta(x-a)\Delta t\right) d\Delta x.$$

We have split the time interval $[t_0,t+\Delta t]$ into two parts $[t_0,t]$ and $[t,t+\Delta t]$ and introduced the intermediate state $x(t) = x - \Delta x$ with Δx distributed according to an infinitesimal Wiener process. It follows that

$$G(s,x,t+\Delta t|x_0,t_0) = \mathrm{e}^{-sn(t)\delta(x-a)\Delta t}\int p(\Delta x)G(s,x-\Delta x,t|x_0,t_0)d\Delta x$$
$$= \mathrm{e}^{-sn(t)\delta(x-a)\Delta t}\left(G(s,x,t|x_0,t_0) - \langle\Delta x\rangle\frac{\partial}{\partial x}G(s,x,t|x_0,t_0) + \langle\Delta x^2\rangle\frac{\partial^2}{\partial x^2}G(s,x,t|x_0,t_0) + \ldots\right).$$

Using the fact that for a Wiener process

$$\lim_{\Delta t\to 0}\frac{\langle\Delta X\rangle}{\Delta t} = 0, \quad \lim_{\Delta t\to 0}\frac{\langle(\Delta X)^2\rangle}{\Delta t} = \frac{1}{2}, \tag{2.8}$$

we obtain the following forward FPE in the limit $\Delta t \to 0$:

$$\frac{\partial G}{\partial t} = \frac{1}{2}\frac{\partial^2 G}{\partial x^2} - sn(t)\delta(x-a)G. \tag{2.9}$$

Note that G satisfies the initial condition

$$G(x,t_0|x_0,t_0) = \delta(x-x_0).$$

In contrast to the standard Feynman-Kac formula, Eq. (2.9) is in the form of a stochastic FPE due to the presence of the gating term $n(t)$. More precisely, Eq. (2.9) is a piecewise deterministic FPE. From a practical perspective, it will be simpler to work with the corresponding backward FPE for $Q(s,t|x_0,t-\tau)$. This takes the form

$$\frac{\partial Q}{\partial \tau} = \frac{1}{2}\frac{\partial^2 Q}{\partial x_0^2} - sn(t-\tau)\delta(x_0-a)Q, \tag{2.10}$$

which is supplemented by the "final" condition $Q(s,t|x_0,t) = 1$.

B. Gated occupation time

Now suppose that the stochastic gate is placed at the origin $x = 0$ and physically affects the motion of the Brownian particle. When the gate is closed ($n(t) = 0$) the particle cannot cross between the domains $x < 0$ and $x > 0$ and is reflected if it hits the gate, whereas the particle passes freely between the two domains when the gate is open ($n(t) = 1$). Given a particular realization of the gate over the time interval on $[t_0,t]$, $\sigma = \{n(s), t_0 \le s \le t\}$, let $P_\sigma[X]$ denote the corresponding probability density functional of the stochastically-gated Brownian trajectories. Let $P(\mathcal{T},t|x_0,t_0)$ be the corresponding probability density for the occupation time $\mathcal{T}$ defined by Eq. (1.8). As in the analysis of local time, we introduce the moment generating function or Laplace transform (2.4) with

$$Q(s,t|x_0,t_0)=\int_0^\infty \mathrm{e}^{-s\mathcal{T}}\int_{-\infty}^\infty\left[\int_{x(t_0)=x_0}^{x(t)=x}\delta\left(\mathcal{T}-\int_{t_0}^t\Theta(x(\tau))d\tau\right)P_\sigma[x]\mathcal{D}[x]\right]dxd\mathcal{T}$$
$$=\int_{-\infty}^\infty\left[\int_{x(t_0)=x_0}^{x(t)=x}\exp\left(-s\int_{t_0}^t\Theta(x(\tau))d\tau\right)P_\sigma[x]\mathcal{D}[x]\right]dx \tag{2.11}$$

We are using a path-integral representation of the gated Brownian motion. Introducing the function G according to Eq. (2.6), we have

$$G(s,x,t|x_0,t_0)=\int_{x(t_0)=x_0}^{x(t)=x}\exp\left(-s\int_{t_0}^t\Theta(x(\tau))d\tau\right)P_\sigma[x]\mathcal{D}[x]\equiv\left\langle\exp\left(-s\int_{t_0}^t\Theta(x(\tau)))d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t)=x} \tag{2.12}$$

where $\langle\cdots\rangle$ denotes averaging over realizations of the gated Brownian motion. G changes over an infinitesimal time interval Δt as

$$G(s,x,t+\Delta t|x_0,t_0)=\left\langle\exp\left(-s\int_{t_0}^{t+\Delta t}\Theta(x(\tau))d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t+\Delta t)=x}$$
$$\approx\int p(\Delta x)\left\langle\exp\left(-s\int_{t_0}^t\Theta(x(\tau))d\tau\right)\right\rangle_{x(t_0)=x_0}^{x(t)=x-\Delta x}\exp\left(-s\Theta(x(t))\Delta t\right)d\Delta x,$$

Again we have split the time interval $[t_0,t+\Delta t]$ into two parts $[t_0,t]$ and $[t,t+\Delta t]$ and introduced the intermediate state $x(t)=x-\Delta x$ with Δx distributed according to an infinitesimal Wiener process with density $p(\Delta x)$, at least outside a neighborhood of the switching gate at $x=0$. It follows that

$$G(s,x,t+\Delta t|x_0,t_0)=\mathrm{e}^{-s\Theta(x(t))\Delta t}\int p(\Delta x)G(s,x-\Delta x,t|x_0,t_0)d\Delta x$$
$$=\mathrm{e}^{-s\Theta(x(t))\Delta t}\left(G(s,x,t|x_0,t_0)-\langle\Delta x\rangle\frac{\partial}{\partial x}G(s,x,t|x_0,t_0)+\langle\Delta x^2\rangle\frac{\partial^2}{\partial x^2}G(s,x,t|x_0,t_0)+\ldots\right).$$

First suppose that the gate is open at time t, $n(t)=1$, so that Eq. (2.8) holds for all $x\in\mathbb{R}$. We thus obtain the following forward Fokker-Planck equation (FPE) in the limit $\Delta t\to 0$:

$$\frac{\partial G}{\partial t}=\frac{1}{2}\frac{\partial^2 G}{\partial x^2}-s\Theta(x)G,\quad x\in\mathbb{R},t>t_0 \tag{2.13}$$

provided that $n(t)=1$. On the other hand, if $n(t)=0$, then the infinitesimal Wiener process will be reflected at $x=0$ so that in the limit $\Delta t\to 0$ we have

$$\frac{\partial G}{\partial t}=\frac{1}{2}\frac{\partial^2 G}{\partial x^2}-s\Theta(x)G,\quad x\neq 0,t>t_0 \tag{2.14}$$

for $n(t)=0$, supplemented by the non-flux boundary condition

$$\left.\frac{\partial G(s,x,t|x_0,t_0)}{\partial x}\right|_{x=0^+}=0=\left.\frac{\partial G(s,x,t|x_0,t_0)}{\partial x}\right|_{x=0^-}. \tag{2.15}$$

In both cases G satisfies the initial condition

$$G(s,x,t_0|x_0,t_0)=\delta(x-x_0).$$

Just like Eq. (2.9), we see that Eqs. (2.13) and (2.14) represent a stochastic FPE due to the dependence on the gating term $n(t)$.

Finally, the corresponding backward FPE for $Q(s,t|x_0,t-\tau)$ takes the form

$$\frac{\partial Q}{\partial\tau}=\frac{1}{2}\frac{\partial^2 Q}{\partial x_0^2}-s\Theta(x(t-\tau))Q, \tag{2.16}$$

with n-dependent boundary conditions at $x=0$:

$$Q(s,t|0^+,t-\tau)=Q(s,t|0^-,t-\tau), \tag{2.17a}$$
$$\left.\frac{\partial Q(s,t|x_0,t-\tau)}{\partial x_0}\right|_{x=0^+}=\left.\frac{\partial Q(s,t|x_0,t-\tau)}{\partial x_0}\right|_{x=0^-} \tag{2.17b}$$

for $n(t-\tau)=1$ and

$$\left.\frac{\partial Q(s,t|x_0,t-\tau)}{\partial x_0}\right|_{x=0^+}=0=\left.\frac{\partial Q(s,t|x_0,t-\tau)}{\partial x_0}\right|_{x=0^-} \tag{2.18}$$

for $n(t-\tau)=0$. That is, when the gate is open there is continuity of the concentration and the flux across $x=0$, whereas when the gate is closed the right-hand boundary of $(-\infty,0)$ and the left-hand boundary of $(0,\infty)$ are reflecting. We also have the "final" condition $Q(s,t|x_0,t)=1$. Eq. (2.16) will be analyzed in section IIIB.

