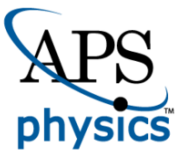



# Interdependent lattice networks in high dimensions

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Phys. Rev. E **94**, 052306 — Published 8 November 2016

DOI: 10.1103/PhysRevE.94.052306

# Interdependent Lattice Networks in High Dimensions

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(Dated: October 10, 2016)

We study the mutual percolation of two interdependent lattice networks ranging from two to seven dimensions, denoted as $D$. We impose that the length (measured as chemical distance) of interdependency links connecting nodes in the two lattices be less than or equal to a certain value, $r$. For each value of $D$ and $r$, we find the mutual percolation threshold, $p_c[D, r]$, below which the system completely collapses through a cascade of failures following an initial destruction of a fraction $(1 - p)$ of the nodes in one of the lattices. We find that for each dimension, $D < 6$, there is a value of $r = r_I > 1$ such that for $r \geq r_I$ the cascading failures occur as a discontinuous first order transition, while for $r < r_I$ the system undergoes a continuous second order transition, as in the classical percolation theory. Remarkably, for $D = 6$, $r_I = 1$ which is the same as in random regular (RR) graphs with the same degree (coordination number) of nodes. We also find that in all dimensions, the interdependent lattices reach maximal vulnerability (maximal $p_c[D, r]$) at a distance $r = r_{max} > r_I$, and for $r > r_{max}$ the vulnerability starts to decrease as $r \to \infty$. However the decrease becomes less significant as $D$ increases, and $p_c[D, r_{max}] - p_c[D, \infty]$ decreases exponentially with $D$. We also investigate the dependence of $p_c[D, r]$ on the system size as well as how the nature of the transition changes as the number of lattice sites, $N \to \infty$.

## I. INTRODUCTION

The behavior of many complex systems in the real world can be better understood and explained through network theory[1–7]. Highway traffic, power outages, the relationship between businesses and many other phenomena can be modeled as networks. Additionally, many of the real networks, such as the communications network and the power grid are interdependent on each other[8–19]. Their behavior can be discussed in terms of mutual percolation: in order to function properly, a node in each network must be connected to the giant component of its own network and must be supported by an interdependent node in the other network. An initial failure of a fraction $(1 - p)$ of nodes in one network will lead to failures in the other network. This will either cause both networks to eventually stabilize, preserving their giant components, or to completely collapse. The communication network and the power grid network are examples of such interdependent networks, embedded in space. A blackout in a city may cause a server operating the power grid to go down, and this may cause further disruption of power stations. Another example is the network of seaports and the network of national highways, which are interdependent on each other. Hurricane Sandy demonstrated that if a seaport gets damaged, the city to which it supplies fuel will become isolated from the highway network. Similarly, a city without fuel for trucks cannot supply a seaport properly, and the seaport will not be able to function well [19]. So, most real world interdependent networks contain nodes which are embedded in a two dimensional surface or in a three-dimensional space. Moreover, it is reasonable to assume that the interdependent nodes in the two networks are not located far away from each other. [17, 18].

Li *et. al* [17] introduced the concept of a dependence on distance, according to which a node in network A can be interdependent with a node in network B, only if the distance between these two nodes does not exceed a value, $r$. The definition of distance used by those authors differs slightly from the chemical distance[20] (sometimes referred as Manhattan metric or Taxicab geometry) used in the present work, and the effect of this difference will be discussed below, in Section V. From here on, the word distance or length in this work will always refer to this chemical distance.

In [17] it was shown that the constraint on the length of the interdependency links significantly affects the mutual percolation of the two networks and alters the properties of the system's collapse. It was found that for $r = 0$, the collapse transition in two interdependent two-dimensional lattice networks is identical to the classical percolation problem in a two dimensional lattice[21, 22]. As $r$ increases, the critical percolation threshold, $p_c$, increases, but the transition remains a second order transition, in which the size of the surviving mutual giant component of the system gradually approaches zero as the fraction $p$ approaches $p_c$. Interestingly, when $r$ reaches a critical value, $r_I \approx 8$, the transition suddenly becomes a first order transition, in which either the majority of the nodes survive, or the networks are completely destroyed. As $r$ increases further, $p_c$ starts to decrease until, for $r \to \infty$, it reaches the value characteristic of the mutual percolation on the lattices with random interdependency links. In the interval $r_I \leq r < \infty$, the cascading failures lead to a small hole which starts to grow circularly until all the nodes of both lattices are wiped out. The explanation of this phenomenon [17] was based on the idea that cascading failures in this regime propagate by the destruction of nodes close to the perimeter of the hole that is larger than $r$. This will happen because such nodes have lost their supporting nodes in the other network, previously located in the hole. For small $r$, $p_c$ is close enough to the critical threshold of classical perco-

lation, $p_c^p$, at which the size of the holes diverge, so that holes larger than $r$ appear at the first stage of the cascade. However as $r$ grows, $p_c$ also grows and eventually the typical size of the holes, dictated by the correlation length of the classical percolation becomes equal to $r$. When this happens, the system becomes metastable: a random formation of a hole of a sufficient size by a local density fluctuation causes the circular growth of such a hole, destroying the entire system. As $r$ increases in the vicinity of $r_I$, a smaller value of $p$ is needed to produce a hole of size $r$. Therefore, $p_c$ starts to decrease for $r > r_I$.

This behavior should be contrasted with that of a random-regular (RR) graph of degree $k$. An RR graph can be regarded as an infinitely dimensional lattice, where the surface is not a well defined concept, because its dimensionality is equal to the dimensionality of the entire graph. Thus, for two interdependent identical RR networks, using the shortest path between a pair of nodes as the distance for the interdependency links, $p_c$ should monotonically increase with $r$, and the transition should switch from second order to first order as $r$ increases. Indeed, this has been shown numerically (and analytically in some of the cases) by Kornbluth *et al.*[19], who found for $k > 8$, $r_I$ becomes 1. For the case ($k = 8$, $r = 1$), a first order phase transition is closely followed by a second order phase transition at a smaller $p$, and for $k \leq 7$, $r_I$ becomes greater or equal than 2. We expect that in our case, when the dimensionality of the lattice increases, the behavior observed for the $D$-dimensional interdependent lattices studied here should converge to the behavior of the interdependent RR graphs. Additionally, there is the possible existence of an upper critical dimension, [21, 22] above which the fractal dimension of the percolation cluster and the fractal dimension of its surface (accessible perimeter) both become equal to 4 and, hence, the propagation of an interface becomes ill-defined. For classical percolation the upper critical dimension is known to be six[24]. Thus one can hypothesize that for the mutual percolation of two 6D lattice networks, the behavior will be similar to that of infinite dimensional networks, for which the interface of a percolation cluster coincides with the cluster itself. The goal of this paper is to test all of the hypotheses discussed above.

