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Comments on “Locomotion of a microorganism in weakly viscoelastic liquids”

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We point out, and show the implications of resolving, an apparent conceptual difficulty in a recent article by De Corato *et al.* [Phys. Rev. E **92**, 053008 (2015)] on the locomotion of certain microorganisms in a second grade fluid. The difficulty arises due to the assumption that $\alpha_1 < 0$, where α_1 is the first normal stress modulus of the (non-Newtonian) liquid, was chosen for this study. In particular, this choice of sign for α_1 is inconsistent with thermodynamics, and as such casts **considerable** doubt on De Corato *et al.*’s assumption regarding the existence of a steady state solution of the equations of motion of the fluid.

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In [1], De Corato *et al.* presented the results of their theoretical study of the locomotion of ciliated microorganisms (“squirmers”) suspended in an incompressible fluid of second grade, where “second grade” refers to a class of non-Newtonian fluids that exhibit visco-retardation phenomena [2]. Employing a regular perturbation expansion, with the Deborah number, De , as the expansion parameter, and assuming low-Reynolds-number (time-independent) flow, De Corato *et al.* derived analytical expressions for the velocity field and the “swimming efficiency” of, in addition to the power dissipated by, the model microorganism.

While the literature of continuum mechanics is replete with works on the subject, it is important to realize that the theory of second grade fluids is *not* a settled one; for over 40 years, two schools of thought have been vying for the right to claim this title [3]. Thus, before presenting our critique of [1] it is instructive to briefly review the tenets of both schools.

The first of these, which we label as “S1” here, is based on the work of Dunn and Fosdick [4],¹ who put forth a theory of incompressible second grade fluids in 1974. Under their formulation, the total stress, $\mathbf{T}$, in such a fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \quad (1)$$

a relation which Dunn and Fosdick regard as *exact*, as denoted by the “=” sign in the above equation. Here, as in [4], $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors, which are denoted by $\mathbf{A}$ and $\mathbf{B}$, respectively, in [1]; α_1 and α_2 denote the two normal stress moduli of the fluid; μ is the (usual) shear viscosity coefficient; p is the pressure; and $\mathbf{I}$ represents the identity tensor.

Dunn and Fosdick then show [4, Thm. 6, Eq. (5-10)] that, in addition to the well known restriction $\mu \geq 0$,

which of course also applies to Newtonian fluids, a necessary condition for the compatibility of the second grade fluid theory with the Clausius–Duhem inequality (*i.e.*, the second law of thermodynamics) is that

$$\alpha_1 + \alpha_2 = 0. \quad (2)$$

Under S1, there is an additional requirement, *beyond* the Clausius–Duhem inequality, for the compatibility of second grade fluid theory with thermodynamics; it is that the fluid’s free energy attains its minimum when the fluid is in its equilibrium state. For constant α_1 , this takes the form [4, Eq. (5-15)]

$$\alpha_1 \geq 0, \quad (3)$$

where it should be noted that subsequent to assuming it to be constant, Dunn and Fosdick relabel α_1 as “ α ” [4, p. 216]. While the constraint in Eq. (2) applies to all motions (*both* steady and unsteady), the constraint in Eq. (3) ensures the existence of stable solutions that are bounded in time. Note also that the case of strict equality implies, by Eq. (2), that $\alpha_2 = 0$ as well; *i.e.*, under this special case, Eq. (1) reduces to the usual constitutive relation for an incompressible Newtonian fluid.

According to the second school of thought [5, §9.3.2.1], which we label as “S2,” Eq. (1) is to be regarded as an approximation to the full constitutive relation for an unspecified viscoelastic fluid.² As such, it is assumed (without proof) that the constraints given by Eqs. (2) and (3) above need not necessarily hold. In support of this, adherents to the S2 school point out that early **experimental** studies are consistent with $\alpha_1 < 0$; see, *e.g.*, [6, 7]; but also see [3, 8], wherein cases against such interpretations are made. Moreover, the nature of the aforementioned experiments has prompted the S2 community to contend

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¹ See also [9, §6.5], and the literature cited therein.

² Note that this interpretation necessitates that the “=” sign in Eq. (1) be replaced by “ $\approx$.”

that use of the second-grade model with $\alpha_1 < 0$ is generally justifiable when describing steady or “creeping” flow situations [3].

An example of this modeling approach, applied to a problem that is closely related to the one at hand, can be found in the work of Sturges [10]. In 1981, Sturges carried out a perturbation analysis of the impact of a second grade fluid with $\alpha_1 < 0$ [10, p. 359] on the swimming velocity of a waving sheet, which we note is an early (simpler) model of a swimmer due to G. I. Taylor [11]. Sturges was, among other things, able to recover Taylor’s (Newtonian fluid-based) solution from the low-frequency approximation of [10, Eq. (28)], thereby showing the leading order term of his solution to be independent of α_1 (and α_2) in the low-frequency regime (see [10, Eq. (29)]).