III. DYNAMICAL EQUATIONS FOR $\mathbb{E}_\sigma[Q]$

From the definition of Q, see Eq. (2.4), it is clear that solving Eq. (2.10) or Eq. (2.16) for fixed σ allows us to determine the k-th moment of the local time density (1.7) or occupation time (1.8) averaged with respect to the gated Brownian motion $X(t)$:

$$\begin{aligned}\langle \mathcal{T}^k\rangle &\equiv \int_0^\infty \mathcal{T}^k P(\mathcal{T},t|x_0,t-\tau)d\mathcal{T}\\ &= (-1)^k \left.\frac{d^k}{ds^k}Q(s,t|x_0,t-\tau)\right|_{s=0}.\end{aligned} \tag{3.1}$$

However, in order to determine statistics of the doubly stochastic process, we also need to take expectations with respect to realizations σ of the gate:

$$\langle\langle \mathcal{T}^k(\tau)\rangle\rangle = \mathbb{E}_\sigma[\langle \mathcal{T}^k\rangle] = (-1)^k \left.\frac{d^k}{ds^k}\mathbb{E}_\sigma[Q(s,t|x_0,t-\tau)]\right|_{s=0}, \tag{3.2}$$

assuming we can reverse the order of expectation and differentiation. Hence, calculating the moments of $\mathcal{T}$ with respect to the doubly stochastic process, requires determining $\mathbb{E}_\sigma[Q]$. The latter is the generator of moments of $\mathcal{T}$ averaged with respect to realizations of the stochastic gate. We will proceed by adapting our recent work on stochastic diffusion equations in randomly switching environments [19]. The basic idea of our approach is to note that since Q is a random field with respect to realizations of the stochastic gate, there exists a probability density functional ϱ that determines the distribution of the densities $q(x_0,\tau) = Q(s,t|x_0,t-\tau)$ for fixed s,t. The expectation $\mathbb{E}_\sigma[Q]$ then corresponds to the first moment of this density functional. (This is distinct from the first moment of $\mathcal{T}$ generated by Q.) Rather than dealing with the probability density functional directly, we spatially discretize the piecewise deterministic backward FPE (2.10) or (2.16) using a finite-difference scheme and use this to derive corresponding differential equations for $\mathbb{E}_\sigma[Q]$. More precisely, we will derive equations for $E_\sigma[Q]$ conditioned on the initial state of the gate.

A. Gated local time density

We begin by discretizing the backward FPE (2.10) for local time. Introduce the lattice spacing ℓ and set $x_0 = j\ell, \ell \in \mathbb{Z}$. Let $Q_j(\tau) = Q(s,t|t-\tau,j\ell)$, $j\in\mathbb{Z}$, and $a = J\ell$. This yields the piecewise deterministic ODE (for fixed s,t)

$$\frac{dQ_i}{d\tau} = \sum_{j\in\mathbb{Z}}\Delta_{ij}Q_j - sn(t-\tau)\delta_{i,J}Q_i, \tag{3.3}$$

with $Q_j(s,0)=1$. Here

$$\Delta_{ij} = \frac{1}{2\ell^2}[\delta_{i,j+1}+\delta_{i,j-1}-2\delta_{i,j}] \tag{3.4}$$

is the discrete Laplacian. Let $\mathbf{Q}(\tau) = (Q_j(\tau), j\in\mathbb{Z})$ and introduce the probability density

$$\text{Prob}\{\mathbf{Q}(\tau)\in(\mathbf{Q},\mathbf{Q}+d\mathbf{Q}), n(t-\tau)=n\} = \varrho_n(\mathbf{Q},\tau)d\mathbf{Q}, \tag{3.5}$$

The probability density evolves according to the following infinite-dimensional differential Chapman-Kolmogorov (dCK) equation:

$$\frac{\partial\varrho_n}{\partial\tau} = -\sum_{i\in\mathbb{Z}}\frac{\partial}{\partial Q_i}\left[\left(\sum_{j\in\mathbb{Z}}\Delta_{ij}Q_j - \frac{s}{\ell}n\delta_{i,J}Q_i\right)\varrho_n(\mathbf{Q},t)\right] + \sum_{m=0,1}W_{nm}\varrho_m(\mathbf{Q},\tau), \tag{3.6}$$

where W is the generator of the two-state Markov process underlying the stochastic gate, see Eq. (2.1). Since the dCK equation (3.6) is linear in the Q_j, it follows that we can obtain a closed set of equations for the first-order (and higher-order) moments of the density ϱ_n.

Let

$$\mathcal{Q}_{n,k}(s,\tau) = \mathbb{E}[Q_k(s,\tau)1_{n(t-\tau)=n}] = \int \varrho_n(\mathbf{Q},\tau)Q_k d\mathbf{Q}, \tag{3.7}$$

where

$$\int F(\mathbf{Q})d\mathbf{Q} = \left[\prod_j \int_0^\infty dQ_j\right]F(\mathbf{Q}).$$

Multiplying both sides of Eq. (3.6) by Q_k and integrating with respect to $\mathbf{Q}$ gives (after integrating by parts and assuming that $\varrho_n(\mathbf{Q},\tau)\to 0$ as $\mathbf{Q}\to\infty$)

$$\frac{d\mathcal{Q}_{n,k}}{dt} = \sum_{j\in\mathbb{Z}}\Delta_{kj}\mathcal{Q}_{n,j} - \frac{s}{\ell}n\delta_{k,J}\mathcal{Q}_{n,k} + \sum_{m=0,1}W_{nm}\mathcal{Q}_{m,k}. \tag{3.8}$$

We have assumed that the final discrete state $n(t)$ is distributed according to the stationary distribution ρ_n so that

$$\int \varrho_n(\mathbf{Q},0)d\mathbf{Q} = \rho_n. \tag{3.9}$$

If we now retake the continuum limit $\ell\to 0$ and set

$$\mathcal{Q}_n(x;s,\tau) = \mathbb{E}_\sigma[Q(s,t|x,t-\tau)1_{n(t-\tau)=n}] \tag{3.10}$$

for fixed t, then we obtain the system of equations

$$\frac{\partial \mathcal{Q}_0}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \mathcal{Q}_0}{\partial x^2} - \beta \mathcal{Q}_0 + \beta \mathcal{Q}_1 \tag{3.11a}$$

$$\frac{\partial \mathcal{Q}_1}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \mathcal{Q}_1}{\partial x^2} - s\delta(x-a)\mathcal{Q}_1 + \alpha \mathcal{Q}_0 - \alpha \mathcal{Q}_1 \tag{3.11b}$$

We have dropped the subscript on the initial position x_0. In the above derivation, we have assumed that integrating with respect to $\mathbf{Q}$ and taking the continuum limit commute. (One can also avoid the issue that $\mathbf{Q}$ is an infinite-dimensional vector by carrying out the discretization over a finite domain $[-L, L]$, and taking the limit $L \to \infty$ once the moment equations have been derived.) Finally, applying the final condition $Q(s,t|x_0,t) = 1$ implies that $\mathcal{Q}_n(x;s,0) = 1$.

Note that in the remainder of the paper, we will modify the definition of the moments of $\mathcal{T}$ given by Eq. (3.2) under the assumption that the initial state of the gate, $n(t-\tau)$, is determined by the stationary distribution ρ_n:

$$\begin{aligned}\langle\langle \mathcal{T}^k(\tau)\rangle\rangle &= (-1)^k \frac{d^k}{ds^k}\mathbb{E}_\sigma[Q(s,t|x_0,t-\tau)_0 1_{n(t-\tau)=0}]\rho_0 \\ &\quad + (-1)^k \frac{d^k}{ds^k}\mathbb{E}_\sigma[Q(s,t|x_0,t-\tau)_0 1_{n(t-\tau)=1}]\rho_1 \\ &= (-1)^k \frac{d^k}{ds^k}\left[\rho_0 \mathcal{Q}_0(x_0;s,\tau) + \rho_1 \mathcal{Q}_1(x_0;s,\tau)\right]_{s=0}\end{aligned} \tag{3.12}$$

B. Gated occupation time

We proceed in a similar fashion for the occupation time (1.8) by spatially discretize the piecewise deterministic backward FPE (2.16). The resulting piecewise deterministic ODE (for fixed s,t) now takes the form

$$\frac{dQ_i}{d\tau} = \sum_{j\in\mathbb{Z}} \Delta^n_{ij} Q_j - sQ_i, \quad i > 0 \tag{3.13a}$$

$$\frac{dQ_i}{d\tau} = \sum_{j\in\mathbb{Z}} \Delta^n_{ij} Q_j \quad i < 0 \tag{3.13b}$$

for $n(t-\tau) = n$. Away from the origin ($i \neq \pm 1$), $\Delta^n_{ij} = \Delta_{ij}$ where Δ_{ij} is the discrete Laplacian (3.4). If the gate is open then the particle can freely hop across the origin so $\Delta^1_{\pm 1,j} = \Delta_{\pm 1,j}$, whereas if the gate is closed then the particle is reflected at the origin, which means

$$\Delta^0_{1,j} = \frac{1}{\ell^2}[\delta_{j,2} - \delta_{j,1}], \quad \Delta^0_{-1,j} = \frac{1}{\ell^2}[\delta_{j,-2} - \delta_{j,-1}].$$

The probability density of Eq. (3.5) evolves according to the following infinite-dimensional differential Chapman-Kolmogorov (dCK) equation:

$$\frac{\partial \varrho_n}{\partial \tau} = -\sum_{i\neq 0}\frac{\partial}{\partial Q_i}\left[\left(\sum_{j\neq 0}\Delta^n_{ij}Q_j - s\Theta_i Q_i\right)\varrho_n(\mathbf{Q},t)\right] + \sum_{m=0,1} W_{nm}\varrho_m(\mathbf{Q},\tau), \tag{3.14}$$

where $\Theta_i = 1$ if $i > 0$ and zero otherwise, and W is the generator (2.1). Again the dCK equation (3.14) is linear in the Q_j, so we can obtain a closed set of equations for the first-order (and higher-order) moments of the distribution ϱ_n.