## II. THE MODEL

We study the mutual percolation[8] of two interdependent hypercubic lattice networks in several dimensions. We create two identical networks A and B, whose nodes are labeled $1, 2, ...N = L^D$ where $D$ is the dimension of the lattice and $L$ is the number of nodes along each of its dimensions. Each node is connected with edges to exactly $k = 2D$ nearest-neighbor nodes. We then introduce one-to-one bidirectional interdependency links, such that the shortest path between any two interdependent nodes is not greater than $r$. In order to decrease computation time and define how the network is built, we introduce two isomorphisms between networks A and B. These isomorphisms, the topological isomorphism, $\mathcal{T}$, and the dependency isomorphism, $\mathcal{D}$, are those which were defined in Kornbluth, *et. al* [19]. The topological isomorphism is defined for each node $A_i$ as $\mathcal{T}(A_i) = B_i$ and verifies that if $A_i$ and $A_j$ are first neighbors in network A, then $\mathcal{T}(A_i)$ and $\mathcal{T}(A_j)$ are first neighbors in network B and vice versa. For the case of lattices, the topological isomorphism is automatically established due to the identical lattice structure. The dependency isomorphism, establishes the interdependency links, and we create it following the restriction that $B_k = \mathcal{D}(A_i)$ only if the shortest path connecting $A_i$ and $A_k = \mathcal{T}(B_k)$ is of a length $r_{ik} \leq r$. Since our goal is to compare the behavior of $D$-dimensional hypercubic lattices to the RR graphs with $k = 2D$, for which the concept of coordinates is not applicable, we choose our definition of distance as one that is identical to that used for RR graphs (i.e. the chemical distance which is the smallest number of edges connecting the two sites). In the context of hypercubes this metric is the Manhattan metric, which slightly differs from both the Euclidian metric and the cubic metric used in Li *et al.*[17], $r = \max_{i=1}^{D} |\Delta x_i|$, where $\Delta x_i$ are the coordinate differences of the two interdependent nodes.

In order to establish the dependency isomorphism, while still satisfying the shortest path restriction, Li *et al.*[17] created a random permutation of the indices of all the nodes that fulfilled the distance restriction. However, in our case we followed the procedure developed by Kornbluth *et al.*[19], namely we set $\mathcal{D}(A_i) = B_i$ only if there are no other possibilities for $\mathcal{D}(A_i)$. Additionally, we require that if $\mathcal{D}(A_i) = B_k$, then $\mathcal{D}(B_i) = A_k$. This further restriction decreases the time required for computation, without affecting the results in any essential way.

Initially, a fraction $(1 - p)$ of randomly selected nodes in the first network are destroyed. Any node in the second network whose interdependent node in the first network has been destroyed, or who lost its connectivity to the largest percolation cluster (the largest group of nodes, connected to each other) will also be destroyed. We return to the first network and further destroy all the nodes who lost their support in the previous process, or who got disconnected from the largest percolation cluster, as a consequence of the previous stage. This process of destruction continues to alternate between the networks and is referred to as a cascade of failures. The process ends when both networks no longer contain nodes that will fail. The largest mutual cluster of nodes which spans the entire network is called the mutual giant component. In addition to the largest mutual cluster, we also find mutual clusters of smaller sizes as defined in Ref.[8]. In all cases, if the fraction of nodes $p$ surviving the initial attack falls below a certain critical threshold, $p$-critical or $p_c$, the network completely collapses and the largest mutual cluster becomes a negligible fraction of the initial size of the system. We study how $p_c$ changes as a func-

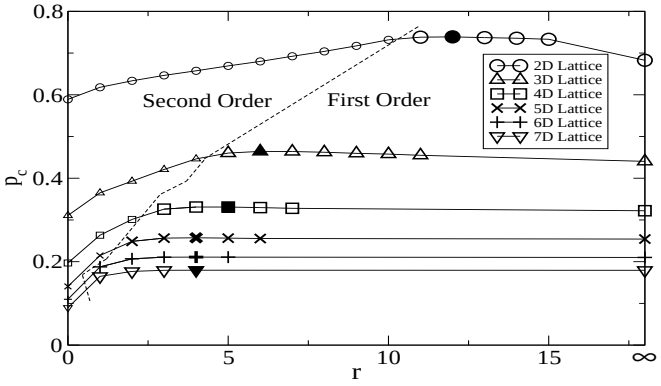



FIG. 1: Plot of $p_c[D,r]$ vs. $r$ for lattices of dimensions ranging from 2 to 7. The smaller symbols correspond to second order transitions, the larger symbols correspond to first order transitions and the bold symbols denote the maximum value of $p_c[D,r]$ for a given dimension. The last value in each plot is the value of $p_c[D,\infty]$

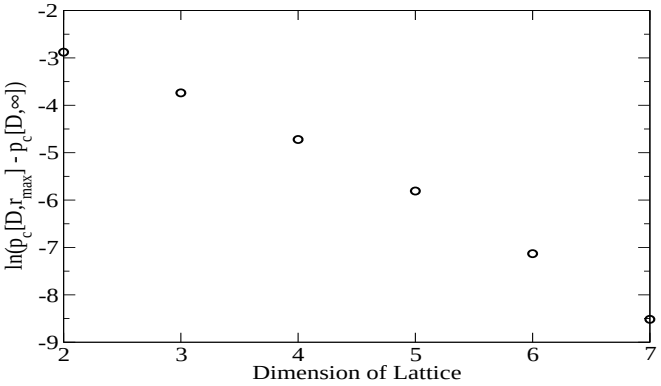



FIG. 2: Plot of the decrease of the difference between $p_c[D,r_{max}]$ and $p_c[D,\infty]$ as the dimension of the lattice increases.