A careful reading of [1] makes clear that, like Sturges [10], De Corato *et al.* have grounded their study exclusively in S2; indeed, they take $\alpha_1 < 0$ ³ and consider parameter values corresponding to $\alpha_1 + \alpha_2 \neq 0$. Given, then, the unsettled state of the theory of second grade fluids, it is our position that De Corato *et al.*’s analysis must be regarded as *inconclusive* since it does not (also) take into account the implications of **the restrictions imposed under S1**.

In what follows, we shall first note the impact of imposing the thermodynamic constraints derived by the S1 school on the findings presented in [1] and then offer what is, in our view, an obvious, if not tedious, action to remedy the aforementioned shortcoming of [1].

To this end, we observe that the most obvious effects of imposing the constraints cited in Eqs. (2) and (3) on the expressions given in [1] are, respectively,

- (i) $b(= \alpha_2/\alpha_1)$ assumes the *fixed* value $b = -1$,
- (ii) the assumption that a steady state solution to the squirmer problem exists is *invalid*.

Thus, by (ii) there is no basis for positing a regular perturbation expansion, as done in [1], for the steady flow of a second grade fluid with $\alpha_1 < 0$, even if $\alpha_1 + \alpha_2 = 0$ holds. Indeed, according to Dunn and Fosdick such a second grade fluid flow always exhibits unbounded growth in time when starting from rest, which precludes the existence of a steady state; see [4, §7]. And although it is possible to achieve $b < 0$ but $b \neq -1$, as assumed in [1], for $\alpha_1 > 0$, doing so requires that the constraint given in Eq. (2) be violated; recall, however, that values other than $b = -1$ correspond to second grade fluids that are inconsistent with the S1 paradigm.

On the other hand, the primary implications of (i) above are the following:

- [1, Eq. (30)] becomes $V_{\text{SOF}} = 0$,
- [1, Eq. (31)] becomes $V = 1 + \mathcal{O}(\text{De}^2)$,
- [1, Eq. (40)] becomes $P = P_N = 6\pi(2 + \beta^2)$,
- [1, Eq. (43)] becomes $\eta = \eta_N = 1/(2 + \beta^2)$,
- [1, Eq. (A8)] becomes $\Psi_{\text{SOF}} = 0$,
- [1, Eq. (A15)] becomes $v_{r,\text{SOF}} = 0$,
- [1, Eq. (A16)] becomes $v_{\theta,\text{SOF}} = 0$.

Here, we should point out that De Corato *et al.* do note that the choice of parameters satisfying $b = -1$ “is interesting” because, then, the “flow field is identically zero” [1, p. 8]. Actually, the above (bulleted) list makes clear that at $\mathcal{O}(\text{De})$, the viscoelastic response of a second grade fluid **that satisfies the constraints of S1** has *no effect* upon the kinematics and dynamics of a swimmer of the type considered in [1]. Additionally, this cancellation serves to eliminate the effect of β (*i.e.*, the specific type of squirmer—“pusher,” “puller,” or “neutral” [1, §II]) from many (most?) of the results derived in [1], thus precluding us from drawing any conclusions regarding the so affected expressions based on the value of β .

Some further implications of (i) above are that the limit as $b \rightarrow -1$ *does* exist in [1, Eqs. (A12d), (A12h)], as can be shown using [1, Eq. (A14)], and that the expression for p_{SOF} [1, Eq. (A18)] is also impacted by the fact that $b = -1$ is the only possible value of b under S1.

Finally, it is appropriate to mention that De Corato *et al.* note that their perturbative results, which we recall are based on a second grade fluid with $b \neq -1$ under steady state flow, are “at variance [sic] with recent numerical simulations” by others [1, p. 6]; however, De Corato *et al.* provide insufficient details to fully judge such a statement vis-à-vis the present critique.

Clearly, it would be of interest to researchers from both the microbiology and the fluid mechanics communities to have De Corato *et al.*’s second grade fluid version of the Lighthill–Blake squirmer problem generalized to include the constraints of the S1 paradigm. This, of course, means computing higher-orders terms in De ; specifically, up to and including (at least) the first term that does *not* vanish on setting $b = -1$. Beyond this, there is the option of considering a different non-Newtonian model⁴—one that might both prove more tractable from the analytical standpoint and better elucidate the effects of fluid viscoelasticity on a squirmer.

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I.C.C. became aware of [10] from the review on locomotion through complex fluids by Elfring and Lauga [13].

³ Neither this assumption, nor that $\alpha_1 \geq 0$ is a possibility, are explicitly stated by De Corato *et al.*; however, $\alpha_1 < 0$ is easily inferred by observing that both $-\alpha_1/\mu$, the flow’s characteristic time scale, and De must be positive.

⁴ Given that the regime of interest is $\text{De} \ll 1$, an obvious choice is the upper-convected Maxwell model; see, *e.g.*, [9, p. 67] and [12].

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