Multiplying both sides of Eq. (3.14) by Q_k and integrating with respect to $\mathbf{Q}$ gives (after integrating by parts and assuming that $\varrho_n(\mathbf{Q},\tau) \to 0$ as $\mathbf{Q} \to \infty$)

$$\frac{d\mathcal{Q}_{n,k}}{dt} = \sum_{j\neq 0}\Delta^n_{kj}\mathcal{Q}_{n,j} - s\Theta_k \mathcal{Q}_{n,k} + \sum_{m=0,1} W_{nm}\mathcal{Q}_{m,k}. \tag{3.15}$$

with $\mathcal{Q}$ defined by Eq. (3.7). We have again assumed that the final discrete state $n(t)$ is distributed according to the stationary distribution ρ_n so that Eq. (3.9) holds. If we now retake the continuum limit $\ell \to 0$, then we obtain the following system of equations for $\mathcal{Q}_n(x;s,\tau)$ defined by Eq. (3.10):

$$\frac{\partial \mathcal{Q}_0}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \mathcal{Q}_0}{\partial x^2} - s\Theta(x)\mathcal{Q}_0 - \beta\mathcal{Q}_0 + \beta\mathcal{Q}_1, \quad x \in \mathbb{R}\backslash 0 \tag{3.16a}$$

$$\frac{\partial \mathcal{Q}_1}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \mathcal{Q}_1}{\partial x^2} - s\Theta(x)\mathcal{Q}_1 + \alpha\mathcal{Q}_0 - \alpha\mathcal{Q}_1, \quad x \in \mathbb{R}\backslash 0 \tag{3.16b}$$

supplemented by the boundary conditions

$$\mathcal{Q}_1(s,t|0^+,t-\tau) = \mathcal{Q}_1(s,t|0^-,t-\tau), \tag{3.17a}$$

$$\left.\frac{\partial \mathcal{Q}_1(s,t|x_0,t-\tau)}{\partial x}\right|_{x=0^+} = \left.\frac{\partial \mathcal{Q}_1(s,t|x_0,t-\tau)}{\partial x}\right|_{x=0^-} \tag{3.17b}$$

and

$$\left.\frac{\partial \mathcal{Q}_0(s,t|x_0,t-\tau)}{\partial x}\right|_{x=0^+} = 0 = \left.\frac{\partial \mathcal{Q}_0(s,t|x_0,t-\tau)}{\partial x}\right|_{x=0^-} \tag{3.18}$$

IV. ANALYSIS OF $\mathcal{Q}_0$ AND $\mathcal{Q}_1$: LOCAL TIME

A. No switching

In the following we will take the location of the gate to be at $a=0$. In the absence of a switching gate ($n(t)=1$ for all t), we have $\mathcal{Q}_0=0$ and $\alpha=0$ so that Eqs. (3.11) reduce to

$$\frac{\partial Q}{\partial \tau} = \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} - s\delta(x)Q \tag{4.1}$$

for $\mathcal{Q}_1 = Q$, where

$$Q(x;s,\tau) = \int_0^\infty \mathrm{e}^{-sT}P(T,t|x,t-\tau)dT. \tag{4.2}$$

with

$$T=T(\tau) = \int_{t-\tau}^t \delta(X(t'))dt' = \int_0^\tau \delta(X(t'))dt'.$$

Since the process is now time-homogeneous, the solutions are independent of t, that is, $P(T,t|x,t-\tau) = P(T,\tau|x,0)$.

The case of no switching has been analyzed in Ref. [23], both for pure Brownian motion and for Brownian motion in the presence of ordered and spatially disordered potentials. We focus on pure Brownian motion here. Laplace transforming the backward FPE (4.1) with respect to time τ yields the ODE

$$z\widetilde{Q}(x;s,z) - 1 = \frac{1}{2}\widetilde{Q}''(x;s,z) - s\delta(x)\widetilde{Q}(x;s,z), \tag{4.3}$$

with $\widetilde{Q}' = d\widetilde{Q}/dx$ and

$$\begin{aligned}\widetilde{Q}(x;s,z) &= \int_0^\infty \mathrm{e}^{-z\tau}Q(x,s,\tau)d\tau \\ &= \int_0^\infty\int_0^\infty \mathrm{e}^{-z\tau-sT}P(T,\tau|x,0)dT\,d\tau.\end{aligned}$$

If a particle starts at $x=\pm\infty$ then it will never cross the origin a finite time τ in the future, that is, $P(T,\tau|\pm\infty,0) = \delta(T)$. Substituting this into the definition of $\widetilde{Q}$ shows that $\widetilde{Q}(\pm\infty;s,z) = z^{-1}$. Following the standard analysis of 1D Green's functions, we have to solve Eq. (4.3) separately for $\widetilde{Q} = u_+(x;s,z)+z^{-1}$ in $x\in(0,\infty)$ and $\widetilde{Q} = u_-(x;s,z)+z^{-1}$ in $x\in(-\infty,0)$, and then match the solutions at $x=0$. (We have performed a uniform shift of the solutions for convenience.) That is, $u_\pm$ satisfy the Eqs.

$$\frac{1}{2}u_\pm''(x;s,z) - zu_\pm(x;s,z) = 0, \tag{4.4}$$

with corresponding boundary conditions $u_\pm(\pm\infty;s,z) = 0$. Note that these equations are independent of the Laplace variable s. The s-dependence emerges from the matching conditions, which are obtained by (i) imposing continuity of the solution at $x=0$ and (ii) integrating Eq. (4.3) across $x=0$:

$$u_+(0;s,z) = u_-(0;s,z) = U(s,z) - z^{-1}, \tag{4.5a}$$

$$u_+'(0;s,z) - u_-'(0;s,z) = 2sU(s,z), \tag{4.5b}$$

for some unknown U. Rearranging this pair of equations shows that

$$U = U(s,z) = \frac{\lambda(z)}{z(s+\lambda(z))}, \tag{4.6}$$

where

$$\lambda(z) = \frac{u_-'(0;s,z)/u_-(0;s,z) - u_+'(0;s,z)/u_+(0;s,z)}{2}. \tag{4.7}$$

Using the fact that $\lambda(z)$ is independent of s, we can perform the inverse Laplace transform $\mathcal{L}^{-1}$ with respect to s, which shows that

$$\mathcal{L}^{-1}[U](T,z) = \frac{\lambda(z)}{z}\mathrm{e}^{-\lambda(z)T}.$$

Noting that

$$\mathcal{L}^{-1}[U](T,z) = \mathcal{L}^{-1}[\widehat{Q}](0;T,z) = \int_0^\infty \mathrm{e}^{-z\tau}P(T,\tau|0,0)d\tau,$$

we conclude that

$$F(T,z) \equiv \int_0^\infty \mathrm{e}^{-z\tau}P(T,\tau|0,0)d\tau = \frac{\lambda(z)}{z}\mathrm{e}^{-\lambda(z)T}. \tag{4.8}$$

The function $F(T,z)$ can be calculated explicitly in the case of pure Brownian motion. In particular, Eqs. (4.4) have the solutions

$$u_\pm(x;s,z) = -\frac{sU(s,z)}{\sqrt{2z}}\mathrm{e}^{\mp\sqrt{2z}x}. \tag{4.9}$$

Substituting into the continuity condition (4.5a) implies

$$U(s,z) - z^{-1} = -\frac{sU(s,z)}{\sqrt{2z}},$$

that is,

$$U(s,z) = \frac{1}{z}\frac{\sqrt{2z}}{s+\sqrt{2z}}.$$

Comparison with Eq. (4.6) establishes that $\lambda(z) = \sqrt{2z}$ and hence

$$F(T,z) = \sqrt{\frac{2}{z}}\mathrm{e}^{-\sqrt{2z}T}.$$

Inverting the inverse Laplace transform with respect to z then shows that the distribution of local times around the origin is a Gaussian,

$$P(T,t|0,t-\tau) = \sqrt{\frac{2}{\pi\tau}}\mathrm{e}^{-T^2/2\tau}. \tag{4.10}$$

It follows that the first and second moments of the local time density (starting at $x = 0$) are

$$\langle T(\tau)\rangle = \int_0^\infty TP(T,\tau|0,0)dT = \sqrt{\frac{2\tau}{\pi}}, \tag{4.11}$$

and

$$\langle T^2(\tau)\rangle = \int_0^\infty T^2P(T,\tau|0,0)dT = \tau. \tag{4.12}$$

B. Stochastically-gated local time

Let us now turn to the full switching model given by Eqs. (3.11). Laplace transforming these equation with respect to τ gives

$$-1 = \frac{1}{2}\frac{\partial^2\widetilde{\mathcal{Q}}_0}{\partial x^2} - (z+\beta)\widetilde{\mathcal{Q}}_0 + \beta\widetilde{\mathcal{Q}}_1 \tag{4.13a}$$

$$-1 = \frac{1}{2}\frac{\partial^2\widetilde{\mathcal{Q}}_1}{\partial x^2} - s\delta(x)\widetilde{\mathcal{Q}}_1 + \alpha\widetilde{\mathcal{Q}}_0 - (z+\alpha)\widetilde{\mathcal{Q}}_1 \tag{4.13b}$$

Adding the pair of equations, after multiplying the first by ρ_0 and the second by ρ_1, and setting $\widetilde{\mathcal{Q}} = \rho_0\widetilde{\mathcal{Q}}_0 + \rho_1\widetilde{\mathcal{Q}}_1$ yields

$$-1 = \frac{1}{2}\frac{\partial^2\widetilde{\mathcal{Q}}}{\partial x^2} - z\widetilde{\mathcal{Q}} - s\rho_1\delta(x)\widetilde{\mathcal{Q}}_1. \tag{4.14}$$

Eq. (4.14) can be analyzed along similar lines to Eq. (4.3). In particular, we solve Eq. (4.14) separately for $\widetilde{\mathcal{Q}} = u_+(x;s,z) + z^{-1}$ in $x \in (0,\infty)$ and $\widetilde{\mathcal{Q}} = u_-(x;s.z) + z^{-1}$ in $x \in (-\infty,0)$ using the boundary conditions $u_\pm(\pm\infty;s,z) = 0$, and then match the solutions at $x = 0$. The matching conditions now take the form

$$u_+(0;s,z) = u_-(0;s,z) = \rho_0U_0(s,z) + \rho_1U_1(s,z) - z^{-1},$$
$$u_+'(0;s,z) - u_-'(0;s,z) = 2\rho_1 sU_1(s,z),$$

with $\widetilde{\mathcal{Q}}_n = U_n$ at $x = 0$. Rearranging this pair of equations shows that

$$\rho_1U_1(s,z) = \frac{\lambda(z)(1-z\rho_0U_0(s,z))}{z(s+\lambda(z))}, \tag{4.15}$$

where $\lambda(z)$ is again given by Eq. (4.7) so that $\lambda(z) = \sqrt{2z}$.