tion of the maximum length $r$ of interdependent links, as well as the dimensionality of the networks $D$. We denote the $p$-critical value for a network of dimension, $D$, and distance, $r$, as $p_c[D,r]$. In all cases we run our simulations for lattices of at least $N = 10^6$ nodes. To estimate the finite size effect, we perform additional simulations for several system sizes up to $N = 6.4 \times 10^7$. In Section IV B, we discuss the finite size effects. In particular we explore the cases in which changing the system size leads to the change in the order of the transition. For each combination of $[D,r]$ and several values of $p$ in the vicinity of $p_c[D,r]$ we conduct $M = 1000$ independent simulations and compute the size of the largest cluster and that of the second largest cluster. We then construct a histogram of the sizes of the largest cluster and compute the average size of the second largest cluster.

## III. SIMULATION RESULTS

### A. Main results

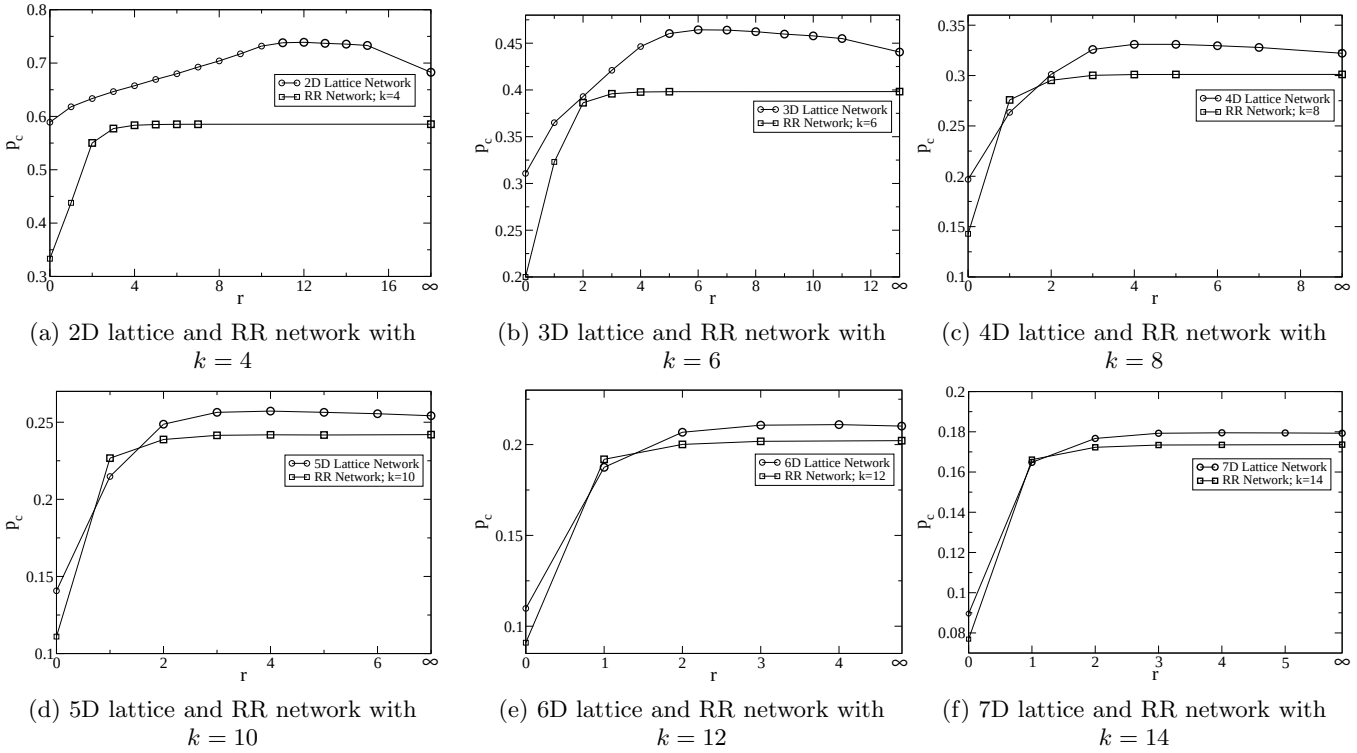
We run simulations to determine the value of $p_c$ for lattices of two through seven dimensions. For these lattices, we find that the value of $p_c$ increases with $r$, reaches a maximum at $r = r_{max}$, and then slowly converges to $p_c[D,\infty]$, which is the value of $p_c$ for random interdependency links (Figure 1). For low values of $r < r_I$, the transition is second order, while for higher values of $r \geq r_I$ the transition is first order. Additionally, we find that $r_{max} > r_I$ for all $D$. For example, in a two-dimensional lattice for $0 \leq r \leq 10$, the transition is second order and, for $r \geq r_I = 11$, the transition is first order, while the maximum value of $p_c$ occurs when $r = r_{max} = 12$.

The trend of the maximum value of $p_c$ occurring after the change from second to first order transitions is present through all dimensions, including the seven-dimensional lattice. However, the difference, $p_c[D,r_{max}] - p_c[D,\infty]$, decreases exponentially with $D$ (Fig. 2). It is also interesting that the difference between $r_{max}$ and $r_I$ increases with $D$. Thus, the case of RR graphs in which the maximum of $p_c$ is reached only for $r = \infty$ [19], is the limiting case of the behavior of finite lattices when $D \to \infty$.

Additionally, as $D$ increases, the difference between the individual values of $p_c$ for the lattice and RR networks with $k = 2D$ decreases (Fig. 3). Figure 4 shows a comparison between the $p_c[D,\infty]$ for our simulation results and the analytical results for $p_c$ for a RR network with $k = 2D$ and random interdependent links. The $p_c$ of the lattice network slowly approaches that of the RR network as the number of neighboring nodes (degree), $k$, increases.