Eq. (4.15) determines U_1 in terms of U_0. It remains to find U_0, which can be achieved by substituting $\beta\widetilde{\mathcal{Q}}_1 = (\alpha+\beta)\widetilde{\mathcal{Q}} - \alpha\widetilde{\mathcal{Q}}_0$ in Eq. (4.13a):

$$\frac{1}{2}\frac{\partial^2\widetilde{\mathcal{Q}}_0}{\partial x^2} - (z+\alpha+\beta)\widetilde{\mathcal{Q}}_0 = -1 - (\alpha+\beta)\widetilde{\mathcal{Q}} \tag{4.16}$$

This equation can be solved in terms of the Green's function $H(x,x';z)$ defined according to

$$\frac{1}{2}\frac{\partial^2 H}{\partial x^2} - (z+\alpha+\beta)H = -\delta(x-x') \tag{4.17}$$

which yields

$$H(x,x';z) = \frac{1}{\lambda(z+\alpha+\beta)}\mathrm{e}^{-\lambda(z+\alpha+\beta)|x-x'|} \tag{4.18}$$

Having obtained H, it follows that

$$\widetilde{\mathcal{Q}}_0(x;s,z) = \int_{-\infty}^\infty H(x,x';z)[1+(\alpha+\beta)\widetilde{\mathcal{Q}}(x';s,z)]dx'. \tag{4.19}$$

Now setting $x = 0$ in Eq. (4.19) gives

$$\begin{aligned}
U_0(s,z) &= \frac{1}{\lambda(z+\alpha+\beta)}\int_{-\infty}^\infty \mathrm{e}^{-\lambda(z+\alpha+\beta)|x'|}\left[1+(\alpha+\beta)\left(\frac{1}{z} - \frac{s\rho_1U_1(s,z)}{\lambda(z)}\mathrm{e}^{-\lambda(z)|x'|}\right)\right]dx' \\
&= \frac{2}{\lambda(z+\alpha+\beta)}\int_0^\infty \mathrm{e}^{-\lambda(z+\alpha+\beta)x'}\left[1+(\alpha+\beta)\left(\frac{1}{z} - \frac{s\rho_1U_1(s,z)}{\lambda(z)}\mathrm{e}^{-\lambda(z)x'}\right)\right]dx' \\
&= \frac{2}{\lambda(z+\alpha+\beta)}\left[\frac{z+\alpha+\beta}{z\lambda(z+\alpha+\beta)} - \frac{s\beta U_1(s,z)}{\lambda(z)[\lambda(z+\alpha+\beta)+\lambda(z)]}\right]
\end{aligned} \tag{4.20}$$

Eqs. (4.15) and (4.20) yield a pair of equations for the two unknowns $U_0(s,z)$ and $U_1(s,z)$.

Rather than obtaining the distribution of the local time, it is simpler to focus on the moments of the local time defined according to (3.12). Laplace transforming with respect to τ gives

$$\langle\!\langle \widetilde{\mathcal{T}}^k(z)\rangle\!\rangle = (-1)^k\frac{d^k}{ds^k}\left[\rho_0U_0(s,z) + \rho_1U_1(s,z)\right]_{s=0}. \tag{4.21}$$

Let us first consider the mean local time density $\langle\langle \mathcal{T}(\tau)\rangle\rangle$. Differentiating Eq. (4.15) with respect to s shows that

$$\rho_1[s+\lambda(z)]\partial_s U_1(s,z)+\rho_1 U_1(s,z) = -\rho_0\lambda(z)\partial_s U_0(s,z). \tag{4.22}$$

Now setting $s=0$ in Eqs. (4.15) and (4.20) gives

$$\rho_1 U_1(0,z) = z^{-1}-\rho_0 U_0(0,z)$$
$$U_0(0,z) = \frac{1}{z},$$

where we have used the result $\lambda(z+\alpha+\beta) = \sqrt{2[z+\alpha+\beta]}$. Eq. (4.22) thus shows that

$$\langle\langle \widetilde{\mathcal{T}}(z)\rangle\rangle = \frac{\rho_1}{z\sqrt{2z}}. \tag{4.23}$$

Therefore, in the presence of a switching gate

$$\langle\langle \mathcal{T}(\tau)\rangle\rangle = \rho_1\sqrt{\frac{2\tau}{\pi}}, \quad \tau>0 \tag{4.24}$$

This agrees with the result obtained by taking the expectation of Eq. (1.7) with respect to realizations of the stochastic gate directly, see section IIA. It reflects the fact that since the switching gate and the Brownian motion are independent stochastic processes, the expected local time is a product of the local time without switching multiplied by the fraction of time that the gate is open.

However, the presence of the stochastic gate has a nontrivial affect on higher-order moments of the local time due to temporal correlations between the state of the gate at different times. In order to illustrate this, we consider the second-order moment $\langle\langle \mathcal{T}^2(\tau)\rangle\rangle$. Differentiating Eq. (4.22) with respect to s gives

$$\rho_1[s+\lambda(z)]\partial_s^2 U_1(s,z)+2\rho_1\partial_s U_1(s,z) = -\rho_0\lambda(z)\partial_s^2 U_0(s,z). \tag{4.25}$$

Hence, in Laplace space

$$\langle\langle \widetilde{\mathcal{T}}^2(z)\rangle\rangle = -\frac{2\rho_1}{\lambda(z)}\partial_s U_1(0,z) = \frac{2\rho_1}{z\lambda^2(z)}+\frac{2\rho_0}{\lambda(z)}\partial_s U_0(0,z), \tag{4.26}$$

where we have used Eq. (4.22). Differentiating Eq. (4.20) with respect to s and setting $s=0$ gives

$$\partial_s U_0(0,z) = -\frac{\beta\lambda(z)F(z)}{z^2},$$
$$F(z) = \frac{1}{\lambda(z+\alpha+\beta)[\lambda(z+\alpha+\beta)+\lambda(z)]}$$

Therefore,

$$\langle\langle \widetilde{\mathcal{T}}^2(z)\rangle\rangle = \frac{\rho_1}{z^2}-\frac{2\beta\rho_0 F(z)}{z^2} \tag{4.27}$$

Finally, setting $\lambda(z)=\sqrt{2z}$ and inverting the Laplace transforms using the convolution theorem,

$$\langle\langle \mathcal{T}^2(\tau)\rangle\rangle = \rho_1\tau-2\beta\rho_0\int_0^\tau f(\tau-t)tdt, \tag{4.28}$$

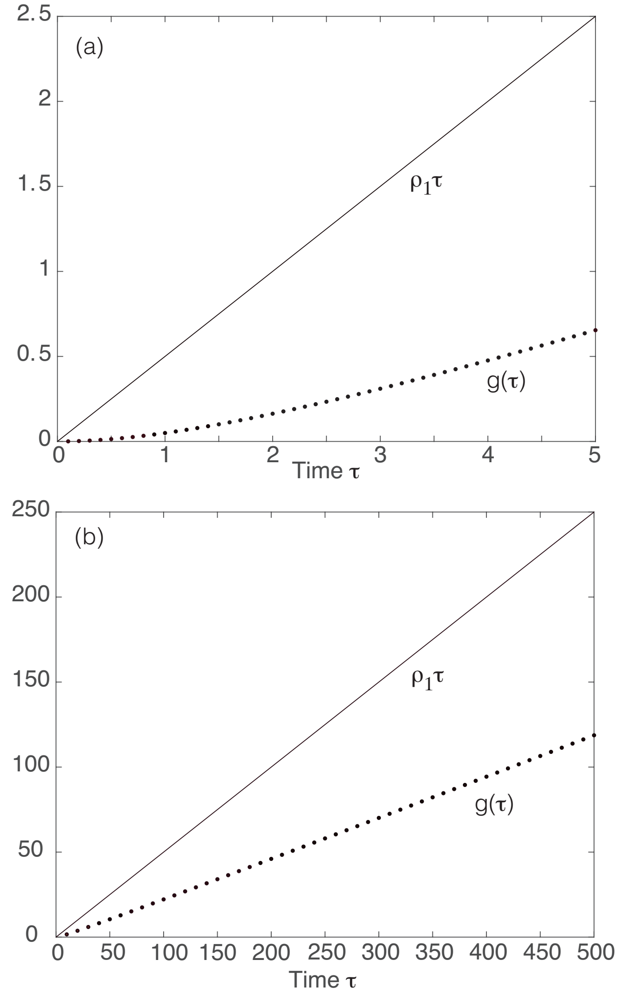

FIG. 3. Comparison of functions $g(\tau)=2\beta\rho_0\int_0^\tau f(\tau-t)tdt$ and $\rho_1\tau$ for $\alpha=\beta=\rho_0=\rho_1=0.5$. (a) $g(\tau)\ll\rho_1\tau$ at relatively short times. (b) On longer time-scales $g(\tau)\to\rho_1\tau/2$.