### B. Types of collapses

As discussed in connection with Figure 1, the size of the surviving fraction of the networks at the end of the cascade of failures experiences a transition as a function of the size of the initial attack. When this attack is small the network survives almost intact, but if this attack is large enough the final largest mutual cluster will become a negligible fraction of the initial size of the system. For each dimension, the nature of this transition can be first order or second order, depending on the value of the distance $r$ of the interdependency links. We can clearly distinguish the nature of the transition by examining the cumulative distribution of the fraction of nodes, $\mu$, in the largest mutual cluster of the networks for different realizations of the initial attack at criticality, $p = p_c$. As seen in Fig. 5, for the case of a first order phase transition the values of $\mu$ fall into two very well separated ranges. In the case illustrated there, for $D = 3$ and $r = 5$, the




(a) 2D lattice and RR network with $k = 4$

(b) 3D lattice and RR network with $k = 6$

(c) 4D lattice and RR network with $k = 8$

(d) 5D lattice and RR network with $k = 10$

(e) 6D lattice and RR network with $k = 12$

(f) 7D lattice and RR network with $k = 14$

FIG. 3: Comparison of $p_c[D, r]$ vs. $r$ for lattices of varying dimensions and the corresponding RR network. The smaller symbols denote second order transitions and the larger symbols denote first order transitions.

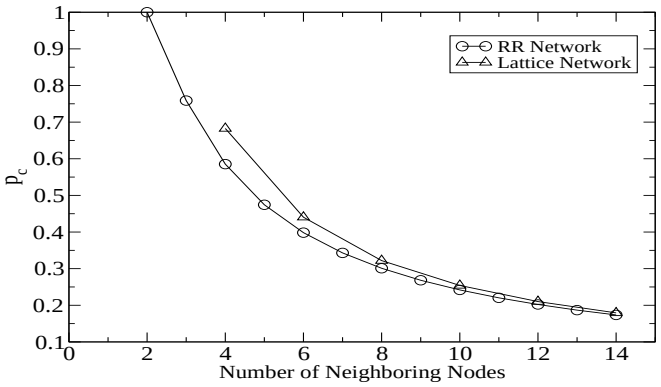



FIG. 4: Comparison between our simulation results for $p_c$ of lattice networks with random interdependent links and the analytical results for $p_c$ for RR networks with $k = 2D$ and random interdependent links

values of $\mu$ are above the value $\alpha \approx 0.35$, or below the value $\beta \approx 0.02$. There are no simulations which result in $\alpha > \mu > \beta$. Accordingly, we define a transition as first order, if there is this clear gap in the plot of the cumulative distribution of the largest cluster. For a second order transition, as seen in the same figure for the case of $[D = 3, r = 4]$, the graph of the cumulative distribution of the mutual largest cluster of the networks looks significantly different from that of a first order transition. The cumulative distribution of $\mu$ decreases continuously, and the size of the largest mutual cluster can take many values with no discontinuous jump in the middle of the distribution as in the previous case.

### C. Determination of $p_c$

In finite networks, there is always an uncertainty in the size of the largest mutual cluster, which makes the precise determination of $p_c$ a formidable problem. Accordingly, we use two different methods for determining the approximate values of $p_c$ for the cases of the first and the second order transitions, which have been developed in Refs. [10, 19, 22]

In case of the first order transition, for each value of $p$ we study, we first define $q(p)$ as a fraction of realizations that result in $\mu \geq \alpha$. Accordingly, the fraction of realizations that result with $\mu \leq \beta$ is $1 - q(p)$. Following Ref. [19] we define $p_c$ as the value of $p$ such that $q(p) = 1/2$. However, for finite size networks, the size of the largest mutual largest cluster is subject to statistical fluctuations. Therefore, the value of $q(p)$ is defined with a certain statistical error. Using the Law of Large Numbers, we find the upper bound for its standard deviation, $\sigma_q \leq \frac{1}{2\sqrt{M}}$, where $M$ is the number of independent realization of the system. In first order transitions, a small change in $p$ will result in a dramatic change in the largest

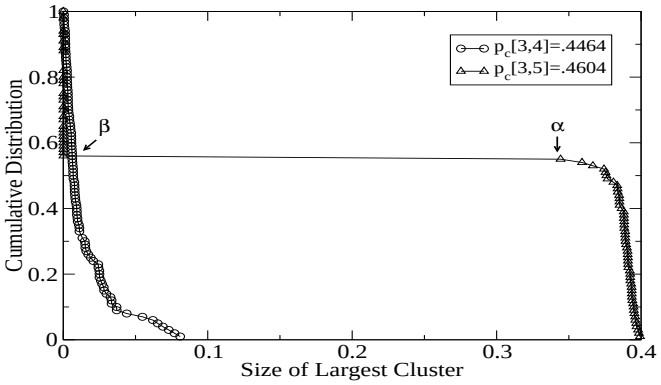



FIG. 5: Plot of the cumulative distribution of the largest cluster for the last second order transition of a 3D network with $N = 10^6$, at $p_c[3, 4] = .4464$ and the first order transition at $p_c[3, 5] = .4604$. There is a clear large gap in the plot of the size of the largest cluster for the first order transition, which is absent in the case of the second order phase transition

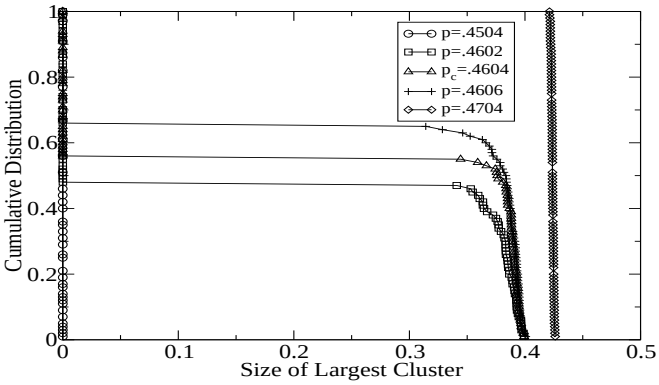
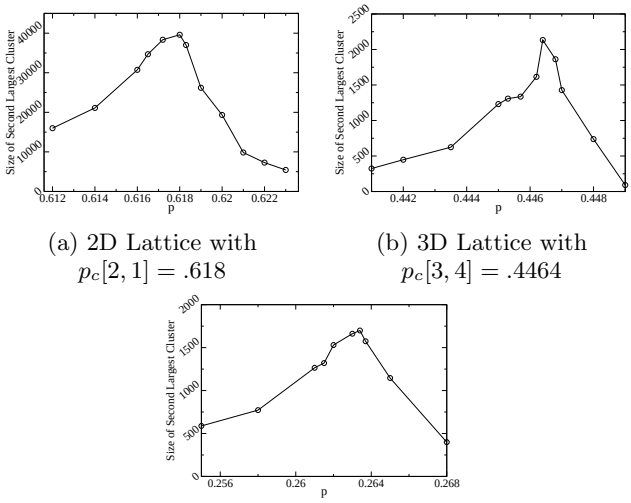



FIG. 6: Plot of largest cluster size at and near $p_c[3, 5] = .4604$. It can be seen that a very small change in $p$ leads to drastically different largest cluster size distributions.




(a) 2D Lattice with $p_c[2, 1] = .618$

(b) 3D Lattice with $p_c[3, 4] = .4464$

(c) 4D Lattice with $p_c[4, 1] = .2634$

FIG. 7: Plot of the average second largest cluster size as a function of $p$ for 2, 3 and 4 dimensional lattices

mutual cluster distribution (Fig. 6). A slight increase in $p$ will lead to $q(p) = 1$ and a slight decrease will lead to $q(p) = 0$. Obviously $p_c$ must belong to this interval. Therefore, in order to determine $p_c$, we produce several simulations for values of $p$ which belong to the interval in which $0 < q < 1$, such that we get at least one point $p_1$ with $q_1 = q(p_1)$ belongs to the interval $[0.1, 0.5]$ and at least one point $p_2$ with $q_2 = q(p_2)$ belongs to the interval $[0.5, 0.9]$. We next find $p_c$ by linear interpolation

$$p_c = p_1 + (\frac{1}{2} - q_1)\frac{p_2 - p_1}{q_2 - q_1}. \quad (1)$$

Based on $\sigma_q$, we can estimate the 95% confidence error bar in $p_c$ as

$$\Delta p = \frac{p_2 - p_1}{(q_2 - q_1 - 2\sigma_q)\sqrt{N}}. \quad (2)$$

Usually we obtain more than one pair of points $p_1, p_2$, which satisfy the above conditions. In this case we can construct a linear least square fit of $q(p)$ and solve $q(p) = 1/2$. The values of $p_c$ obtained by this method are always within $\Delta p$ from $p_c$ found by linear interpolation. This observation suggests that the actual error bar is smaller than the estimate given by Eq.(2), which in all cases studied does not exceed 0.0004. This error bar is sufficient to precisely determine $r_{\max}$ for each studied dimension $D$ and system size $N$.

For the case of the second order transition the method described above cannot be used because $\alpha$ and $\beta$ cannot be clearly defined. Instead, we use the average second largest mutual cluster to determine the value of $p_c$ [10, 22]. As discussed in Kornbluth *et al.* [19], when $p > p_c$ the largest mutual cluster spans the network, preventing other large clusters from forming. When $p < p_c$ the network is very fragmented and large clusters are, therefore, not able to form. However, when $p \approx p_c$, the average size of the second largest cluster develops a sharp peak (Fig. 7). We verified this in cases of second order transitions of all lattices, regardless of dimension. Thus, for the case of second order transitions, we determine $p_c$ by finding the value of $p$ for which the average size of the second largest cluster reaches its maximum. The error bar in $p_c$ in this case is defined as the difference between the two values of $p$ one above and another below the value providing the maximum. We make sure, that the step in $p$ is large enough, so that a single maximum is observed. In this case the error bar in $p_c$ also never exceeds 0.0004. Thus in all cases studied, the error bar of $p_c$ is much smaller than the symbol size in Fig. 1.

## IV. DISCUSSION OF THE RESULTS

### A. Propagation of the interface of the hole

It was noted in Ref. [17] that these interesting phenomena (the existence of $r_I$ and $r_{\max}$) are related to the presence of the surface of a hole in the mutual cluster which is valid only in a system of a finite dimension. Indeed, one can specifically study the problem of a propagating $D-1$ dimensional interface on a D-dimensional lattice based on the mutual percolation rules discussed above, with the maximal interdependence distance $r$ and the initial density of surviving sites $p$. The process of this propagation is similar to the various models of fluid propagation in disordered media [23] which are characterized by the depinning transition: i.e. there is a critical threshold $p = p_c^f$ above which the interface is completely blocked by the obstacles, but below which the velocity of the interface propagation is finite, and gradually decreases to zero when approaching the critical threshold: $v \sim (p_c^f - p)^\theta$. The depinning transition is a second order transition characterized by several critical exponents, one of which is $\theta > 0$. The fluid propagation near $p = p_c^f$ is characterized by avalanches: one remaining active site in a completely blocked interface can create an avalanche of propagation. The size distribution of the avalanches obeys a power law similar to the distribution of the cluster sizes in percolation theory. In the mutual percolation model, $p_c^f(r)$ of the interface propagation increases with $r$ from the value $p_c^p$ of classical percolation theory at $r = 0$, to the value 1 at a certain $r = r_f$. If $r > r_f$ the interface propagates freely through the system even if the lattice is completely intact.