where

$$f(\tau) = \int_0^\tau \mathcal{F}(\tau-\tau')\mathcal{G}(\tau')d\tau'$$

and

$$\mathcal{F}(\tau) = \frac{1}{\sqrt{2\pi\tau}}\mathrm{e}^{-(\alpha+\beta)\tau},$$
$$\mathcal{G}(\tau) = \frac{1}{\sqrt{2(\alpha+\beta)}}\frac{1}{2\tau}\mathrm{e}^{-(\alpha+\beta)\tau/2}I_{1/2}\left(\frac{(\alpha+\beta)\tau}{2}\right).$$

Here $I_\nu(x)$ denotes the modified Besel function of order ν. The first term on the right-hand side of Eq. (4.28) is the product of the result for no switching and the fraction of time the gate is open. The second term specifies the reduction in the second-order moment of the local times due to the effects of temporal correlations in the gate dynamics. In the limit $z\to 0$, $F(z)\sim 1/(2(\alpha+\beta))$, which means that

$$\langle\!\langle \widetilde{\mathcal{T}}^2(z)\rangle\!\rangle \sim \frac{\rho_1}{z^2} - \frac{\rho_1\rho_0}{z^2},$$

and

$$\langle\!\langle \mathcal{T}^2(\tau)\rangle\!\rangle \sim \rho_1^2\tau, \quad \tau\rightarrow\infty. \tag{4.29}$$

Our results are confirmed by numerical plots of the inverse Laplace transforms in Fig. 3.

V. ANALYSIS OF $\mathcal{Q}_0$ AND $\mathcal{Q}_1$: OCCUPATION TIME

A. Ungated Brownian motion

In the absence of a switching gate ($n(t)=1$ for all t), we have $\mathcal{Q}_0 = 0$ and $\alpha = 0$ so that Eqs. (3.11) reduce to

$$\frac{\partial Q}{\partial \tau} = \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} - s\Theta(x)Q \tag{5.1}$$

for $\mathcal{Q}_1 = Q$, where Q is given by Eq. (4.2) with

$$T = T(\tau) = \int_{t-\tau}^t \Theta(X(t'))dt' = \int_0^\tau \Theta(X(t'))dt'.$$

Since the process is now time-homogeneous, the solutions are independent of t, that is, $P(T,t|x,t-\tau) = P(T,\tau|x,0)$.

The case of no switching was originally analyzed for pure Brownian motion by Levy [1] and has also been extended to Brownian motion in the presence of ordered and spatially disordered potentials, see Ref. [23]. Laplace transforming the backward FPE (5.1) with respect to time τ yields the ODE

$$z\widetilde{Q}(x;s,z) - 1 = \frac{1}{2}\widetilde{Q}''(x;s,z) - s\Theta(x)\widetilde{Q}(x;s,z), \tag{5.2}$$

with $\widetilde{Q}' = d\widetilde{Q}/dx$ and

$$\begin{aligned}\widetilde{Q}(x;s,z) &= \int_0^\infty \mathrm{e}^{-z\tau}Q(x,s,\tau)d\tau \\ &= \int_0^\infty\int_0^\infty \mathrm{e}^{-z\tau-sT}P(T,\tau|x,0)dT\,d\tau. \end{aligned}\tag{5.3}$$

Following the standard analysis of 1D Green's functions, we have to solve Eq. (5.2) separately for $\widetilde{Q} = u_+(x;s,z)$ in $x\in(0,\infty)$ and $\widetilde{Q} = u_-(x;s,z)$ in $x\in(-\infty,0)$, and then match the solutions at $x=0$. That is, $u_\pm$ satisfy the equations

$$\frac{1}{2}u_+'' - (z+s)u_+ = -1, \quad x>0 \tag{5.4a}$$

$$\frac{1}{2}u_-'' - zu_- = -1, \quad x<0 \tag{5.4b}$$

The matching conditions are obtained by (i) imposing continuity of the solution at $x=0$ and (ii) integrating Eq. (5.2) across $x=0$:

$$u_+(0;s,z) = u_-(0;s,z) = U, \quad u_+'(0;s,z) = u_-'(0;s,z).$$

for $U = \widetilde{Q}(0;s,z)$. In order to determine the far-field boundary conditions for $x\rightarrow\pm\infty$, we note that if a particle starts at $x=\pm\infty$ then it will never cross the origin a finite time τ in the future, that is,

$$P(T,\tau|\infty,0) = \delta(t-T), \quad P(T,\tau|-\infty,0) = \delta(T).$$

Substituting this into the definition of $\widetilde{Q}$ shows that

$$u_+(\infty;s,z) = \frac{1}{z+s}, \quad u_-(-\infty;s,z) = \frac{1}{z}. \tag{5.5}$$

If we now perform the shifts

$$u_+(x;s,z) = \frac{1}{z+s} + B_+y_+(x;s,z), \tag{5.6a}$$

$$u_-(x;s,z) = \frac{1}{z} + B_-y_-(x;s,z), \tag{5.6b}$$

then we obtain the homogeneous equations

$$\frac{1}{2}y_+'' - (z+s)y_+ = 0, \quad x>0 \tag{5.7a}$$

$$\frac{1}{2}y_-'' - zy_- = 0, \quad x<0 \tag{5.7b}$$

with $y_+(\infty;s,z)=0$ and $y_-(-\infty;s,z)=0$. The constants $B_\pm$ can be found by imposing the matching conditions, which take the explicit form

$$\frac{1}{z+s} + B_+y_+(0;s,z) = \frac{1}{z} + B_-y_-(0;s,z) = U \tag{5.8a}$$

$$B_+y_+'(0;s,z) = B_-y_-'(0;s,z). \tag{5.8b}$$

Eqs. (5.8) can be used to express $U(s,z)$ in terms of $\lambda_\pm(s,z) = y_\pm'(0;s,z)/y_\pm(0;s,z)$:

$$U(s,z) = \frac{L_1(s,z)}{z} + \frac{L_2(s,z)}{z+s}, \tag{5.9}$$

where

$$L_1(s,z) = \frac{\lambda_-(s,z)}{\lambda_-(s,z)-\lambda_+(s,z)},\; L_2(s,z) = \frac{-\lambda_+(s,z)}{\lambda_-(s,z)-\lambda_+(s,z)}. \tag{5.10}$$

Note that $L_1(s,z)+L_2(s,z) = 1$. Finally, explicitly solving Eqs. (5.7) yields

$$\lambda_+(s,z) = -\sqrt{2(z+s)}, \quad \lambda_-(s,z) = \sqrt{2z}, \tag{5.11}$$

so that combining Eqs. (5.9) and (5.10), and using the definition (5.3), we have

$$\begin{aligned}U(s,z) &\equiv \int_0^\infty\int_0^\infty \mathrm{e}^{-z\tau-sT}P(T,\tau|0,0)dT\,d\tau \\ &= \frac{1}{\sqrt{z(s+z)}}.\end{aligned} \tag{5.12}$$

Inverting the double Laplace transform with respect to s and then z recovers the well known "arcsine' law [1] for the probability density of the occupation time for pure Brownian motion starting at the origin:

$$P(T,\tau|0,0) = \frac{1}{\pi\sqrt{T(\tau-T)}}, \quad 0<T<\tau. \tag{5.13}$$

B. Stochastically-gated Brownian motion

Let us now turn to the occupation time of stochastically-gated Brownian motion and Eqs. (3.11). Laplace transforming these equation with respect to τ gives

$$-1 = \frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{Q}}_0}{\partial x^2} - s\Theta(x)\widetilde{\mathcal{Q}}_0 - (z+\beta)\widetilde{\mathcal{Q}}_0 + \beta\widetilde{\mathcal{Q}}_1 \quad (5.14a)$$

$$-1 = \frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{Q}}_1}{\partial x^2} - s\Theta(x)\widetilde{\mathcal{Q}}_1 + \alpha\widetilde{\mathcal{Q}}_0 - (z+\alpha)\widetilde{\mathcal{Q}}_1 \quad (5.14b)$$

supplemented by the boundary conditions

$$\widetilde{\mathcal{Q}}_1(0^+;s,z) = \widetilde{\mathcal{Q}}_1(0^-;s,z), \quad (5.15a)$$

$$\left.\frac{\partial\widetilde{\mathcal{Q}}_1(x;s,z)}{\partial x}\right|_{x=0^+} = \left.\frac{\partial\widetilde{\mathcal{Q}}_1(x;s,z)}{\partial x}\right|_{x=0^-} \quad (5.15b)$$

and

$$\left.\frac{\partial\widetilde{\mathcal{Q}}_0(x;s,z)}{\partial x}\right|_{x=0^+} = 0 = \left.\frac{\partial\widetilde{\mathcal{Q}}_0(x;s,z)}{\partial x}\right|_{x=0^-} \quad (5.16)$$

From the boundary conditions (5.15) and (5.16), we set

$$\left.\frac{\partial\widetilde{\mathcal{Q}}_1(x;s,z)}{\partial x}\right|_{x=0^+} = \left.\frac{\partial\widetilde{\mathcal{Q}}_1(x;s,z)}{\partial x}\right|_{x=0^-} = K_1,$$

with K_1 determined later by imposing (5.16).