When $p_c^f$ is close to the percolation threshold $p_c^p$, the correlation length of percolation, $\xi(p_c^f)$, is greater than $r$. This means that there are always holes of size greater than $r$, and the interface is always spontaneously created. The interface will start to propagate from many different places. However, if $p$ is close to $p_c^f$ from above, the propagation will stop leaving a sponge-like mutual giant component with holes of all possible sizes. The destruction of a single node may disconnect a huge portion of the mutual giant component and may dramatically reduce its size. Hence there is a broad distribution of the sizes of the mutual giant component, which is one of the characteristics of a second order phase transition. In contrast, when $p_c^f$ is far above $p_c$, for large values of $r$, the size of the holes are smaller than $r$ and the interface cannot be created spontaneously. Therefore, one must reduce $p$ in order for a hole of size $r$ to be created. As we consider larger values of $r$, the value of $p$ required to create such a hole decreases. Once the hole is created, its interface starts to propagate freely because $p < p_c^f$, and it will wipe out the entire lattice. In this scenario, for small $r$ the critical threshold of the mutual percolation is $p_c = p_c^f(r)$, which increases with $r$, until $\xi(p_c^f) = r$. In that interval, the transition is second order. But for $r > \xi(p_c^f)$, $p_c$ starts to decrease, following the equation $\xi(p_c) = r$, and the transition becomes first order. In fact, the values of $r = r_I$ at which the transition becomes first order for the first time and $r = r_{max}$ at which $p_c$ starts to decrease may not exactly coincide. There is always a probability that a hole of size $r > \xi(p)$ may appear in a large enough system. Thus for $r > r_I$, one can expect the average $p_c$ to be in between the increasing function $p_c^f(r)$ and the decreasing function $p(r)$ defined by the equation $\xi[p(r)] = r$, and hence may still increase until it reaches its maximum at $r = r_{max}$.

### B. Dependence of the transition order and $r_{\max}$ on the system size

One would expect that the values of $r_I$ and $r_{max}$ should depend on the system size. Moreover, as shown in the previous section, for $r \geq r_I$ the system becomes metastable with respect to formation of a hole, in a similar way a superheated liquid is metastable to spontaneous nucleation of a gas bubble. The larger the system, the greater the probability that the critical hole will spontaneously form. We notice that if the interdependence distance $r$ is chosen in the vicinity of $r_I$, the system exhibits the strongest finite size effects in terms of $p_c[D, r]$. However, based on our studies of networks of different sizes, we find that, in all dimensions except $D = 2$ and $D = 6$, the values of $r_{\max}$ and $r_I$ do not depend on the system size if $N \geq 10^6$. The case $D = 2$, $r = 10$ is presented in Figs. 8 and 9 and the case $D = 6$ and $r = 1$ is presented in Fig. 11. These two examples are very different from each other and the increasing importance of the finite size effects which mask the nature of the transition are caused by different mechanisms. In the low dimensional system, the dependence is caused by the increasing probability of formation of holes. However, in the high dimensional system it is caused by the statistical uncertainty in the value of the largest mutual percolation cluster, which becomes a small fraction of the entire network.

The formation of the large holes as the mechanism for network collapse is especially important in low dimensional systems in which the dimensionality of the interior of the hole and its perimeter are significantly different. As mentioned in the previous section, the critical threshold of the moving free interface, $p_c^f(r)$, linearly increases with $r$. However, the probability of the spontaneous formation of a hole of size $r$ at $p_c^f(r)$, decreases with $p_c^f$. This is because the probability, per one lattice site, of the formation of a hole of size $r$ decreases exponentially with $r$, $p_h(r) \sim \exp(-r/\xi)$ where $\xi \sim (p_c^f - p_c^p)^{-\nu}$ is the percolation correlation length, $p_c^p$ is the percolation critical threshold and $\nu$ is a critical exponent. The total probability of the formation of a hole in a lattice of size $N$ is $Np_h(r)$. We can expect the formation of the hole in a given instance of the lattice if $Np_h(r) = 1$. Thus the fraction, $p_h$, of survived nodes for which the hole of size $r$

will be formed can be found from the following equations:

$$\begin{aligned} L^D \exp[-ar(p_h - p_c^p)^\nu] &= 1 \\ ar(p_h - p_c^p)^\nu &= D\ln(L) \\ p_h(r) &= (D\ln(L)/ar)^{1/\nu} + p_c^p \end{aligned} \tag{3}$$

where $a$ is a proportionality coefficient. If $p_h < p_c^f(r)$, then the system is metastable and the hole of size $r$ certainly eliminates the entire system. If $p_h > p_c^f(r)$ then the interface of the hole will grow unpredictably, as in the second order phase transition. Thus if $L$ is large enough we can expect the transition to be second order (Figs. 8 and 9) and follow the increasing function, $p_c = p_c^f(r)$ for larger and larger $r$ (Fig. 10). For a fixed $L$, as soon as $p_h(r) < p_c^f(r)$, the type of the transition will change to first order. Moreover, $p_c$ will switch to follow the graph of $p_h(r)$. Thus, at a fixed $r$, $p_c$ will increase logarithmically with the system size until it reaches $p_c^f(r)$, after which the dependence on $L$ stops (Fig. 10). We also observe the logarithmic dependence of $p_c$ in the vicinity of $r_{\max}$ on the system size for all other dimensions, but the strength of the dependence becomes weaker as $D$ increases and for $D \geq 3$ it is not sufficient to change the value of $r_{\max}$ and $r_I$.

### C. The effect of statistical fluctuations

The other interesting case is the 6-D lattice with $r = 1$. For $L = 10$ ($N = 10^6$), the transition looks second order (Fig. 11). As we increase $L$, the transition begins to slowly shift from second order to first order, and only when $L = 20$ ($N = 4 \times 10^6$), the transition becomes distinctly first order. The explanation of this fact is based on the statistical errors in finite systems. If we remove exactly $N(1-p)$ random sites from the system in an initial attack, it does not mean that the size of the giant component in the lattice after the first stage of the cascade will be exactly $Ng(p)$, where $g(p)$ is the expected value of the giant component in a percolation problem. According to the law of large numbers the size of the giant component will be distributed around $g(p)$ with a standard deviation $\sigma_g \sim 1/\sqrt{N}$. Moreover, the long cascade of failures at $p = p_c$ can be viewed as sequence or iterations approaching the tangential point between the curve $y = pg(x)$ and $y = x$ [8]. If $g(x)$ is changed by an error $\sigma_g$, the root of the equation $x = pg(x)$ will change as $\sqrt{\sigma_g}$, because at the tangential point this equation becomes a quadratic equation with zero discriminant and hence changes in discriminant of the order of $\sigma_g$ will result in the change of the root of the order of $\sqrt{\sigma_g} \sim N^{-1/4}$. Thus, we can expect that the statistical error of the mutual giant component as well as its mean value near the first order transition will decrease with the system size as $N^{-1/4}$. We observe this behavior for all $r$ and $D$, but only for $D = 6$ are these effects are strong enough to affect the apparent order of the collapse transition. For

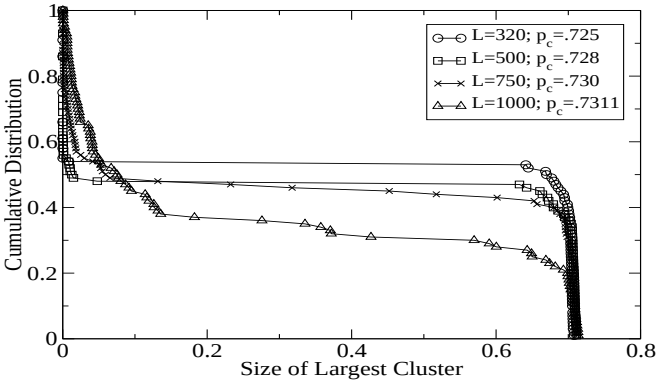



FIG. 8: Plot of the largest cluster distribution for 2D lattice networks of increasing size (with $r = 10$). It can be seen that as the size of the network increases, the type of transition becomes more second order. When $L \geq 750$, the transition becomes second order and approaches the true transition of the 2D lattice.

all other dimensions, the order of the transitions remains the same even when the system size is increased.

Indeed, the probability density function (PDF) of the mutual giant component near the first order phase transition is the derivative of the cumulative distribution and hence the inflection point of the plateau of the cumulative distribution corresponds to the minimum of the PDF. Thus, the PDF of $\mu$ near the first order phase transition is a bimodal distribution with a left peak corresponding to the collapsed states of the system, and a right peak corresponding to the survived states of the system. Figure 12 shows the PDF of $\mu$ (for $D = 6, r = 1$) for various values of $L$. One can see that the right peak becomes sharper as $L$ increases. As the right peak becomes narrower, for $L = 20$ it practically stops overlapping with the left peak, making the distribution clearly first order-like.

Figure 13 shows the standard deviation and mean of the right peak as function of $1/N^{-1/4}$ (equal to $1/L^{-3/2}$ in six dimensions). One can easily see an approximately linear behavior confirming our theory. The different curves correspond to different methods of estimating $\sigma$ and $\mu$. The first method is the direct computation of the average $\mu$ and the variance from realizations of $\mu > \mu_{min}$, where $\mu_{min}$ is the value of the minimum of the PDF. The second method consists of doing a Gaussian fit near the maximum of the right peak of the PDF. In this last case $\sigma$ can be computed from the maximum of the PDF and from the coefficient of the second power of the quadratic polynomial fitting the logarithm of the PDF.

### D. The upper critical dimension

The upper critical dimension of the classical percolation might play an important role in the mutual percolation problem with distance restriction as well. This means that, qualitatively, the behavior of our model for

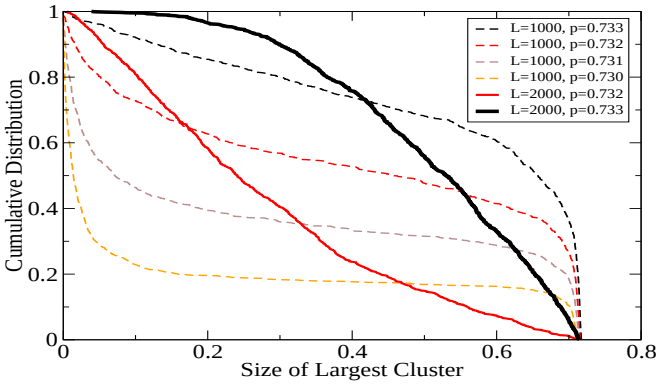



FIG. 9: (Color online) Plot of the largest cluster distribution for a 2D lattice network of size $L = 1000$ and $L = 2000$, with $r = 10$, for different values of $p$. When $L = 2000$, denoted by the thicker lines, the transition is completely second order.

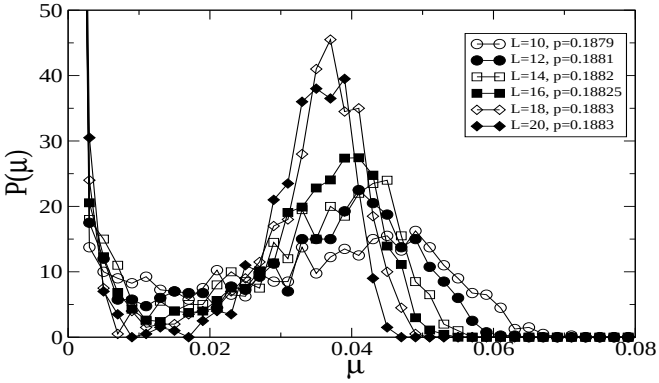



FIG. 12: PDF of $\mu$ for $D = 6$, $r = 1$ for increasing values of $L$ from 10 to 20. One can see that the right peak, corresponding to survived giant component, becomes sharper as $L$ increases.

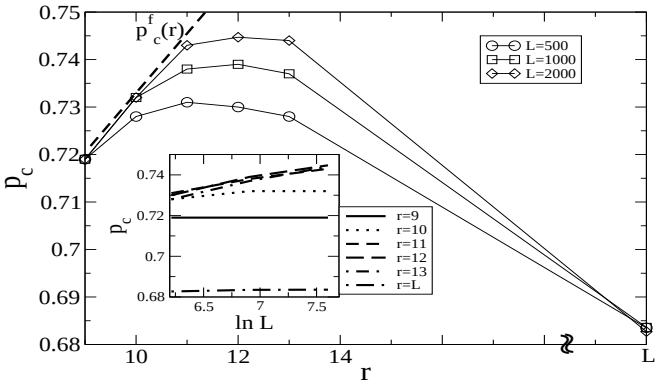



FIG. 10: Behavior of $p_c$ for $D = 2$, as function of $r$ for different system sizes: $L = 500, 1000$ and 2000. The inset shows $p_c$ as function of $\ln(L)$, to test Eq. (3).