Adding the pair of equations (5.14), after multiplying the first by ρ_0 and the second by ρ_1, and setting $\widetilde{\mathcal{Q}} = \rho_0\widetilde{\mathcal{Q}}_0 + \rho_1\widetilde{\mathcal{Q}}_1$ yields

$$-1 = \frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{Q}}}{\partial x^2} - s\Theta(x)\widetilde{\mathcal{Q}} - z\widetilde{\mathcal{Q}}, \quad (5.17)$$

supplemented by the boundary condition

$$\left.\frac{\partial\widetilde{\mathcal{Q}}(x;s,z)}{\partial x}\right|_{x=0^+} = \left.\frac{\partial\widetilde{\mathcal{Q}}(x;s,z)}{\partial x}\right|_{x=0^-} = \rho_1 K_1.$$

Eq. (5.17) can be analyzed along similar lines to Eq. (5.2). In particular, we solve Eq. (5.17) separately for $\widetilde{Q} = u_+(x;s,z)$ in $x \in (0,\infty)$ and $\widetilde{Q} = u_-(x;s,z)$ in $x \in (-\infty,0)$, and then match the solutions at $x = 0$ by setting

$$u_+'(0;s,z) = u_-'(0;s,z) = \rho_1 K_1$$

The resulting solution is

$$u_+(x;s,z) = \frac{1}{z+s} - \frac{\rho_1 K_1}{\sqrt{2(z+s)}}\mathrm{e}^{-\sqrt{2(z+s)}x}, \quad (5.18a)$$

$$u_-(x;s,z) = \frac{1}{z} + \frac{\rho_1 K_1}{\sqrt{2z}}\mathrm{e}^{\sqrt{2z}x}, \quad (5.18b)$$

Since $\beta\widetilde{\mathcal{Q}}_1 = (\alpha+\beta)\widetilde{\mathcal{Q}} - \alpha\widetilde{\mathcal{Q}}_0$, we can rewrite equation (5.14a) as

$$\frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{Q}}_0}{\partial x^2} - s\Theta(x)\widetilde{\mathcal{Q}}_0 - (z+\alpha+\beta)\widetilde{\mathcal{Q}}_0 = -1 - (\alpha+\beta)\widetilde{\mathcal{Q}} \quad (5.19)$$

with $\partial_x\widetilde{\mathcal{Q}}_0(0^-;s,z) = 0 = \partial_x\widetilde{\mathcal{Q}}_0(0^+;s,z)$. This equation can be solved in terms of the Green's function $K(x,x';s,z)$ defined according to

$$\frac{1}{2}\frac{\partial^2 K}{\partial x^2} - s\Theta(x)K - (z+\alpha+\beta)K = -\delta(x-x') \quad (5.20)$$

which yields

$$K(x,x';s,z) = \frac{1}{\mu_+(z)}\mathrm{e}^{-\mu_+(z)|x-x'|} \quad (5.21)$$

for $x > 0$ and

$$K(x,x';s,z) = \frac{1}{\mu_-(z)}\mathrm{e}^{-\mu_-(z)|x-x'|} \quad (5.22)$$

for $x < 0$. Here

$$\mu_\pm(z) = |\lambda_\pm(z+\alpha+\beta)|. \quad (5.23)$$

Having obtained K, it follows that

$$\widetilde{\mathcal{Q}}_0(x;s,z) = \left[A\mathrm{e}^{-\mu_+(z)x}\Theta(x) + B\mathrm{e}^{\mu_-(z)x}\Theta(-x)\right] + \int_{-\infty}^{\infty} K(x,x';s,z)[1+(\alpha+\beta)\widetilde{\mathcal{Q}}(x';s,z)]dx'. \quad (5.24)$$

Substituting for $\widetilde{\mathcal{Q}}$ using Eqs. (5.18) we obtain a complicated expression for $\widetilde{\mathcal{Q}}_0$ that involves the unknown constants A, B, K_1. The first two can be expressed in terms of K_1 by imposing $\partial_x\widetilde{\mathcal{Q}}_0(0^-;s,z) = 0 = \partial_x\widetilde{\mathcal{Q}}_0(0^+;s,z)$. Finally, K_1 is determined by imposing the boundary conditions (5.15a). The details are presented in the appendix.

The main result we obtain in the appendix is that if $\rho_0 < 1$ (the gate is open at least some of the time), then

$$\widetilde{\mathcal{Q}}(0^\pm,s,z) \approx \frac{1}{\sqrt{z(s+z)}}, \quad s,z \to 0. \quad (5.25)$$

Performing a double inverse Laplace transform then implies that in the limit $T, \tau \to \infty$, we have

$$\rho_0\mathbb{E}_\sigma[P(\mathcal{T},t|0,t-\tau)1_{n(t-\tau)=0}] + \rho_1\mathbb{E}_\sigma[P(\mathcal{T},t|0,t-\tau)1_{n(t-\tau)=1}] \approx \frac{1}{\pi\sqrt{\mathcal{T}(\tau-\mathcal{T})}} \quad (5.26)$$

for $0 \ll T < \tau$. This result has the following interpretation. Although the closing of the gate reduces the number of times the particle crosses the origin from left to right, it also reduces the number of times it crosses from right to left. Hence in the large-time limit these two affects cancel out. On the other hand, if $\rho_0 = 1$ (gate always closed) then either $P = \delta(\mathcal{T}-\tau)$ when the particle starts in the positive x-axis or $P = \delta(\mathcal{T})$ when the particle starts in the negative x-axis.

VI. HIGHER-ORDER MOMENTS OF Q

So far we have focused on first moments of the generator Q with respect to realizations of the gate. Here we briefly discuss the interpretation of higher-order moments of Q. For the sake of simplicity, we focus on the case of local time, although similar conclusions hold for the occupation time. Equations for these higher-order moments can be derived using the discretization scheme of section III and, in particular, the dCK equation (3.6). We will illustrate this by considering second-order moments. Let

$$\begin{aligned} C_{n,kl}(s,\tau) &= \mathbb{E}_\sigma[Q_k(s,\tau)Q_l(s,\tau)1_{n(t-\tau)=n}] \\ &= \int \varrho_n(\mathbf{Q},t)Q_k(s,t)Q_l(s,t)d\mathbf{Q}. \end{aligned}$$

Multiplying both sides of Eq. (3.6) by $Q_k(s,\tau)Q_l(s,\tau)$ and integrating with respect to $\mathbf{Q}$ gives (after integration by parts)

$$\frac{dC_{n,kl}}{d\tau} = \sum_j [\Delta_{kj}C_{n,jl} + \Delta_{lj}C_{n,jk}] - \frac{s}{\ell}n\delta_{k,J}C_{n,Jl} - \frac{s}{\ell}n\delta_{l,J}C_{n,kJ} + \sum_{m=0,1} W_{nm}C_{m,kl}. \tag{6.1}$$

If we now retake the continuum limit $a \to 0$, we obtain a system of parabolic equations for the equal-time two-point correlations

$$C_n(x,y;s,\tau) = \mathbb{E}_\sigma[Q(s,t|x,t-\tau)Q(s,t|y,t-\tau)1_{n(t-\tau)=n}], \tag{6.2}$$

given by

$$\frac{\partial C_0}{\partial \tau} = \frac{1}{2}\frac{\partial^2 C_0}{\partial x^2} + \frac{1}{2}\frac{\partial^2 C_0}{\partial y^2} - \beta C_0 + \beta C_1 \tag{6.3a}$$

$$\begin{aligned} \frac{\partial C_1}{\partial \tau} &= \frac{1}{2}\frac{\partial^2 C_1}{\partial x^2} + \frac{1}{2}\frac{\partial^2 C_1}{\partial y^2} + \alpha C_0 - \alpha C_1 \\ &\quad - s\delta(x-a)C_1(a,y;s,\tau) - s\delta(y-a)C_1(x,a;s,\tau) \end{aligned} \tag{6.3b}$$

As we will show below C_n determines statistical correlations between the local times of two independent Brownian particles moving in the same switching environment. The existence of such correlations reflects the fact that, in general, solutions of Eqs. (6.3) cannot be expressed as the product of first-order moments.

From the definition of Q in Eq. (2.5), we see that

$$\begin{aligned} C(x,y;s,\tau) = \mathbb{E}_\sigma\bigg[&\int_0^\infty e^{-s\mathcal{T}}P(\mathcal{T},t|x,t-\tau)d\mathcal{T} \\ &\times \int_0^\infty e^{-s\mathcal{T}'}P(\mathcal{T}',t|y,t-\tau)d\mathcal{T}'\bigg]. \end{aligned} \tag{6.4}$$

for $C = C_0 + C_1$. It follows that

$$(-1)^k\,\frac{d^k}{ds^k}C(x,y;s,\tau)\bigg|_{s=0} = \mathbb{E}_\sigma\left[\mathcal{T}_k(x,y,\tau)\right]$$

where

$$\begin{aligned} &\mathcal{T}_k(x,y,\tau) \\ &= \int_0^\infty\int_0^\infty (\mathcal{T}+\mathcal{T}')^k P(\mathcal{T},t|x,t-\tau)P(\mathcal{T}',t|y,t-\tau)d\mathcal{T}d\mathcal{T}'. \end{aligned}$$

For $k = 1$, we simply have the sum of the local times averaged with respect to the Brownian motion and the stochastic gate,

$$\mathbb{E}_\sigma\left[\mathcal{T}_1(x,y,\tau)\right] = \langle\!\langle \mathcal{T}(x,\tau)\rangle\!\rangle + \langle\!\langle \mathcal{T}(y,\tau)\rangle\!\rangle.$$

whereas for $k = 2$, we have the expression

$$\begin{aligned} \mathbb{E}_\sigma\left[\mathcal{T}_2(x,y,\tau)\right] &= \langle\!\langle \mathcal{T}(x,\tau)^2\rangle\!\rangle + \langle\!\langle \mathcal{T}(y,\tau)^2\rangle\!\rangle \\ &\quad + \mathbb{E}_\sigma[\langle \mathcal{T}(x,\tau)\rangle\langle \mathcal{T}(y,\tau)\rangle]. \end{aligned}$$

Let us now define

$$\overline{C}(x,y;s,\tau) = \mathbb{E}_\sigma[Q(s,t|x,t-\tau)]\mathbb{E}_\sigma[Q(s,t|y,t-\tau)]$$

and set

$$(-1)^k\,\frac{d^k}{ds^k}\overline{C}(x,y;s,\tau)\bigg|_{s=0} = \mathcal{S}_k(x,y,\tau)$$

It is straightforward to show that

$$\begin{aligned} &\mathbb{E}_\sigma\left[\mathcal{T}_1(x,y,\tau)\right] - \mathcal{S}_1(x,y,\tau) = 0, \\ &\mathbb{E}_\sigma\left[\mathcal{T}_2(x,y,\tau)\right] - \mathcal{S}_2(x,y,\tau) \\ &\quad = \mathbb{E}_\sigma[\langle \mathcal{T}(x,\tau)\rangle\langle \mathcal{T}(y,\tau)\rangle] - \langle\!\langle \mathcal{T}(x,\tau)\rangle\!\rangle\langle\!\langle \mathcal{T}(y,\tau)\rangle\!\rangle. \end{aligned}$$

Since $C(x,y;s,\tau) \neq \overline{C}(x,y;s,\tau)$, it follows that

$$\mathbb{E}_\sigma[\langle \mathcal{T}(x,\tau)\rangle\langle \mathcal{T}(y,\tau)\rangle] \neq \langle\!\langle \mathcal{T}(x,\tau)\rangle\!\rangle\langle\!\langle \mathcal{T}(y,\tau)\rangle\!\rangle.$$

In other words, $C - \overline{C}$ is a generator for statistical correlations between the expected local times of two independent Brownian particles moving in the same switching environment but starting from different points x, y. A similar argument shows that r-th order moments of ϱ generate statistical correlations between r independent Brownian particles with initial positions $x_1, \ldots x_r$ all moving in the same randomly switching environment.

VII. DISCUSSION

In this paper we have shown how to extend the probabilistic notions of local and occupation times to the case of a 1D Brownian particle moving in a stochastically switching environment. We derived a generalized Feynman-Kac formula for the moment generating function of the local time density or occupation time in the

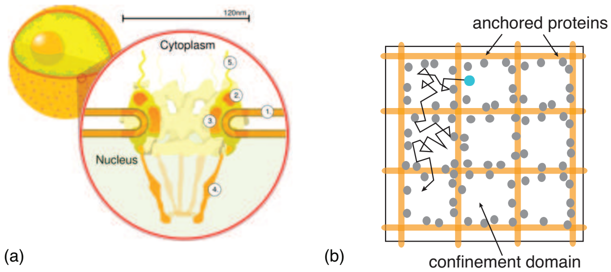

FIG. 4. Examples of gated diffusion. (a) The nuclear envelope with inset showing components of a single nuclear pore. 1. Nuclear envelope. 2. Outer ring. 3. Spokes. 4. Basket. 5. Filaments. Each of the eight protein subunits surrounding the actual pore (the outer ring) projects a spoke-shaped protein into the pore channel. (Public domain figure from Wikimedia. (b) Picket-fence model of membrane diffusion. The plasma membrane is parceled up into compartments whereby both transmembrane proteins and lipids undergo short-term confined diffusion within a compartment and long-term hop diffusion between compartments. Transmembrane proteins are confined either by the actin-based membrane skeleton or anchored-proteins.

form of a stochastic, backward Fokker-Planck equation (FPE). We then analyzed the stochastic FPE using a moments method recently developed for diffusion processes in randomly switching environments. In addition to constructing the general mathematical framework, we also obtained illustrated the theory with two simple examples. First, we showed that the expected local time density, averaged with respect to realizations of the Brownian motion and the stochastic gate, is given by the product of the local time density without switching and the fraction of time that the gate is open. On the other hand, temporal correlations of the stochastic gate generate nontrivial corrections to second-order moments of the local time density. Second, we showed that in the large-time limit the presence of a gate does not affect the asymptotic form of the averaged occupation time probability density. This latter result would have been difficult to establish *a priori*.

Although we have focused on a relatively simple model of Brownian motion with a stochastic gate, the mathematical framework for analyzing stochastically-gated Brownian functionals can be applied to a much wider range of problems. One natural extension is to consider the occupation time of a Brownian particle in a bounded domain $\Sigma \subset \mathbb{R}^d$ for $d = 1, 2, 3$, with parts of the boundary $\partial\Sigma$ consisting of stochastically-gated channels. We have recently analyzed such systems in the case of first-passage time problems [22]. Another obvious extension is to consider a Brownian particle moving in some external potential (conservative force). This has been investigated extensively in 1D for both ordered and spatially disordered potentials, but in the absence of a switching gate [23]. It would also be interesting to explore other examples of Brownian functionals as highlighted in the review [12].

As we briefly indicated in the introduction, the general problem of stochastically-gated Brownian functionals is strongly motivated by an important class of problems in cellular biophysics. These involve the exchange of macromolecules such as proteins between subcellular compartments and the cytoplasm via small pores in the membrane of the compartments [13]. If each pore acts as a stochastic gate, then one has the problem of analyzing Brownian motion in a finite domain with a randomly switching boundary (or partial boundary). One important example is the transport of molecules between the cell nucleus and the surrounding cytoplasm. The nucleus of eukaryotes is surrounded by a protective nuclear envelope (NE) within which are embedded nuclear pore complexes (NPCs), see Fig. 4(a), which are the sole mediators of exchange between the nucleus and cytoplasm. In general small molecules of diameter ~ $5nm$ can diffuse through the NPCs unhindered, whereas larger molecules up to around $40nm$ in diameter are excluded unless they are bound to a family of soluble protein receptors known as karyopherins (kaps) [24]. Another classical example is the lateral diffusion of protein receptors within the plasma membrane. It has been observed that transmembrane proteins undergo confined diffusion within, and hopping between, membrane microdomains or corrals [25]; the corraling could be due to "fencing" by the actin cytoskeleton or confinement by anchored protein "pickets", see Fig. 4(b). One subtle feature of these and other biophysical examples, such as gap junctions [16, 17], is that in some cases there is a physical gate that switches between an open and a closed state, whereas in other cases the pore is always open but the Brownian particle switches between different conformational states; only a subset of these states allow the particle to pass through the pore (eg. the role of kaps in nuclear transport). Another characteristic feature of higher-dimensional systems is that the stochastically-gated pores are typically much smaller than the total size of the domain boundary, so that one often has to treat the transport process as a *narrow escape problem* [26].

In conclusion, cellular biophysics provides a wide range of potential applications of the theory developed in this paper, in particular, motivating the types of higher-dimensional geometries and switching mechanisms to consider in future work. The derivation of the stochastically-gated Feynman-Kac formula for occupation times means that one keep keep track of the amount of time a protein complex spends in a given domain, irrespective of the number of times it exits and reenters the domain, in contrast to first passage time problems. Similarly, in the case of local times, one can keep track of multiple interactions between a complex and some subcellular target.

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APPENDIX

In this appendix we provide the details of the evaluation of Eq. (5.24). First we rewrite Eq. (5.24) as

$$\widetilde{\mathcal{Q}}_0(x;s,z)=\left[A\mathrm{e}^{-\mu_+(z)x}\Theta(x)+B\mathrm{e}^{\mu_-(z)x}\Theta(-x)\right]+\mathcal{H}(x;s,z)+(\alpha+\beta)\mathcal{K}(x;s,z), \tag{A.1}$$

where

$$\mathcal{H}(x;s,z)=\int_{-\infty}^{\infty}K(x,x';s,z)dx'. \tag{A.2}$$

and

$$\mathcal{K}(x;s,z)=\int_{-\infty}^{\infty}K(x,x';s,z)\widetilde{\mathcal{Q}}(x';s,z)dx'. \tag{A.3}$$

with K and $\widetilde{\mathcal{Q}}$ given by Eqs. (5.21) and (5.18), respectively. For ease of notation, we drop any arguments in s and z, and write $\mu_\pm=|\lambda_\pm(z+\alpha+\beta)|$ and $\nu_\pm=|\lambda_\pm(z)|$ with $\mu_\pm>\nu_\pm$. If $x>0$ then

$$\mathcal{H}(x)=\mathcal{H}_+=\frac{1}{\mu_+}\left[\int_{-\infty}^{x}\mathrm{e}^{-\mu_+(x-x')}dx'+\int_{x}^{\infty}\mathrm{e}^{-\mu_+(x'-x)}dx'\right]=\frac{2}{\mu_+^2}=\frac{1}{s+z+\alpha+\beta},$$

whereas if $x<0$ then

$$\mathcal{H}(x)=\mathcal{H}_-=\frac{1}{\mu_-}\left[\int_{-\infty}^{x}\mathrm{e}^{-\mu_-(x-x')}dx'+\int_{x}^{\infty}\mathrm{e}^{-\mu_-(x'-x)}dx'\right]=\frac{2}{\mu_-^2}=\frac{1}{z+\alpha+\beta},$$