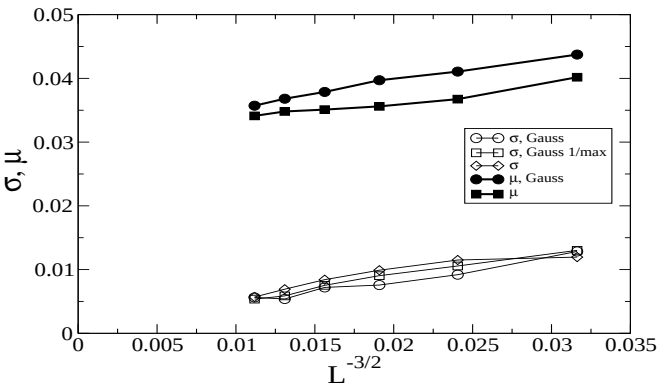



FIG. 13: The average mutual giant component, $\mu$, and its standard deviation $\sigma$, computed for $D = 6$, $r = 1$ as functions of the lattice size $L$ plotted against $N^{-1/4} = L^{-3/2}$. One can see an approximately linear behavior for both quantities. Different curves correspond to different methods of estimating $\sigma$ and $\mu$ as discussed in the text.

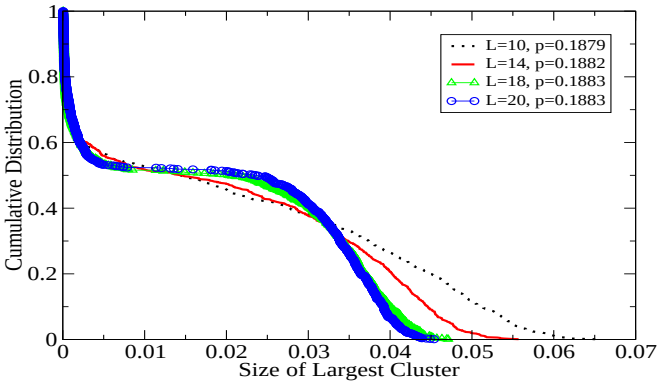



FIG. 11: (Color online) Plot of the largest cluster distribution for a 6D lattice network with $r = 1$ of increasing size. It can be seen that as the size of the network increases, the type of transition becomes more and more first-order-like. When $L = 20$, denoted by the blue circles, the transition becomes completely first order and the finite size effects are no longer present.

$D \geq 6$ should coincide with the behavior of a RR network with $k = 2D = 12$. In this RR network, the first value of $r$ in which there is a first order transition, is $r_I = 1$ [19]. As shown above, when analyzing very large 6D lattice networks for which the finite-size effects become negligible (for $L \geq 20$), the transition at $r = 1$ is first order as well. For $D = 7$ and $L = 10$, the transition for $r = 1$ is a clear first order transition. This supports our hypothesis that the upper critical dimension for percolation plays a role in the problem of mutual percolation with restricted interdependency distance. However, the quantitative difference of the behavior of $p_c$ for lattices and RR graphs gradually decreases with $D$.

## V. CONCLUSION

In our study we confirm that the behavior of the interdependent $D$-dimensional lattices with distance limitation $r$ between the interdependent nodes approaches the behavior of the interdependent RR graphs as $D$ increases. We find that for $D < 6$ there is a value of $r = r_I > 1$ such that for $r \geq r_I$ the cascading failures happen as a discontinuous first order transition, while for $r < r_I$ the transition is a continuous second order transition, as in the classical percolation theory.

We also find that in all dimensions, the interdependent lattices reach maximal vulnerability (largest $p_c$) at a distance $r = r_{max} > r_I$, such that for $r > r_{max}$ the vulnerability starts to decrease as $r \to \infty$. These findings are in qualitative agreement with Li *et al.*[17] who have found that for a lattice of $D = 2$, $r_I = r_{max} = 8$. In this work we find for $D = 2$ $r_I = 11$, $r_{max} = 12$. The quantitative difference between our results can be explained by the fact that we use the chemical distance, or shortest path, as a metric while Li *et al.* use a maximal coordinate difference as a metric. The number of proximal nodes in Li *et al.* for $r = 8$ is hence $(2r+1)^2 = 289$. In our model the number of proximal nodes is $1 + 2r(r+1)$ which for $r = 11$ becomes 265 and for $r = 12$ is 313. Thus in terms of number of proximal nodes the value found Li *et al* for $r_I = r_{max} = 8$ falls exactly in between our values $r_I = 11$ and $r_{max} = 12$.

Note that as $D$ increases, both $r_I$ and $r_{max}$ decrease, but their difference increases. Moreover, the difference between $p_c[D, r_{max}]$ and $p_c[D, \infty]$ decreases exponentially with $D$.

More significantly we find that for $D = 6$ and $r = 1$, the transition is first order. This coincides with RR graphs with $r = 1$ and large $k > 8$. This finding suggests that the upper critical dimension of the classical percolation, $D = 6$, plays an important role in the problem of mutual percolation with distance restrictions.

We also investigate how the nature of the transition change as number of lattice sites $N \to \infty$. We find that when $N$ increases, the value of $p_c$ near the maximum increases logarithmically with $N$, approaching the value of $p_c^f$, the depinning transition of the propagation of the hole perimeter. The problem of the upper critical dimension for this depinning transition and its universality class is an interesting problem, which requires further investigation. $r_I$ and $r_{max}$ have a tendency to increase with $N$; however, this dependence is small and could be observed only for $D = 2$ in our study.

We also showed that when $r$ is close to $r_I$, the value for which the nature of the transition changes, the true order of the transition in the thermodynamic limit can be identified only for very large $N$, which has been determined above. The bimodality of the distribution of the giant component indicated by the inflection point in the cumulative distribution, may either disappear, suggesting that the true nature of the transition for $N \to \infty$ is second order, or can become stronger, indicating that the transition is first order in the thermodynamic limit.

## VI. ACKNOWLEDGMENTS

We wish to thank DTRA #HDTRA1-14-1-0017 for financial support and Shlomo Havlin for stimulating discussions. We acknowledge the partial support of this research through the Dr. Bernard W. Gamson Computational Science Center at Yeshiva College.

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