The calculation of $\mathcal{K}$ is more involved. First, we partition the integral with respect to x',

$$\mathcal{K}(x)=\int_{0}^{\infty}K(x,x')u_+(x')dx'+\int_{-\infty}^{0}K(x,x')u_-(x')dx'. \tag{A.4}$$

Next we partition one of the two integrals in order to take into account the piecewise nature of $K(x,x')$:

$$\mathcal{K}(x)=\int_{0}^{x}K(x,x')u_+(x')dx'+\int_{x}^{\infty}K(x,x')u_+(x')dx'+\int_{-\infty}^{0}K(x,x')u_-(x')dx'. \tag{A.5}$$

for $x>0$ and

$$\mathcal{K}(x)=\int_{0}^{\infty}K(x,x')u_+(x')dx'+\int_{-\infty}^{x}K(x,x')u_-(x')dx'+\int_{x}^{0}K(x,x')u_-(x')dx'. \tag{A.6}$$

for $x<0$. For convenience, denote the six integrals in Eqs. (A.5) and (A.6) by $I_n(x)$, $n=1,\ldots,6$. We evaluate each of the integrals in turn.

$$I_1(x)=\frac{1}{\mu_+}\int_0^x\mathrm{e}^{-\mu_+(x-x')}\left[\frac{1}{z+s}-\frac{\rho_1K_1}{\sqrt{2(z+s)}}\mathrm{e}^{-\nu_+x'}\right]dx'=\frac{1}{\mu_+}\left[\frac{1-\mathrm{e}^{-\mu_+x}}{\mu_+(z+s)}-\frac{\rho_1K_1}{\sqrt{2(z+s)}}\frac{\mathrm{e}^{-\nu_+x}-\mathrm{e}^{-\mu_+x}}{\mu_+-\nu_+}\right]$$

$$I_2(x)=\frac{1}{\mu_+}\int_x^\infty\mathrm{e}^{\mu_+(x-x')}\left[\frac{1}{z+s}-\frac{\rho_1K_1}{\sqrt{2(z+s)}}\mathrm{e}^{-\nu_+x'}\right]dx'=\frac{1}{\mu_+}\left[\frac{1}{\mu_+(z+s)}-\frac{\rho_1K_1}{\sqrt{2(z+s)}}\frac{\mathrm{e}^{-\nu_+x}}{\mu_++\nu_+}\right]$$

$$I_3(x) = \frac{1}{\mu_+}\int_{-\infty}^{0} \mathrm{e}^{-\mu_+(x-x')}\left[\frac{1}{z}+\frac{\rho_1 K_1}{\sqrt{2z}}\mathrm{e}^{\nu_- x'}\right]dx' = \frac{1}{\mu_+}\left[\frac{\mathrm{e}^{-\mu_+ x}}{\mu_+ z}+\frac{\rho_1 K_1}{\sqrt{2z}}\frac{\mathrm{e}^{-\mu_+ x}}{\mu_+ + \nu_-}\right]$$

$$I_4(x) = \frac{1}{\mu_-}\int_{0}^{\infty} \mathrm{e}^{\mu_-(x-x')}\left[\frac{1}{z+s}-\frac{\rho_1 K_1}{\sqrt{2(z+s)}}\mathrm{e}^{-\nu_+ x'}\right]dx' = \frac{1}{\mu_-}\left[\frac{\mathrm{e}^{\mu_- x}}{\mu_-(z+s)}-\frac{\rho_1 K_1}{\sqrt{2(z+s)}}\frac{\mathrm{e}^{\mu_- x}}{\mu_- + \nu_+}\right]$$

$$I_5(x) = \frac{1}{\mu_-}\int_{-\infty}^{x} \mathrm{e}^{-\mu_-(x-x')}\left[\frac{1}{z}+\frac{\rho_1 K_1}{\sqrt{2z}}\mathrm{e}^{\nu_- x'}\right]dx' = \frac{1}{\mu_-}\left[\frac{1}{\mu_- z}+\frac{\rho_1 K_1}{\sqrt{2z}}\frac{\mathrm{e}^{\nu_- x}}{\mu_- + \nu_-}\right]$$

$$I_6(x) = \frac{1}{\mu_-}\int_{x}^{0} \mathrm{e}^{\mu_-(x-x')}\left[\frac{1}{z}+\frac{\rho_1 K_1}{\sqrt{2z}}\mathrm{e}^{\nu_- x'}\right]dx' = \frac{1}{\mu_-}\left[\frac{1-\mathrm{e}^{\mu_- x}}{\mu_- z}+\frac{\rho_1 K_1}{\sqrt{2z}}\frac{\mathrm{e}^{\nu_- x}-\mathrm{e}^{\mu_- x}}{\mu_- - \nu_-}\right]$$

Given the solution

$$\mathcal{K}(x) = I_+(x) \equiv \sum_{n=1}^{3} I_n(x), \quad x>0, \qquad \mathcal{K}(x) = I_-(x) \equiv \sum_{n=4}^{6} I_n(x), \quad x<0, \tag{A.7}$$

we determine the coefficients A and B in Eq. (5.24) as functions of K_1 by imposing the boundary conditions (5.16):

$$-\mu_+ A + (\alpha+\beta)\left.\frac{\partial I_+(x)}{\partial x}\right|_{x=0^+} = 0, \quad \mu_- B + (\alpha+\beta)\left.\frac{\partial I_-(x)}{\partial x}\right|_{x=0^-} = 0. \tag{A.8}$$

In particular,

$$A = A(K_1) \equiv \frac{\alpha+\beta}{\mu_+^2}\left[\frac{1}{z+s}-\frac{1}{z}-\frac{\mu_+}{\mu_+ + \nu_+}\frac{\rho_1 K_1}{\sqrt{2(z+s)}}-\frac{\mu_+}{\mu_+ + \nu_-}\frac{\rho_1 K_1}{\sqrt{2z}}\right] \tag{A.9}$$

and

$$B = B(K_1) \equiv -\frac{\alpha+\beta}{\mu_-^2}\left[\frac{1}{z+s}-\frac{1}{z}-\frac{\mu_-}{\mu_- + \nu_+}\frac{\rho_1 K_1}{\sqrt{2(z+s)}}-\frac{\mu_-}{\mu_- + \nu_-}\frac{\rho_1 K_1}{\sqrt{2z}}\right] \tag{A.10}$$

Finally, the constant K_1 is obtained by imposing the boundary condition (5.15a):

$$(\alpha+\beta)u_+(0) - \alpha[A(K_1) + (\alpha+\beta)(I_+(0)+\mathcal{H}_+)] = (\alpha+\beta)u_-(0) - \alpha[B(K_1) + (\alpha+\beta)(I_-(0)+\mathcal{H}_-)]. \tag{A.11}$$

Using Eqs. (5.18), (A.7), and

$$A(K_1) + (\alpha+\beta)(I_+(0)+\mathcal{H}_+) = 2\frac{\alpha+\beta}{\mu_+^2}\left[1+\frac{1}{z+s}-\frac{\mu_+}{\mu_+ + \nu_+}\frac{\rho_1 K_1}{\sqrt{2(z+s)}}\right]$$

$$B(K_1) + (\alpha+\beta)(I_-(0)+\mathcal{H}_-) = 2\frac{\alpha+\beta}{\mu_-^2}\left[1+\frac{1}{z}+\frac{\mu_-}{\mu_- + \nu_-}\frac{\rho_1 K_1}{\sqrt{2z}}\right]$$

we find

$$\left[\frac{1}{z+s}-\frac{1}{z}-\frac{\rho_1 K_1}{\sqrt{2(z+s)}}-\frac{\rho_1 K_1}{\sqrt{2z}}\right] = \frac{2\alpha}{\mu_+^2}\left[1+\frac{1}{z+s}-\frac{\mu_+}{\mu_+ + \nu_+}\frac{\rho_1 K_1}{\sqrt{2(z+s)}}\right] - \frac{2\alpha}{\mu_-^2}\left[1+\frac{1}{z}+\frac{\mu_-}{\mu_- + \nu_-}\frac{\rho_1 K_1}{\sqrt{2z}}\right].$$

In the limit $\alpha \to 0$, we recover the results of the no-switching case, since the right-hand side vanishes. That is.

$$\rho_1 K_1 \to \Gamma_0 \equiv \sqrt{\frac{2}{z+s}} - \sqrt{\frac{2}{z}}.$$

Given the complexity of the above equation, we will focus on the large time behavior by taking $s, z \to 0$. To leading order we have $\mu_\pm^2 \to 2(\alpha+\beta)$ and $\mu_\pm/(\mu_\pm+\nu_\pm) \to 1$, so that we have the simplified equation

$$\left[\frac{1}{z+s} - \frac{1}{z} - \frac{\rho_1 K_1}{\sqrt{2(z+s)}} - \frac{\rho_1 K_1}{\sqrt{2z}}\right] \approx \rho_0 \left[\frac{1}{z+s} - \frac{\rho_1 K_1}{\sqrt{2(z+s)}} - \frac{1}{z} - \frac{\rho_1 K_1}{\sqrt{2z}}\right].$$

This implies that provided $\rho_0 < 1$ (the gate is open some of the time), then we obtain the same large time behavior as the case when the gate is always open. That is,

$$\widetilde{\mathcal{Q}}(0^\pm, s, z) \approx \frac{1}{\sqrt{z(s+z)}}, \quad s, z \to 0. \tag{A.12}$$

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